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ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN β-HOMOGENEOUS F-SPACES

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Abstract. In this paper, we solve the additive ρ-functional inequalities

$$\|f(2x-y)+f(y-x)-f(x)\| \le \|\rho\,(f(x+y)-f(x)-f(y))\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho|<1$, and

$$\|f(x+y)-f(x)-f(y)\| \le \|\rho(f(2x-y)+f(y-x)-f(x))\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho|<\frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in β-homogeneous F-spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [22] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain

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E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

Definition 1.1. Let X be a (complex) linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;

(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;

(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ and $(X, \|\cdot\|)$ is called a β-homogeneous F-space (see [16]).

In Section 2, we solve the additive ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.1) in β-homogeneous F-space.

In Section 3, we solve the additive ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (0.2) in β-homogeneous F-space.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1-homogeneous F-space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous F-space with norm $\|\cdot\|$.

2. ADDITIVE ρ-FUNCTIONAL INEQUALITY (0.1) IN β-HOMOGENEOUS F-SPACES

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive ρ-functional inequality (0.1) in β-homogeneous F-spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\|f(2x - y) + f(y - x) - f(x)\| \leq \|\rho\,(f(x + y) - f(x) - f(y))\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = 0$ and $y = 0$ in (2.1), we get $\|f(0)\| \le \|\rho(f(0))\|$ and so $f(0) = 0$ with $|\rho| < 1$.

Letting $x = 0$ in (2.1), we get $\|f(-y) + f(y)\| \le 0$ and so f is an *odd mapping*.

Letting $x = z$ and $y = z - w$ in (2.1), we get

$$\|f(z+w) - f(z) - f(w)\| \le \|\rho(f(2z-w) + f(w-z) - f(z))\| \tag{2.2}$$

for all $z, w \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|f(2x-y) + f(y-x) - f(x)\| &\le \|\rho(f(x+y) - f(x) - f(y))\| \\ &\le |\rho|^2 \|f(2x-y) + f(y-x) - f(x)\|\end{aligned}$$

and so $f(2x - y) + f(y - x) = f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in β-homogeneous F-spaces.

Theorem 2.2. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}&\|f(2x-y) + f(y-x) - f(x)\| \\ &\qquad \le \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r)\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}} \|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$, in (2.3), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = 0$ in (2.3), we get

$$\|f(2x) + f(-x) - f(x)\| \le \theta \|x\|^r \tag{2.5}$$

for all $x \in X$.

Letting $x = 0$ in (2.3), we get

$$\|f(y) + f(-y)\| \le \theta \|y\|^r \tag{2.6}$$

for all $y \in X$.

From (2.5) and (2.6), we get

$$\begin{aligned} \|f(2x)-2f(x)\| &\le \|f(2x)+f(-x)-f(x)\|+\|f(x)+f(-x)\| \\ &\le 2\theta\|x\|^r \end{aligned} \tag{2.7}$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{2^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} \|A(2x-y)+A(y-x)-A(x)\| &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{2x-y}{2^n}\right)+f\left(\frac{y-x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|2^n\rho\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad + \lim_{n\to\infty}\frac{2^{\beta_2 n}}{2^{\beta_1 r n}}\theta(\|x\|^r+\|y\|^r) \\ &= \|\rho\left(A\left(x+y\right)-A(x)-A(y)\right)\| \end{aligned}$$

for all $x, y \in X$. So

$$\|A(2x-y)+A(y-x)-A(x)\| \le \|\rho(A(x+y)-A(x)-A(y))\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned} \|A(x)-T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right)-2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\|+\left\|2^q T\left(\frac{x}{2^q}\right)-2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_1 r}-2^{\beta_2}}\frac{2^{\beta_2 q}}{2^{\beta_1 q r}}\|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired. □

Theorem 2.3. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying* (2.3). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{2.9}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{2}{2^{\beta_2}}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \frac{2}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r j}}{2^{\beta_2 j}}\theta\|x\|^r \qquad (2.10) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Remark 2.4. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β-homogeneous real F-space, then all the assertions in this section remain valid.

3. Additive ρ-functional Inequality (0.2) in β-homogeneous F-spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive ρ-functional inequality (0.2) in β-homogeneous F-spaces.

Lemma 3.1. *If a mapping* $f : X \to Y$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \|\rho(f(2x-y) + f(y-x) - f(x))\| \tag{3.1}$$

for all $x, y \in X$*, then* $f : X \to Y$ *is additive.*

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so

$$2f(x) = f(2x) \tag{3.2}$$

for all $x \in G$.

Letting $y = 2x$ in (3.1), we get $\|f(3x) - f(x) - f(2x)\| \leq 0$ and from (3.2),

$$3f(x) = f(3x) \tag{3.3}$$

for all $x \in X$.

Letting $y = -x$ in (3.1), we get $\|f(x) + f(-x)\| \leq \|\rho(f(3x) + f(-2x) - f(x))\|$. From (3.2) and (3.3), $f(3x)+f(-2x)-f(x) = 2f(x)+2f(-x)$, so $\|f(x) + f(-x)\| \leq 0$, and we get

$$f(x) + f(-x) = 0 \tag{3.4}$$

for all $x \in X$. So f is an *odd mapping*.

Letting $x = z, y = z - w$ in (3.1), we get

$$\|f(2z-w) - f(z) - f(z-w)\| \leq \|\rho(f(z+w) + f(-w) - f(z))\|$$

and from (3.4),

$$\|f(2z-w) + f(w-z) - f(z)\| \leq \|\rho(f(z+w) - f(z) - f(w))\| \tag{3.5}$$

for all $z, w \in X$.

It follows from (3.1) and (3.5) that

$$\begin{aligned} \|f(x+y) - f(x) - (y)\| &\leq \|\rho(f(2x-y) + f(y-x) - f(x))\| \\ &\leq |\rho|^2 \|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So f is *additive*. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in β-homogeneous F-spaces.

Theorem 3.2. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| \\ \leq \|\rho(f(2x-y) + f(y-x) - f(x))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (3.6), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{3.8}$$

for all $x \in X$. So

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta\|x\|^r \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying* (3.4). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{3.10}$$

for all $x \in X$.

Proof. It follows from (3.8) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2}{2^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \frac{2}{2^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{2^{\beta_2 j}} \theta \|x\|^r \end{aligned} \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.11) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Remark 3.4. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a β-homogeneous real F-space, then all the assertions in this section remain valid.

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