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PLANE CURVES MEETING AT A POINT WITH HIGH INTERSECTION MULTIPLICITY

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Abstract. As a generalization of an inflection point, we consider a point P on a smooth plane curve C of degree m at which another curve C' of degree n meets C with high intersection multiplicity. Especially, we deal with the existence of two curves of degree m and n meeting at the unique point.

1. Introduction and Preliminaries

Let C_m and C_n be smooth complex projective plane curve of degree m and n, respectively, with $m, n \in \mathbb{N}$. Let P be an intersection point of C_m and C_n. We denote $I(C_m \cap C_n; P)$ the intersection multiplicity at P of two curves C_m and C_n.

For a point P of $C = C_d$ $(d \geq 3)$ and for a general line L passing through P, the intersection multiplicity $I(C \cap L; P)$ is one. If $L = T_P(C)$ is the tangent line of C_d at P then we have $I(C \cap T_P(C); P) \geq 2$ and equality holds for general point P. If $I(C \cap T_P(C); P) = e > 2$, we call P an inflection point of C_d with intersection multiplicity e. In particular, if $I(C \cap T_P(C); P) = d$, we call P a total inflection point of C_d. In this case the tangent line and the curve meet at only one point P by Bezout's theorem.

Existence of an inflection point of high intersection multiplicity helps us to find Weierstrass points on C([1]). The canonical series of a smooth curve C_d is cut out by the system of degree $d-3$ curves, hence $e(d-3)P$ is a special divisor. Thus, if $d \geq 4$ and $e \geq [\frac{d+1}{2}]$, then an inflection point with multiplicity e is a Weierstrass point. More generally, if there exists a curve of C_{d-3} with $I(C_d \cap C_{d-3}; P) \geq g$ where $g = \frac{(d-1)(d-2)}{2}$, then the point P is a Weierstrass point of the curve C_d.

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With this motivation we generalize the notion of an inflection point. We want to find two curves C_m and C_n with an intersection point P with high intersection multiplicity $I(C_m \cap C_n; P)$.

To construct smooth plane curves satisfying our condition, we use the following theorems frequently.

Theorem 1.1 ([5, Bertini's Theorem])**.** *The genereic element of a linear system is smooth away from the base locus of the system.*

Theorem 1.2 ([4, Namba's Lemma])**.** *Let C, C_1 and C_2 be plane curves. If P is a nonsingular point of C, then we have*

$$I(C_1 \cap C_2; P) \geq \min\{I(C \cap C_1; P), I(C \cap C_2; P)\}.$$

Theorem 1.3 ([2, Bezout's Theorem])**.** *Let C_m and C_n be smooth plane curves of degree m and n. Then we have*

$$\sum_{P \in C_m \cap C_n} I(C_m \cap C_n; P) = mn.$$

2. Plane Curves Meeting with Maximal Intersection Multiplicity

At first we give easy examples of smooth plane curves C_m and C_n with $C_m . C_n = mnP$. Throughout this paper, the point P is the origin $(0,0)$ in the affine plane, i.e., the point $(0,0,1)$ in homogeneous coordinate of the projective plane.

Example 2.1. (1) The case $m = 1$:

Let C_1 and C_n be the curves defined by non-homogeneous equations as follows;

$$\begin{cases} C_1 : & y = 0, \\ C_n : & y - x^n + ay^n = 0. \end{cases}$$

Then for general a, the curve C_n is smooth by 1.1 and we have

$$I(C_1 \cap C_n; P) = n.$$

(2) The case $m = 2$:

Let C_2 and $C_n (n \geq 2)$ be the curves defined by non-homogeneous equations as follows;

$$\begin{cases} C_2 : y - x^2 = 0, \\ C_n : (y - x^2)(1 + x^{n-2} + y^{n-2}) + ay^n = 0. \end{cases}$$

Then for general a, the curve C_n is smooth by Bertini's theorem and we have

$$I(C_2 \cap C_n; P) = 2n.$$

(3) A generalization of (1) and (2) :

Let C_m and C_n with $m \leq n$ be the curves defined by non-homogeneous equations as follows;

$$\begin{cases} C_m : y - x^m + y \cdot k(x,y) = 0, \quad \deg k = m-1 \\ C_n : (y - x^m + y \cdot k(x,y))h(x,y) + y^n = 0, \quad \deg h = n-m. \end{cases}$$

Then for general $k(x,y)$ and $h(x,y)$, the curves C_m and C_n are smooth and we have

$$I(C_m \cap C_n; P) = mn.$$

Remark 2.2. In (3) of above example, the point P is a total inflection point of C_m.

Now we are interested in the point P which is a total inflection point of neither C_m nor C_n. To consider such a problem, we prove some theorems concerning to the existence of such curves.

Theorem 2.3. *Let m, n be positive integers. Suppose that there exist smooth curves C_m and C_n such that $I(C_m \cap C_n; P) = mn$. If k is a positive integer such that $kn \geq m$, then there exists a smooth curve C_{kn} such that $I(C_m \cap C_{kn}; P) = kmn$.*

Proof. Consider a linear system $\lambda C_m(1 + x^{kn-m} + y^{kn-m}) + \mu C_n^k$. If $Q(\neq P)$ is contained in the base locus of the linear system $< C_m(1 + x^{kn-m} + y^{kn-m}), C_n^k >$ then Q lies on the curve $1 + x^{kn-m} + y^{kn-m}$ and does not lie on C_m, since C_m and C_n meet only at P. Since Q is a smooth point of the curve $1 + x^{kn-m} + y^{kn-m}$, it is a smooth point of a general member in the system. On the other hand P is a smooth point of C_m and not on the curve $1+x^{kn-m}+y^{kn-m}$, P is a smooth point of a general member of the system. Let C_{kn} be a general member in the linear system. Then, by Bertini's theorem, C_{kn} is a smooth curve. □

Remark 2.4. In fact, we may obtain Example 2.1 (3) from Example 2.1 (1) and Theorem 2.3.

Corollary 2.5. *Let m be any positive integer and n be a positive even integer with $n \geq m$. Then there exist smooth curves C_m and C_n such that $I(C_m \cap C_n; P) = mn$.*

Proof. If $m = 1$, then it follows from Example 2.1 (1). So we assume that $m \geq 2$. Let C_m and C_2 be curves in (2) of Example 2 which satisfy $I(C_m \cap C_2; P) = 2m$.

Then for any even $n \geq m$, there exists a smooth C_n such that $I(C_m \cap C_n; P) = mn$, by above theorem. □

Theorem 2.6. *Let $m > n \geq k$ be natural integers. If C_m, C_n and C_k are smooth curves such that $I(C_m \cap C_n; P) = mn$ and $I(C_m \cap C_k; P) = mk$, then $n = k$ and C_n and C_k are the same curves.*

Proof. By Namba's lemma, $I(C_n \cap C_k; P) \geq mk$ which is bigger than nk, the product of the degrees of C_n and C_k. By Bezout's theorem, C_k and C_n has a common component. However it is impossible since C_k and C_n are smooth and so irreducible unless $C_n = C_k$. □

Corollary 2.7. *Let C_m be a smooth curve. Then there exists at most one smooth curve C_k with $1 \leq k \leq m-1$ such that $I(C_m \cap C_k; P) = mk$.*

Proof. Obvious. □

Theorem 2.8. *Let C_3 and C_3' be distinct smooth cubics such that $I(C_3 \cap C_3'; P) = 9$. Then $I(C_3 \cap C_2; P) \leq 5$ for any irreducible conic C_2.*

Proof. Suppose $I(C_3 \cap C_2; P) = 6$ for some irreducible conic C_2. Then, by Namba's theorem, the point P can not be an inflection point of C_3, and hence $C_3 \cdot T_P C_3 = 2P + Q$ with $P \neq Q$, where $T_P C_3$ is the tangent line to C_3 at P. Then

$$9P = C_3 \centerdot C_3' \sim C_3 \centerdot (C_2 \cdot T_P C_3) = 8P + Q.$$

Thus we have $P \sim Q$, which is a contradiction, since the genus of C_3 is one. □

Theorem 2.9. *Let $C_m (m \geq 3)$ and C_2 satisfies $I(C_m \cap C_2; P) = 2m$. Then there exists no smooth curve C_n of odd degree n such that $I(C_m \cap C_n; P) = mn$.*

Proof. Note that $T_P C_m = T_P C_2$ and $I(C_2 \cap T_P C_2) = 2$.

If $n = 1$ then for any C_1, $I(C_m \cap C_1; P) \leq I(C_m \cap T_P C_m; P) = I(C_2 \cap T_P C_2; P) = 2$, by Namba's lemma.

Suppose that for $n = 2k+1 (k \leq 1)$ there exists a smooth curve C_{2k+1} such that

$$C_m \centerdot C_{2k+1} = m(2k+1)P.$$

On the other hand, since $T_P C_m = T_P C_2$ and $I(C_2 \cap T_P C_2; P) = 2$, we have

$$C_m \centerdot (C_2^k \cdot T_P C_m) = (2km + 2)P + D,$$

where degD = m − 2. Then this implies the existence of the linear series g^1_{m-2} on C_m, which is a contradiction. □

To state a generalization of the above theorem we need notations. Let $i_s := i_s(C_m) = \max\{I(C_m \cap F; P) \mid \mathrm{degF} = \mathrm{s}\}$, and let $I_s(C_m)$ be a curve of degree s such that $i_s(C_m) = I(C_m \cap I_s(C_m); P)$. Note that $I_1(C_m) = T_P C_m$ and $i_1 = I(C_m \cap T_P C_m; P)$.

Theorem 2.10. *Let $m \geq 3$ and let C_m be smooth. Suppose that there exists a smooth curve $C_r (r \geq 2)$ such that $I(C_m \cap C_r; P) = mr$. If P is not a total inflection point of C_m, then, for $k \geq 1$, there is no smooth curve $C_{kr\pm 1}$ such that $I(C_m \cap C_{kr\pm 1}; P) = m(kr \pm 1)$.*

Proof. Suppose that there exists a smooth curve C_{kr+1}[resp. C_{kr-1}]. Then

$$C_m \centerdot C_{kr+1} = m(kr+1)P.$$
$$[\text{resp. } C_m \centerdot (C_{kr-1} \cdot T_P C_m) = (m(kr-1) + i_1)P + D.]$$

On the other hand,

$$C_m \centerdot (C_r^k \cdot T_P C_m) = (m(kr) + i_1)P + D,$$
$$[\text{resp. } C_m \centerdot C_r^k = m(kr)P,]$$

where D is the divisor such that $C_m \centerdot T_P C_m = i_1 P + D$, hence its degree is $m - i_1$ which satisfies $1 \leq m - i_1 \leq m - 2$. Comparing two divisors, we conclude that there exists a linear series $g^1_{m-i_1}$ on C_m, which is impossible on a smooth plane curve of degree m. □

Theorem 2.11. *Let $m \geq 7$ and let C_m be smooth. Suppose that there exists a smooth curve C_r with $2 < r < \frac{m}{2}$ such that $I(C_m \cap C_r; P) = mr$. Then, for $k \geq 1$, there is no smooth curve $C_{kr\pm 2}$ such that $I(C_m \cap C_{kr\pm 2}; P) = m(kr \pm 2)$.*

Proof. Suppose that there exists such a curve C_{kr+2}[resp. C_{kr-2}]. Then

$$C_m \centerdot C_{kr+2} = m(kr+2)P.$$
$$[\text{resp. } C_m \centerdot (C_{kr-2} \cdot I_2(C_m)) = (m(kr-2) + i_2)P + D.]$$

On the other hand,

$$C_m \centerdot (C_r^k \cdot I_2(C_m)) = (m(kr) + i_2)P + D,$$
$$[\text{resp. } C_m \centerdot C_r^k = m(kr)P,]$$

where D is the divisor such that $C_m \centerdot I_2(C_m) = i_2 P + D$, hence its degree is $2m - i_2$. Comparing two divisors, we conclude that there exists a linear series $g^1_{2m-i_2}$. By Namba's lemma and since the dimension of conics is 5, we have $5 \leq i_2 \leq 2r \leq m$.

By Coppens' results([3]), we can not have such a linear series on a smooth plane curve of degree m. □

Theorem 2.12. *Let $m \geq s^2+2$ and let C_m be smooth. Suppose that there exists a smooth curve C_r with $s < r < \frac{m}{s}$ such that $I(C_m \cap C_r; P) = mr$ and $i_s(C_m) \geq s^2+1$. Then, for $k \geq 1$, there is no smooth curve $C_{kr\pm s}$ such that $I(C_m \cap C_{kr\pm s}; P) = m(kr \pm s)$.*

Proof. Suppose that there exists such a curve C_{kr+s}[resp. C_{kr-s}]. Then

$$C_m \centerdot C_{kr+s} = m(kr+s)P.$$
$$[\text{resp. } C_m \centerdot (C_{kr-s} \cdot I_s(C_m)) = (m(kr-s)+i_s)P + D.]$$

On the other hand,

$$C_m \centerdot (C_r^k \cdot I_s(C_m)) = (m(kr)+i_s)P + D,$$
$$[\text{resp. } C_m \centerdot C_r^k = m(kr)P,]$$

where D is the divisor such that $C_m \centerdot I_s(C_m) = i_sP + D$, hence its degree is $sm - i_s$. Comparing two divisors, we conclude that there exists a linear series $g^1_{sm-i_s}$. By Namba's lemma and our assumption, we have $s^2+1 \leq i_s \leq sr < m$. By Coppens' results([3]), we can not have such a linear series on a smooth plane curve of degree m. □

3. Some Examples

Now we give examples of C_m and C_n meeting at the unique point P which is not an inflection point of any curve and $I(C_m \cap C_n; P) = mn$.

If a smooth curve C passing through the origin $P(0,0)$ is given by the equation

$$C : y - x^2 + y \cdot k(x,y) + h(x) = 0,\ deg(k) \geq 1,\ deg(h) \geq 3$$

then $T_P(C)$ is given by the equation $y = 0$ and $I(C \cap T_P(C); P) = 2$ so P is not an inflection point of C.

We found examples using the mathematics package, Maple.

Example 3.1. The case $m = 3$ and $n = 3, 4$ *or* 6.
(1) $m = n = 3$: Let C_3 and C_3' be given by the equations

$$\begin{cases} C_3: & A(x,y) = y - x^2 + xy^2 = 0 \\ C_3': & B(x,y) = y - x^2 + xy - x^3 + xy^2 + y^3 = 0. \end{cases}$$

Then the equation for C_3 and C_3' satisfies

$$\begin{aligned}B(x,y) =& y - x^2 + xy - x^3 + xy^2 + y^3 \\ =& (y - x^2 + xy^2) + x(y - x^2 + xy^2) + y^2(y - x^2 + xy^2) - xy^4 \\ =& (y - x^2 + xy^2)(1 + x + y^2) - xy^4 \\ =& A(x,y)(1 + x + y^2) - xy^4\end{aligned}$$

so

$$I(C_3 \cap C_3'; P) = I(C_3 \cap xy^4; P) = I(C_3 \cap x; P) + 4I(C_3 \cap y; P) = 9.$$

Also we can represent C_3 and C_3' using a parameter t as follows

$C_3 : \quad (t, t^2 - t^5 + 2t^8 - 5t^{11} + 14t^{14} - 42t^{17} + \dots)$

$C_3' : \quad (t, t^2 - t^5 + 2t^8 + t^9 - t^{10} - 4t^{11} - 7t^{12} + \dots)$

and we get $I(C_3 \cap C_3'; P) = 9$ again.

In fact the resultant of C_3 and C_3' is x^9 so $I(C_3 \cap C_3'; P) = 9$.

(2) $m = 3$ and $n = 4$: Let C_3 and C_4 be given by the equations

$$\begin{cases} C_3 : & A(x,y) = y - x^2 + y^3 = 0 \\ C_4 : & B(x,y) = (y - x^2 + y^3) + (y - x^2)^2 = 0. \end{cases}$$

Then

$$\begin{aligned}B(x,y) &= (y - x^2 + y^3) + (y - x^2)^2 \\ &= (y - x^2 + y^3) + (y - x^2 + y^3)(y - x^2) - (y - x^2 + y^3)y^3 + y^6 \\ &= A(x,y)(1 + y - x^2 - y^3) + y^6\end{aligned}$$

so

$$I(C_3 \cap C_4; P) = I(C_3 \cap y^6; P) = 6I(C_3 \cap y; P) = 12.$$

We can represent C_3 and C_4 with a parameter t as follows

$C_3 : \quad (t, t^2 - t^6 + 3t^{10} - 12t^{14} + 55t^{18} + \dots)$

$C_4 : \quad (t, t^2 - t^6 + 3t^{10} - t^{12} - 12t^{14} + 9t^{16} + 53t^{18} + \dots)$

so $I(C_3 \cap C_4; P) = 12$ again.

In fact the resultant of C_3 and C_4 is x^{12} so $I(C_3 \cap C_4; P) = 12$.

(3) $m = 3$ and $n = 6$: Using C_3 and C_3' in (1), we construct curves for $m = 3$ and $n = 6$.

$$\begin{cases} C_3 : & y - x^2 + xy^2 = 0 \\ C_6 : & (y - x^2 + xy^2)(1 + x^3 + y^3) + (y - x^2 + xy - x^3 + xy^2 + y^3)^2 = 0 \end{cases}$$

With the similar method as above we have

$$I(C_3 \cap C_6; P) = I(C_3 \cap {C_3'}^2; P) = 2I(C_3 \cap C_3'; P) = 18.$$

Also the resultant of C_3 and C_6 is x^{18}.

Example 3.2. The case $m = 3$ and $n = 5$.

Let C_3 and C_5 be given by the equations

$$\begin{cases} C_3: & y - x^2 + xy^2 = 0 \\ C_5: & y - x^2 - 2y^3 + x^4 + 4x^2y^2 + xy^3 - 2y^4 + x^5 - 3x^4y - 4x^3y^2 - y^5 = 0. \end{cases}$$

We can see that C_3 and C_5 are smooth plane curves and the resultant of C_3 and C_5 is x^{15} by Maple so $I(C_3 \cap C_5; P) = 15$.

Example 3.3. The case $m = 3$ and $n = 7$.

Let C_3 and C_7 be given by the equations

$$\begin{cases} C_3: & y - x^2 + (-2 + 2\sqrt{5})y^2 + 2x^3 = 0 \\ C_7: & 80\sqrt{5}xy^2 + 2320\sqrt{5}x^5y^2 - 3240\sqrt{5}xy^4\sqrt{5} + 2760\sqrt{5}x^3y^3 - 6944\sqrt{5}x^2y^4 \\ & -1088\sqrt{5}xy^5 + 384\sqrt{5}xy^6 - 223\sqrt{5}x^2y^2 - 40\sqrt{5}x^2y - 512\sqrt{5}x^2y^5 \\ & -612\sqrt{5}x^2y^3 + 1472\sqrt{5}y^6 - 14y - 384xy^6 + 512y^7 - 28x^3 - 2624y^6 + 14x^2 \\ & +14944x^2y^4 - 10256\sqrt{5}y^5 + 23152y^5 + 88x^2y - 5880x^3y^3 + 495x^2y^2 - 6\sqrt{5}x^2 \\ & +1332x^2y^3 + 6\sqrt{5}x^7 - 176xy^2 - 15\sqrt{5}y^3 + 6\sqrt{5}y + 12\sqrt{5}x^3 + 2176xy^5 \\ & +142\sqrt{5}x^4y - 14x^7 + 37y^3 + 7016xy^4 + 640x^2y^5 - 308x^4y - 5360x^5y^2 = 0. \end{cases}$$

We can see that C_3 and C_7 are smooth and that the resultant of C_3 and C_7 is $33554432x^{21}$ using Maple. So we get C_3 and C_7 with $I(C_3 \cap C_7; P) = 21$ which is the maximal possible intersection multiplicity that two smooth plane curves of degree 3 and 7 can have.

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