ALGEBRAIC ENTROPIES OF NATURAL NUMBERS WITH ONE OR TWO PRIME FACTORS

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Abstract. We formulate the additive entropy of a natural number in terms of the additive partition function, and show that its multiplicative entropy is directly related to the multiplicative partition function. We give a practical formula for the multiplicative entropy of natural numbers with two prime factors. We use this formula to analyze the comparative density of additive and multiplicative entropy, prove that this density converges to zero as the number tends to infinity, and empirically observe this asymptotic behavior.

1. Introduction

In statistical physics, entropy S is a measure of the size of all the possible microstates of a system, and S can be expressed as $k \cdot \ln \Omega$, where k is Boltzmann's constant and Ω is the number of microstates [1]. In information theory, following Shannon [2], entropy $H(X)$ measures the amount of information in a message, and $H(X)$ can be expressed as $-\sum_{i=1}^{n} p(x_i) \cdot \log p(x_i)$, where p is the probability mass function of a discrete random variable X [2]. It is well known that physical and information entropy are equivalent.

In mathematics, the set of natural numbers, $\mathbb{N}$, conceals information in the form of arithmetic relations. Each natural number is endowed with such information by the two fundamental algebraic operations on $\mathbb{N}$, addition and multiplication. This information can be extracted by using mathematical logic to elucidate the relations between natural numbers. Can the concept of entropy be used to measure the amount of mathematical information (or the number of microstates) inherent in natural numbers? In this paper, we answer this question.

Since Patrick Billingsley's lectures in 1973 [3], there have been several efforts [4, 5, 6] to apply the idea of entropy, and other information-theoretic concepts, to number theory. Shannon's entropy function H was recently applied [6] to $\mathbb{N}$, yielding the concept of the entropy of a natural number. However, this form of entropy does not measure the amount of information inherent in natural numbers, but is rather an aid to understanding effects such as the distribution of prime numbers. In this paper we will consider natural numbers as repositories of quantitative information originating from algebraic operations, and explicitly measure the amount of quantitative information stored in these numbers.

The mathematical properties of $\mathbb{N}$ primarily stand on two algebraic operations, addition and multiplication. Addition gives $\mathbb{N}$ a structure of order, in the sense that it arranges natural numbers as follows: for any a, $b \in \mathbb{N}$, $a < b \Leftrightarrow$ if $\exists\, k \in \mathbb{N}$ such that $b = a + k$. Multiplication provides a cladogram of numbers, in the sense that factorization reveals how a natural number is made up of its most elementary factors, which are prime numbers. Thus we can characterize each natural number n in $\mathbb{N}$ in terms of the information incorporated in:

    (i) the possible ways of generating n by adding elements of $\mathbb{N}$, and
    (ii) the possible ways of generating n by multiplying elements of $\mathbb{N}$.

We call the amount of this information, the additive and multiplicative entropy respectively. We will show that additive entropy is easily measured by the additive partition function, and that multiplicative entropy is directly related to the multiplicative partition function. This project is strongly connected to previous work on the additive partition function [7, 8] and on the multiplicative partition function [9, 10, 11, 12, 13]. In particular, our approach is closely related to previous work [11, 12, 13] on the unordered factorization of natural numbers, in the sense that the multiplicative entropy of a natural number has to be computed by such factorization. In this paper we will present a computable formula for the multiplicative entropy of natural numbers with two prime factors, and introduce the comparative density of two algebraic entropies, and use this to analyze their asymptotic behavior. Moreover, we will prove that this density asymptotically converges to zero as the number approaches infinity, and empirically analyze the speed of this convergence.

The remainder of this paper is organized as follows: In Section 2, we introduce the additive entropy and multiplicative entropy of a natural number and their comparative density. In Section 3, we compute the multiplicative entropy and the

comparative density of natural numbers with two prime factors. We also prove that this density converges to zero as the number approaches infinity. We demonstrate that additive and multiplicative entropy have a logarithmic relationship, and observe the asymptotic behavior of their comparative density. Finally in Section 4, we summarize our results and propose some directions for further study.

2. Two Fundamental Entropies of a Natural Number

For a natural number n in $\mathbb{N}$, we can introduce two algebraic entropies, additive entropy and multiplicative entropy.

Let $p(n)$ be the partition function representing the number of possible partitions of a natural number n by addition, which is to say the number of distinct ways of producing n by adding natural numbers. Then we can define additive entropy:

Definition 1. For a natural number n, $A(n)$ is given by

$$(1) \qquad A(n) = k_a \ln p(n), \text{ where } k_a \text{ is a constant.}$$

Let $\mathcal{M}(n)$ be the number of distinct ways of producing n by multiplying natural numbers. Then we can define multiplicative entropy $P(n)$ as follows:

Definition 2. For a natural number n, $P(n)$ is given by

$$(2) \qquad P(n) = k_m \ln \mathcal{M}(n), \text{ where } k_m \text{ is a constant.}$$

The comparative density $d_{rc}(n)$ of the algebraic entropies $A(n)$ and $P(n)$ can be expressed as follows:

Definition 3. For a natural number n, $d_{rc}(n)$ is given by

$$(3) \qquad d_{rc}(n) = \frac{P(n)}{\ln A(n)} = \frac{k_m \ln \mathcal{M}(n)}{\ln(k_a \ln p(n))}.$$

Remark 1. From now on, following Landau's treatment of the constant k in the definition of thermodynamic entropy $k \ln \Omega$, we will assume that $k_a = k_m = 1$.

3. Entropies of Natural Numbers with Few Prime Factors

3.1. Entropies of natural numbers with one prime factor

Lemma 1. *Let $N = q^m$, where q is a prime number and $m \in \mathbb{N}$. Then the multiplicative entropy of N is given by*

(4) $$P(N) = \ln p(m).$$

Proof. $N = N_1 \cdot N_2 = q^{k_1} \cdot q^{k_2}$, with $k_1 + k_2 = m$, which implies that $\mathcal{M}(N) = p(m)$. Consequently, we obtain $P(N) = \ln p(m)$. □

Using Definition 1, it follows from Eq. (4) that $P(N) = A(m)$, when $N = q^m$. Moreover, since $m = \frac{\ln N}{\ln q}$, we additionally obtain $P(N) = A\left(\frac{1}{\ln q} \cdot \ln N\right)$. This implies that, in the simplest case where $N = q^m$, our two algebraic entropies are logarithmically related.

Theorem 1. *Let $N = q^m$, where q is a prime number and $m \in \mathbb{N}$. Then the comparative density of N is given by*

(5) $$d_{rc}(N) = \frac{\ln p(m)}{\ln(\ln p(q^m))} = \frac{\ln p\left(\frac{1}{\ln q} \cdot \ln N\right)}{\ln(\ln p(N))}.$$

Proof. It is apparent from Lemma 1 and Definition 3. □

Theorem 2. *Let $N = q^m$, where q is a prime number and $m \in \mathbb{N}$. Then*

$$\lim_{N \to \infty} d_{rc}(N) = 0 \ \text{ for any prime number } q, \ \text{ and}$$

(6) $$\ln d_{rc}(N) \sim -\frac{1}{2}\ln m + \ln\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\ln q}\right) \ \text{ for a sufficiently large } m.$$

Proof. From Eq. (5) we have

(7) $$\lim_{N \to \infty} d_{rc}(N) = \lim_{N \to \infty} \frac{\ln p\left(\frac{1}{\ln q} \cdot \ln N\right)}{\ln(\ln p(N))}$$
$$= \lim_{m \to \infty} \frac{\ln p(m)}{\ln(\ln p(q^m))}.$$

Since $p(s) \sim \frac{1}{4s\sqrt{3}}\exp\left(\pi\sqrt{\frac{2s}{3}}\right)$ for a sufficiently large $s \in \mathbb{N}$, we can compute

$$\lim_{m \to \infty} \frac{\ln p(m)}{\ln(\ln p(q^m))} = 0, \ \text{ which further implies that } \ \lim_{m \to \infty} d_{rc}(N) = 0, \ \text{ and}$$

$$\ln d_{rc}(N) \sim -\frac{1}{2}\ln m + \ln\left(\frac{2\sqrt{\pi}}{\sqrt{3}\ln q}\right) \ \text{ for a sufficiently large } m.$$

□

Figure 1 shows the behavior of $d_{rc}(N)$ and $\ln(d_{rc}(N))$ when $q = 2$, which provides numerical confirmation of Theorem 2.
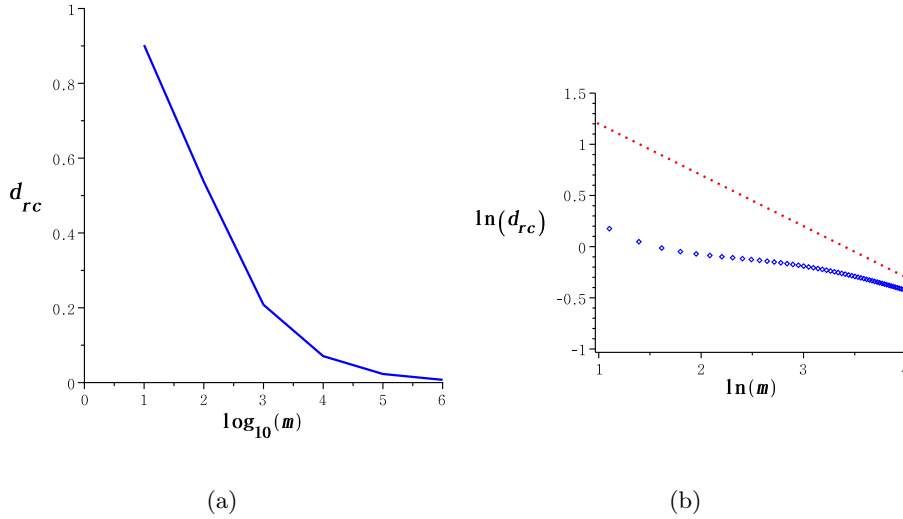
Figure 1.   Behavior of d_{rc} and $\ln(d_{rc})$, when $q = 2$.

We can also deduce from Eq. (6) that, for a sufficiently large m,

$$A(N) \sim \exp\left(P(N) \cdot \exp\left(\frac{1}{2} \ln m - \ln\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\ln q} \right) \right) \right)$$

(8)
$$= \exp\left(P(N) \cdot \left(\frac{\sqrt{3}\ln q}{2\sqrt{2}\pi} \cdot \sqrt{m} \right) \right),$$

which relates the additive and multiplicative entropy for the class of natural numbers given by $N = q^m$.

3.2. Entropies of natural numbers with two prime factors

Now we consider the more complicated case of a natural number $N = q_1^m \cdot q_2^n$, where q_1, q_2 are prime numbers and $m, n \in \mathbb{N}$.

Definition 4. Let N be a natural number given by $N = q_1^m \cdot q_2^n$, where q_1, q_2 are prime numbers and $m, n \in \mathbb{N}$. A factor $q_1^i \cdot q_2^j$ of N, denoted by $[q_1^i, q_2^j]$, is said to be *hybrid* if $i \neq 0$ and $j \neq 0$.

Definition 5. Consider the sets of natural numbers $A = \{a_1, a_2, \cdots, a_{m-1}, a_m\}$ and $B = \{b_1, b_2, \cdots, b_{n-1}, b_n\}$, and let k be a number. Then we define the product $k \cdot (A \times B)$ as follows:

$$k \cdot (A \times B) = \{k \cdot a_{s_1} \cdot b_{s_2}\}, \ \text{where} \ 1 \leq s_1 \leq m, \ 1 \leq s_2 \leq n.$$

Lemma 2. *Let* $N = q_1^m \cdot q_2^n$, *where* q_1 *and* q_2 *are prime numbers and* $m, n \in \mathbb{N}$, *and let* $\mathcal{H}_0(N)$ *be the number of distinct ways of producing* N *by multiplying non-hybrid factors of* N. *Then*

(9) $$\mathcal{H}_0(N) = p(m) \cdot p(n).$$

Proof. Let Q_1 and Q_2 be sets of distinct ways of producing $N_1 = q_1^m$ and $N_2 = q_2^n$ by multiplication. Then, by Lemma 1, we have $\mathcal{M}(N_1) = |Q_1| = p(m)$ and $\mathcal{M}(N_2) = |Q_2| = p(n)$. Since there are $Q_1 \times Q_2$ distinct ways of producing $N = N_1 \cdot N_2$ by multiplying non-hybrid factors of N, we obtain $\mathcal{H}_0(N) = |Q_1 \times Q_2| = |Q_1| \cdot |Q_2| = p(m) \cdot p(n)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 3. *For a natural number* $N = q_1^m \cdot q_2^n$, *we define*

$$d_{rc}^0(N) = \frac{\ln(\mathcal{H}_0(N))}{\ln(A(N))}.$$

Then, $\lim_{N \to \infty} d_{rc}^0(N) = 0$, *and moreover*

$$d_{rc}^0(N) \sim \pi\sqrt{\frac{2}{3}} \cdot \frac{(\sqrt{m} + \sqrt{n})}{\left(\frac{m}{2} \cdot \ln q_1 + \frac{n}{2} \cdot \ln q_2\right)} \quad \text{for sufficiently large } m \text{ and } n.$$

Proof. Since

(10)
$$\begin{aligned} d_{rc}^0(N) &= \frac{\ln(\mathcal{H}_0(N))}{\ln(A(N))} = \frac{\ln(p(m) \cdot p(n))}{\ln(\ln p(q_1^m \cdot q_2^n))} \\ &= \frac{\ln(p(m))}{\ln(\ln p(q_1^m \cdot q_2^n))} + \frac{\ln(p(n))}{\ln(\ln p(q_1^m \cdot q_2^n))}, \end{aligned}$$

and using $p(s) \sim \frac{1}{4s\sqrt{3}} \exp\left(\pi\sqrt{\frac{2s}{3}}\right)$ with a sufficiently large s, we obtain

$$\lim_{N \to \infty} d_{rc}^0(N) = 0.$$

In addition, for sufficiently large value of m and n, we can also write

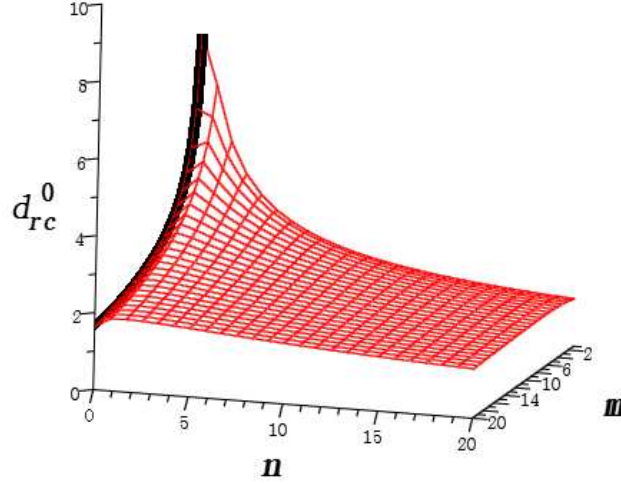$$\frac{\ln(p(m))}{\ln(\ln p(q_1^m \cdot q_2^n))} + \frac{\ln(p(n))}{\ln(\ln p(q_1^m \cdot q_2^n))}$$

Figure 2. Behavior of d_{rc}^0 when $q_1 = 2$ and $q_2 = 3$. The black curve is the graph of d_{rc} from Figure 1 (a).

$$\sim \frac{\ln(\frac{1}{4m\sqrt{3}} \exp\left(\pi\sqrt{\frac{2m}{3}}\right))}{\ln\left(\ln\left(\frac{1}{4\sqrt{3}q_1^m \cdot q_2^n} \exp\left(\pi\sqrt{\frac{2}{3}} \cdot q_1^{\frac{m}{2}} \cdot q_2^{\frac{n}{2}}\right)\right)\right)}$$

$$+ \frac{\ln(\frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right))}{\ln\left(\ln\left(\frac{1}{4\sqrt{3}q_1^m \cdot q_2^n} \exp\left(\pi\sqrt{\frac{2}{3}} \cdot q_1^{\frac{m}{2}} \cdot q_2^{\frac{n}{2}}\right)\right)\right)}$$

$$\sim \pi\sqrt{\frac{2}{3}} \cdot \frac{\sqrt{m}}{\left(\frac{m}{2} \cdot \ln q_1 + \frac{n}{2} \cdot \ln q_2\right)} + \pi\sqrt{\frac{2}{3}} \cdot \frac{\sqrt{n}}{\left(\frac{m}{2} \cdot \ln q_1 + \frac{n}{2} \cdot \ln q_2\right)}.$$

And hence, from Eq. (10), we finally obtain

$$(11) \qquad\qquad d_{rc}^0(N) \sim \pi\sqrt{\frac{2}{3}} \cdot \frac{(\sqrt{m} + \sqrt{n})}{\left(\frac{m}{2} \cdot \ln q_1 + \frac{n}{2} \cdot \ln q_2\right)}.$$

$\square$

$d_{rc}^0(N)$ can be considered as a *partial* comparative density of algebraic entropies of N, because $d_{rc}^0(N)$ is the ratio of the multiplicative entropy of a particular subset of all the distinct ways of producing N by multiplication to the logarithm of the additive entropy of N. Figure 2 illustrates that $d_{rc}^0(N)$ is a natural extension of $d_{rc}(N_1 = q_1^m)$, and we can formalize this:

Remark 2. If the magnitude of m dominates that of n, then we can rewrite Eq. (11) as follows:

$$\ln(d_{rc}^0(N)) \sim \ln\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\ln q_1} \cdot m^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\ln m + \ln\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\ln q_1}\right),$$

which is the result established by Theorem 2. (Conversely, when n dominates m, we obtain $\ln(d_{rc}^0(N)) \sim -\frac{1}{2}\ln n + \ln\left(\frac{2\sqrt{2}\pi}{\sqrt{3}\ln q_2}\right)$.) Furthermore, Eq. (11) also implies

$$A(N) = \exp\left(P(N) \cdot \left(\frac{\sqrt{3}}{2\sqrt{2}\pi}\right) \cdot \left(\frac{m\ln q_1 + n\ln q_2}{\sqrt{m} + \sqrt{n}}\right)\right),$$

which is an extension of Eq. (8). These results signify that Eq. (11) is a natural extension of Eq. (6).

Lemma 3. *For a natural number $N = q_1^m \cdot q_2^n$, let $\mathcal{H}_1^{(i,j)}(N)$ be the number of distinct ways of producing N by multiplication with only one hybrid factor $[q_1^i, q_2^j]$ of N. Then $\mathcal{H}_1^{(i,j)}(N)$ is given by*

$$(12) \qquad \mathcal{H}_1^{(i,j)}(N) = p(m-i) \cdot p(n-j).$$

Proof. Let Q_1^i and Q_2^j be sets of distinct ways of producing $N_1^i = q_1^{m-i}$ and $N_2^j = q_2^{n-j}$ by multiplication. Then we have $\mathcal{M}(N_1^i) = |Q_1^i| = p(m-i)$ and $\mathcal{M}(N_2^j) = |Q_2^j| = p(n-j)$. Note that the set of distinct ways of producing N by multiplying non-hybrid factors of N_1^i and N_2^j together with the hybrid factor $[q_1^i, q_2^j]$ is given by $[q_1^i, q_2^j] \cdot (Q_1^i \times Q_2^j)$. Thus, we obtain $\mathcal{H}_1^{(i,j)}(N) = |Q_1^i \times Q_2^j| = |Q_1^i| \cdot |Q_2^j| = p(m-i) \cdot p(n-j)$. $\qquad \square$

Theorem 4. *For a natural number $N = q_1^m \cdot q_2^n$, let $\mathcal{H}_1^*(N)$ be the number of possible ways of producing N distinctly by multiplication with only one hybrid factor of N. Then $\mathcal{H}_1^*(N)$ is given by*

$$(13) \qquad \mathcal{H}_1^*(N) = \sum_{i=1}^{m} p(m-i) \cdot \sum_{j=1}^{n} p(n-j).$$

Proof. Since $\mathcal{H}_1^*(N) = \sum_{i=1}^m \sum_{j=1}^n \mathcal{H}_1^{(i,j)}(N)$, by the previous lemma, we have

$$\mathcal{H}_1^*(N) = \sum_{i=1}^m \sum_{j=1}^n \mathcal{H}_1^{(i,j)}(N)$$

$$= \sum_{i=1}^m \sum_{j=1}^n p(m-i) \cdot p(n-j)$$

$$= \sum_{i=1}^m p(m-i) \cdot \sum_{j=1}^n p(n-j).$$

$\square$

We will now introduce some necessary new notation.

Definition 6. Given two sets of indices $I = \{i_1, i_2, \cdots, i_l\}$ and $J = \{j_1, j_2, \cdots, j_l\}$, we define the operator $\odot_l$ such that

$$I \odot_l J = \{(i_1, j_1;\ i_2, j_2;\ \cdots\ ;\ i_l, j_l)\}.$$

In addition, consider $\lambda_1, \lambda_2 \in I \odot_l J$ given by $\lambda_1 = (i_1, j_1;\ i_2, j_2;\ \cdots\ ;\ i_l, j_l)$ and $\lambda_2 = (i_1^*, j_1^*;\ i_2^*, j_2^*;\ \cdots\ ;\ i_l^*, j_l^*)$. Then $\lambda_1 = \lambda_2$ if and only if $i_i = i_i^*$ and $j_i = j_i^*$ for all $i = 1, 2, \cdots, l$.

Lemma 4. *Let* $\lambda = (i_1, j_1;\ i_2, j_2) \in I \odot_2 J$. *Then, the number of distinct ways of producing* $N = q_1^m \cdot q_2^n$ *by multiplication, with no hybrid factor other than* $[q_1^{i_1}, q_2^{j_1}]$ *and* $[q_1^{i_2}, q_2^{j_2}]$, $\mathcal{H}_2^\lambda(N)$, *is given by*

$$(14) \qquad\qquad \mathcal{H}_2^\lambda(N) = p(m - i_1 - i_2) \cdot p(n - j_1 - j_2).$$

Proof. Let $Q_1^{i_1, i_2}$ and $Q_2^{j_1, j_2}$ be sets of distinct ways of producing $N_1^{i_1, i_2} = q_1^{m - i_1 - i_2}$ and $N_2^{j_1, j_2} = q_2^{n - j_1 - j_2}$ by multiplication. Then we have $\mathcal{M}(N_1^{i_1, i_2}) = |Q_1^{i_1, i_2}| = p(m - i_1 - i_2)$ and $\mathcal{M}(N_2^{j_1, j_2}) = |Q_2^{j_1, j_2}| = p(n - j_1 - j_2)$. Note that the set of distinct ways of producing N by multiplying non-hybrid factors of $N_1^{i_1, i_2}$ and $N_2^{j_1, j_2}$ together with the hybrid factors $[q_1^{i_1}, q_2^{j_1}]$ and $[q_1^{i_2}, q_2^{j_2}]$ is given by $[q_1^{i_1}, q_2^{j_1}] \cdot [q_1^{i_2}, q_2^{j_2}] \cdot (Q_1^{i_1, i_2} \times Q_2^{j_1, j_2})$. Thus we can write

$$\mathcal{H}_2^\lambda(N) = |Q_1^{i_1, i_2} \times Q_2^{j_1, j_2}|$$

$$= |Q_1^{i_1, i_2}| \cdot |Q_2^{j_1, j_2}|$$

$$= p(m - i_1 - i_2) \cdot p(n - j_1 - j_2).$$

$\square$

From now on, we will assume that the elements of the set $[q_1^{i_1}, q_2^{j_1}] \cdot [q_1^{i_2}, q_2^{j_2}] \cdot (Q_1^{i_1,i_2} \times Q_2^{j_1,j_2})$ are well ordered through constraint by the inequality $i_1 < i_2$, or by $j_1 \leq j_2$ if $i_1 = i_2$. Then, we will describe two sets $A_{\lambda_1} = [q_1^{i_1}, q_2^{j_1}] \cdot [q_1^{i_2}, q_2^{j_2}] \cdot (Q_1^{i_1,i_2} \times Q_2^{j_1,j_2})$ and $A_{\lambda_2} = [q_1^{\tilde{i_1}}, q_2^{\tilde{j_1}}] \cdot [q_1^{\tilde{i_2}}, q_2^{\tilde{j_2}}] \cdot (Q_1^{\tilde{i_1},\tilde{i_2}} \times Q_2^{\tilde{j_1},\tilde{j_2}})$, with $\tilde{i_1} \leq \tilde{i_2}$ and $\tilde{j_1} \leq \tilde{j_2}$, as *mutually disjoint*, if $\lambda_1 = (i_l, i_2; j_1, j_2) \neq \lambda_2 = (\tilde{i_1}, \tilde{i_2}; \tilde{j_1}, \tilde{j_2})$. We will use $\sum_{I \odot_2 J} |A_{\lambda_l}|$ to denote $\sum_{\lambda_l} |A_{\lambda_l}|$ when any two sets A_{λ_l} and $A_{\lambda_l'}$ are mutually disjoint for $\lambda_l, \lambda_l' \in I \odot_2 J$.

Theorem 5. *Let $\mathcal{H}_2^*(N)$ be the number of possible ways of producing a natural number $N = q_1^m \cdot q_2^n$ by the multiplication of just two hybrid factors. Then $\mathcal{H}_2^*(N)$ is given as follows;*

$$(15) \qquad \mathcal{H}_2^*(N) = \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} p(m - i_1 - i_2)|_{i_2=i_1} \cdot \sum_{j_1=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j_2=j_1}^{n-j_1} p(n - j_1 - j_2)$$

$$(16) \qquad + \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1+1}^{m-i_1} p(m - i_1 - i_2) \cdot \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{n-j_1} p(n - j_1 - j_2).$$

Proof. Since $\mathcal{H}_2^*(N) = \sum_{I \odot_2 J} \mathcal{H}_2^\lambda(N)$, by the previous lemma, we have

$$\mathcal{H}_2^*(N) = \sum_{I \odot_2 J} \mathcal{H}_2^\lambda(N)$$

$$(17) \qquad\qquad = \sum_{I \odot_2 J} p(m - i_1 - i_2) \cdot p(n - j_1 - j_2).$$

In addition, since the factors in the disjoint sets are well-ordered, we can rewrite Eq. (17) as follows:

$$\sum_{I \odot_2 J} p(m - i_1 - i_2) \cdot p(n - j_1 - j_2)$$

$$= \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} p(m - i_1 - i_2)|_{i_2=i_1} \cdot \sum_{j_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{j_2=j_1}^{n-j_1} p(n - j_1 - j_2)$$

$$(18) \qquad + \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1+1}^{m-i_1} p(m - i_1 - i_2) \cdot \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{n-j_1} p(n - j_1 - j_2).$$

Note that the first and second terms of Eq. (18) denote the number of all the elements in the mutually disjoint sets, when $i_1 = i_2$ and $i_1 < i_2$ respectively. This completes the proof. $\qquad\qquad \square$

Lemma 5. *For a natural number $N = q_1^m \cdot q_2^n$, let $\lambda = (i_1, j_1; i_2, j_2; \cdots ; i_k, j_k) \in I \odot_k J$. Then, the number of distinct ways of producing N by multiplication only the hybrid factors $[q_1^{i_1}, q_2^{j_1}], \cdots$, and $[q_1^{i_k}, q_2^{j_k}]$, $\mathcal{H}_k^{\lambda}(N)$, is given by*

$$(19) \qquad \mathcal{H}_k^{\lambda}(N) = p(m - i_1 - i_2 - \cdots - i_k) \cdot p(n - j_1 - j_2 - \cdots - j_k).$$

Proof. Let $Q_1^{i_1, \cdots, i_k}$ and $Q_2^{j_1, \cdots, j_k}$ be sets of distinct ways of producing $N_1^{i_1, \cdots, i_k} = q_1^{m - i_1 - \cdots - i_k}$ and $N_2^{j_1, \cdots, j_k} = q_2^{n - j_1 - \cdots - j_k}$ by multiplication. Then we have $\mathcal{M}(N_1^{i_1, \cdots, i_k}) = |Q_1^{i_1, \cdots, i_k}| = p(m - i_1 - \cdots - i_k)$ and $\mathcal{M}(N_2^{j_1, \cdots, j_k}) = |Q_2^{j_1, \cdots, j_k}| = p(n - j_1 - \cdots - j_k)$. Note that the set of distinct ways of producing N by multiplying non-hybrid factors of $N_1^{i_1, \cdots, i_k}$ and $N_2^{j_1, \cdots, j_k}$ together with the hybrid factors $[q_1^{i_1}, q_2^{j_1}], \cdots, [q_1^{i_k}, q_2^{j_k}]$ is given by $\prod_{l=1}^k [q_1^{i_l}, q_2^{j_l}] \cdot (Q_1^{i_1, \cdots, i_k} \times Q_2^{j_1, \cdots, j_k})$. Thus we obtain

$$\begin{aligned}
\mathcal{H}_k^{\lambda}(N) &= |Q_1^{i_1, \cdots, i_k} \times Q_2^{j_1, \cdots, j_k}| \\
&= |Q_1^{i_1, \cdots, i_k}| \cdot |Q_2^{j_1, \cdots, j_k}| \\
&= p(m - i_1 - \cdots - i_k) \cdot p(n - j_1 - \cdots - j_k).
\end{aligned}$$

$\square$

Theorem 6. *For a natural number $N = q_1^m \cdot q_2^n$, let $\mathcal{H}_k^*(N)$ be the number of possible ways of producing N by multiplication with no hybrid factor other than the k hybrid factors of N. Then $\mathcal{H}_k^*(N)$ is given as follows:*

$$\mathcal{H}_k^*(N) = \left\{ \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1}^{\lfloor \frac{m-i_1}{2} \rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\lfloor \frac{m - \sum_{l=1}^{k-2} i_l}{2} \rfloor} p(m - \sum_{l=1}^k i_l)|_{i_k=i_{k-1}} \right\}$$

$$\times \left\{ \sum_{j_1=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j_2=j_1}^{\lfloor \frac{n-j_1}{2} \rfloor} \cdots \sum_{j_{k-1}=j_{k-2}}^{\lfloor \frac{n - \sum_{l=1}^{k-2} j_l}{2} \rfloor} \sum_{j_k=j_{k-1}}^{n - \sum_{l=1}^{k-1} j_l} p(n - \sum_{l=1}^k j_l) \right\}$$

$$+ \left\{ \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1}^{\lfloor \frac{m-i_1}{2} \rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\lfloor \frac{m - \sum_{l=1}^{k-2} i_l}{2} \rfloor} \sum_{i_k=i_{k-1}+1}^{m - \sum_{l=1}^{k-1} i_l} p(m - \sum_{l=1}^k i_l) \right\}$$

$$\times \left\{ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1}^{n-j_1} \cdots \sum_{j_{k-1}=j_{k-2}}^{n - \sum_{l=1}^{k-2} j_l} \sum_{j_k=1}^{n - \sum_{l=1}^{k-1} j_l} p(n - \sum_{l=1}^k j_l) \right\}.$$

Proof. Since $\mathcal{H}_k^*(N) = \sum_{I \odot_k J} \mathcal{H}_k^\lambda(N)$, by the previous lemma, we have

$$\mathcal{H}_k^*(N) = \sum_{I \odot_k J} \mathcal{H}_k^\lambda(N)$$

$$(20) \qquad\qquad = \sum_{I \odot_k J} p(m - i_1 - \cdots - i_k) \cdot p(n - j_1 - \cdots - j_k),$$

where $\lambda = (i_1, j_1; i_2, j_2; \cdots ; i_k, j_k) \in I \odot_k J$. In addition, since the factors in the disjoint sets are well-ordered, according to the cases whether $i_k = i_{k-1}$ or $i_k > i_{k-1}$, we can rewrite Eq. (20) as follows:

$$\sum_{I \odot_k J} p(m - i_1 - i_2) \cdot p(n - j_1 - j_2)$$

$$= \left\{ \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1}^{\lfloor \frac{m-i_1}{2} \rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\lfloor \frac{m-\sum_{l=1}^{k-2} i_l}{2} \rfloor} p\left(m - \sum_{l=1}^{k} i_l\right)\Big|_{i_k=i_{k-1}} \right\}$$

$$\times \left\{ \sum_{j_1=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j_2=j_1}^{\lfloor \frac{n-j_1}{2} \rfloor} \cdots \sum_{j_{k-1}=j_{k-2}}^{\lfloor \frac{n-\sum_{l=1}^{k-2} j_l}{2} \rfloor} \sum_{j_k=j_{k-1}}^{n-\sum_{l=1}^{k-1} j_l} p\left(n - \sum_{l=1}^{k} j_l\right) \right\}$$

$$+ \left\{ \sum_{i_1=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{i_2=i_1}^{\lfloor \frac{m-i_1}{2} \rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\lfloor \frac{m-\sum_{l=1}^{k-2} i_l}{2} \rfloor} \sum_{i_k=i_{k-1}+1}^{m-\sum_{l=1}^{k-1} i_l} p\left(m - \sum_{l=1}^{k} i_l\right) \right\}$$

$$(21) \qquad \times \left\{ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1}^{n-j_1} \cdots \sum_{j_{k-1}=j_{k-2}}^{n-\sum_{l=1}^{k-2} j_l} \sum_{j_k=1}^{n-\sum_{l=1}^{k-1} j_l} p\left(n - \sum_{l=1}^{k} j_l\right) \right\}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Theorem 7. *The multiplicative entropy $P(N)$ of a natural number $N = q_1^m \cdot q_2^n$, is given by*

$$(22) \qquad\qquad P(N) = \ln \mathcal{M}(N) = \ln \left(\sum_{k=0} \mathcal{H}_k^*(N) \right).$$

Proof. It is apparent from the definition $\mathcal{M}(N) = \sum_{k=0} \mathcal{H}_k^*(N)$. $\qquad\qquad \square$

Tables 1 and 2 show the multiplicative entropies of $N = 2^m \cdot 3^n$ for $1 \leq m \leq 20$ and $1 \leq n \leq 20$, obtained from Eq. (22). Note that $P(2^m \cdot 3^n) = P(3^m \cdot 2^n)$, and the symmetric entries are omitted.

		n									
		1	2	3	4	5	6	7	8	9	10
	1	0.693									
	2	1.386	2.197								
	3	1.946	2.773	3.434							
	4	2.485	3.367	4.043	4.691						
	5	2.944	3.850	4.575	5.236	5.811					
	6	3.401	4.344	5.088	5.768	6.354	6.915				
	7	3.807	4.771	5.549	6.242	6.846	7.415	7.932			
	8	4.205	5.198	5.994	6.702	7.315	7.893	8.416	8.914		
	9	4.575	5.587	6.410	7.127	7.752	8.338	8.869	9.371	9.842	
m	10	4.934	5.971	6.810	7.538	8.171	8.764	9.300	9.808	10.283	10.735
	11	5.273	6.328	7.189	7.926	8.568	9.166	9.708	10.220	10.699	11.155
	12	5.606	6.681	7.557	8.302	8.951	9.554	10.101	10.617	11.099	11.558
	13	5.922	7.013	7.908	8.660	9.317	9.925	10.476	10.995	11.481	11.943
	14	6.230	7.340	8.249	9.009	9.671	10.284	10.839	11.361	11.850	12.314
	15	6.528	7.653	8.578	9.344	10.013	10.629	11.188	11.713	12.204	12.671
	16	6.819	7.960	8.898	9.671	10.344	10.964	11.526	12.054	12.548	13.016
	17	7.100	8.255	9.208	9.986	10.665	11.288	11.854	12.384	12.880	13.351
	18	7.376	8.545	9.510	10.295	10.978	11.605	12.172	12.705	13.203	13.676
	19	7.643	8.826	9.804	10.594	11.282	11.911	12.482	13.017	13.517	13.991
	20	7.906	9.102	10.091	10.886	11.578	12.211	12.784	13.321	13.823	14.299

Table 1. Multiplicative entropies of $N = q_1^m \cdot q_2^n$, when $q_1 = 2$ and $q_2 = 3$, for $1 \le m \le 20$ and $1 \le n \le 10$.

		n									
		11	12	13	14	15	16	17	18	19	20
	10										
	11	11.588									
	12	11.993	12.412								
	13	12.381	12.801	13.203							
m	14	12.754	13.177	13.580	13.971						
	15	13.113	13.537	13.943	14.335	14.712					
	16	13.460	13.887	14.294	14.687	15.066	15.434				
	17	13.796	14.225	14.633	15.028	15.408	15.777	16.134			
	18	14.123	14.553	14.963	15.359	15.740	16.110	16.468	16.816		
	19	14.440	14.871	15.283	15.680	16.063	16.434	16.792	17.141	17.481	
	20	14.749	15.175	15.594	15.993	16.377	16.749	17.108	17.458	17.736	18.130

Table 2. Multiplicative entropies of $N = q_1^m \cdot q_2^n$, when $q_1 = 2$ and $q_2 = 3$, for $11 \le m \le 20$ and $11 \le n \le 20$.

From Tables 1 and 2, we observe that multiplicative entropy increases very slowly with m and n, whereas the additive entropy $\ln p(N)$ increases quickly with N. This suggests that $d_{rc}^*(N) \to 0$ as N tends to ∞, a conjecture supported by Figure 3.
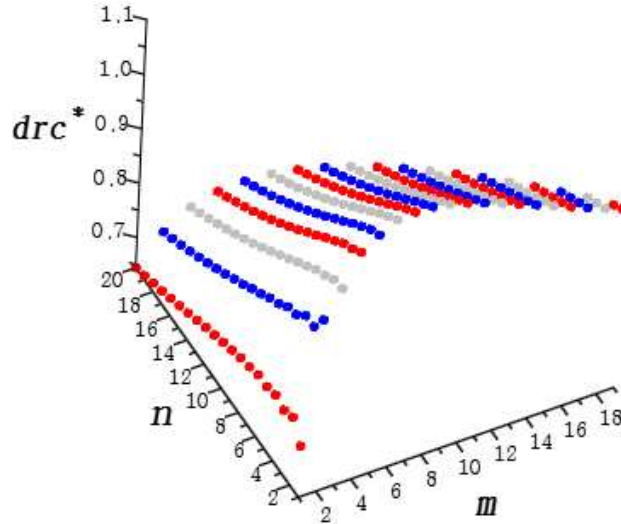
Figure 3. Behavior of d_{rc}^*, when $q_1 = 2$ and $q_2 = 3$, for $1 \leq m \leq 20$ and $1 \leq n \leq 20$: each row of dots of the same color shows the behavior of d_{rc}^* when m is fixed.

Next, we compute the comparative density $d_{rc}^*(N)$ of algebraic entropies of N, and show that this density converges to zero as m, n approach ∞.

Let $\Phi(n) = \sum_{i=0}^{n-1} p(i)$, $n \in \mathbb{N}$; and we now introduce ${}_a T_b$ to denote the number of distinct ways of producing a natural number a by adding b natural numbers. Then we have the following lemma:

Lemma 6. *For a natural number $N = q_1^m \cdot q_2^n$,*

$$(23) \qquad \mathcal{H}_k^*(N) \leq {}_m T_k \cdot \Phi(m) \times {}_n T_k \cdot \Phi(n), \quad 0 \leq k \leq n.$$

Proof. We can rewrite Eq. (21) as follows:

$$\mathcal{H}_k^*(N) = \left(\sum_{t=k}^{m} \lambda_1(t)\, p(m-t) \right) \cdot \left(\sum_{t=k}^{n} \delta_1(t)\, p(n-t) \right)$$
$$+ \left(\sum_{t=k}^{m} \lambda_2(t)\, p(m-t) \right) \cdot \left(\sum_{t=k}^{n} \delta_2(t)\, p(n-t) \right)$$
$$= \sum_{t_1=k}^{m} \sum_{t_2=k}^{n} \lambda^*(t_1)\, \delta^*(t_2) \cdot p(m-t_1)\, p(n-t_2).$$

Let $\Lambda_k = \max_{t_1=k,\cdots,m} \{\lambda^*(t_1)\}$ and $\Delta_k = \max_{t_2=k,\cdots,n} \{\delta^*(t_2)\}$. Then we can write

$$\mathcal{H}_k^*(N) \leq \Lambda_k \, \Delta_k \cdot \left(\sum_{l=0}^{m-k} p(l) \right) \cdot \left(\sum_{l=0}^{n-k} p(l) \right)$$

$$\leq \Lambda_k \, \Delta_k \cdot \left(\sum_{l=0}^{m-1} p(l) \right) \cdot \left(\sum_{l=0}^{n-1} p(l) \right)$$

$$= \Lambda_k \, \Delta_k \cdot \Phi(m) \cdot \Phi(n).$$

Moreover, since $\Lambda_k \leq {}_m\mathrm{T}_k$ and $\Delta_k \leq {}_n\mathrm{T}_k$, we obtain

$$\mathcal{H}_k^*(N) \leq {}_m\mathrm{T}_k \cdot \Phi(m) \times {}_n\mathrm{T}_k \cdot \Phi(n).$$

$\square$

Theorem 8. *For a natural number* $N = q_1^m \cdot q_2^n$,

$$(24) \qquad \lim_{m,\,n \to \infty} d_{rc}^*(N) = \lim_{m,\,n \to \infty} \frac{\ln \left(\sum_{k=0} \mathcal{H}_k^*(N) \right)}{\ln(\ln p(N))} = 0.$$

Proof. From Eq. (23) we obtain

$$(25) \qquad \sum_{k=0} \mathcal{H}_k^*(N) \leq \left(\sum_{k=0}^m {}_m\mathrm{T}_k \right) \times \left(\sum_{k=0}^n {}_n\mathrm{T}_k \right) \Phi(m)\Phi(n).$$

In addition, since

$$p(n) = \frac{1}{n} \sum_{k=1}^n \sigma(k) p(n-k),$$

where $\sigma(k)$ is the sum of the divisors of k, and so

$$(26) \qquad n\,p(n) \geq \sum_{k=0}^{n-1} p(k) \; = \; \Phi(n).$$

Note that

$$(27) \qquad \sum_{k=0}^m {}_m\mathrm{T}_k = p(m) \quad \text{and} \quad \sum_{k=0}^n {}_n\mathrm{T}_k = p(n).$$

Combining Eqs. (25) , (26) and (27), we can write:

$$\mathcal{H}_k^*(N) \leq m\,n\,p(m)^2\,p(n)^2.$$

Consequently we obtain

$$d_{rc}^*(N) = \frac{\ln \left(\sum_{k=0} \mathcal{H}_k^*(N) \right)}{\ln(\ln p(N))} \leq \frac{\ln(m\,n\,p(m)^2\,p(n)^2)}{\ln(\ln p(N))}.$$

For sufficiently large s, we can write $p(s) \sim \frac{1}{4s\sqrt{3}} \exp\left(\pi\sqrt{\frac{2s}{3}}\right)$, and hence

$$\lim_{m,n\to\infty} d_{rc}^*(N) = \lim_{m,n\to\infty} \frac{\ln\left(\sum_{k=0} \mathcal{H}_k^*(N)\right)}{\ln(\ln p(N))} = 0.$$

$\square$

4. Concluding Remarks and Suggestions for Further Study

We have introduced the additive and multiplicative entropy of a natural number, and defined their comparative density. We have showed that, for natural numbers with one prime factor, the additive and multiplicative entropy are logarithmically related, by computing the relative density of the algebraic entropies of these numbers and analyzing its asymptotic behavior. We have also presented a practical formula for computing the relative density of natural numbers with two prime factors. We observed that the logarithmic relation between additive and multiplicative entropy applies to larger numbers in a special subclass of natural numbers which have no hybrid factor in any of the possible multiplicative partitions. We also proved that their relative density converges to zero as the number tends to infinity. In addition, we empirically showed that the asymptotic behavior of the relative density of numbers with two prime factors is more complicated and quite different from the behavior of numbers with a single prime factor, as shown in Figures 1 and 3. We proved that this relative density also converges to zero as the number tends to infinity.

We propose two avenues for further study: firstly, to complete the analysis of the asymptotic behavior of the relative density for natural numbers with two prime factors, and explore this behavior empirically in more detail; and, secondly, to extend our work to natural numbers with more than two prime factors. In particular, a fuller understanding of the multiplicative entropy of an arbitrary natural number might provide useful information on the distribution of prime numbers, which must be related to the distribution of the multiplicative entropies of natural numbers.

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