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MINIMAL QUASI-F COVERS OF REALCOMPACT SPACES

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Abstract. In this paper, we show that every compactification, which is a quasi-F space, of a space X is a Wallman compactification and that for any compactification K of the space X, the minimal quasi-F cover QFK of K is also a Wallman compactification of the inverse image $\Phi_K^{-1}(X)$ of the space X under the covering map $\Phi_K : QFK \longrightarrow K$. Using these, we show that for any space X, $\beta QFX = QF\beta\upsilon X$ and that a realcompact space X is a projective object in the category $\mathbf{Rcomp}_{\#}$ of all realcompact spaces and their $z^{\#}$-irreducible maps if and only if X is a quasi-F space.

1. Introduction

All spaces in this paper are Tychonoff spaces and $(\beta X, \beta_X)$ (resp. $(\upsilon X, \upsilon_X)$) denotes the Stone-Čech compactification (resp. Hewitt realcompactification) of a space X.

Gleason in [5] showed that the projective objects in the category of all compact spaces and continuous maps are precisely the extremally disconnected spaces and that each compact space has a unique projective cover, namely its absolute. Iliadis (resp. Banaschewski) proved similar results for the category of all Hausdorff spaces (resp. regular spaces) and their perfect continuous maps [12].

In order to generalize extremally disconnected spaces, the notions of basically disconnected spaces, quasi-F spaces, and cloz-spaces have been introduced, and their minimal covers have been studied by various authors [6, 8, 9, 10, 12, 15]. In particular, Henriksen, Vermeer, and Woods in [10] showed that every space X has the minimal quasi-F cover (QFX, Φ_X). Indeed, if X is a compact space, then QFX is given by the Wallman cover $(\mathcal{L}(Z(X)^{\#}), \Phi_X)$ which in turn is the projective maximum of X in the category of all compact spaces and their $z^{\#}$-irreducible maps

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[10]. Also the relation of QFX and $QF\beta X$ was investigated [6, 10, 11]. Among other results, they showed that $\beta QFX = QF\beta X$ if and only if X has the projective object in the category of all spaces and their $z^{\#}$-irreducible maps and that if X is a weakly Lindelöf space, then $\beta QFX = QF\beta X$ and Φ_X is a $z^{\#}$-irreducible map.

Here, we investigate the relation of QFX and $QF\beta X$ for an arbitrary realcompact space X. We first show that every compactification, which is a quasi-F space, of a space X is a Wallman compactification (Theorem 3.2) and that for any compactification K of X, the minimal quasi-F cover QFK is also a Wallman compactification of $\Phi_K^{-1}(X)$ (Corollary 3.3). Next, using these results, we establish the equality $\beta QFX = QF\beta\upsilon X$ for any space X (Proposition 3.4). Finally, we show that a realcompact space X is the projective object in the category $\mathbf{Rcomp}_{\#}$ of all realcompact spaces and their $z^{\#}$-irreducible maps if and only if X is a quasi-F space (Corollary 3.6). For the terminology, we refer to [1, 4, 12].

2. Covers and Extensions

We recall that a space X is called *realcompact* if each z-ultrafilter on X with the countable intersection property is fixed and that a pair (Y, j) or simply Y is called *a compactification* (resp. *realcompactification*) of X if $j : X \hookrightarrow Y$ is a dense embedding and Y is a compact (resp. realcompact) space. The following notion due to E. F. Steiner is the basic device in the present setting [13, 14].

Definition 2.1. Let X be a space and $\mathcal{F}$ a family of closed sets in X. Then $\mathcal{F}$ is called a *separating nested generated intersection ring* on X if

(1) it is closed under finite unions and countable intersections,
(2) for any closed set H in X and $x \notin H$, there are disjoint A, B in $\mathcal{F}$ such that $x \in A$ and $H \subseteq B$, and
(3) for any $F \in \mathcal{F}$, there are sequences (A_n) and (B_n) in $\mathcal{F}$ such that for each $n \in \mathbb{N}$, $X - A_{n+1} \subseteq B_{n+1} \subseteq X - A_n \subseteq B_n$ and $F = \bigcap\{B_n \mid n \in \mathbb{N}\}$.

For any space X, we denote by $\mathcal{L}(X)$ the set of all separating nested generated intersection rings on X. It is then well-known that the set $Z(X)$ of all zero-sets in a space X belongs to $\mathcal{L}(X)$ [4] and that for any $\mathcal{F} \in \mathcal{L}(X)$ and $S \subseteq X$, the set

$$\mathcal{F}_S = \{F \cap S \mid F \in \mathcal{F}\},$$

called the *trace* of $\mathcal{F}$ on the subspace S of X, belongs to $\mathcal{L}(S)$ [13, Lemma 1.3]. For any $\mathcal{F} \in \mathcal{L}(X)$, let $(\omega(X, \mathcal{F}), w_X)$ be the Wallman compactification of X associated with $\mathcal{F}$ [12, 13]. Concerning the Wallman compactifications, we will often use the following fact: If (Y, j) is a compactification of X such that $Z(Y)_X \subseteq \mathcal{F}$, then there is a continuous map $f : \omega(X, \mathcal{F}) \longrightarrow Y$ such that $f \circ \omega_X = j$ [12, Theorems 4.2(h), 4.4(g)].

Let $\upsilon(X, \mathcal{F})$ be the set of all $\mathcal{F}$-ultrafilters on X with the countable intersection property. The topology on $\upsilon(X, \mathcal{F})$, taking sets of the form

$$F^* = \{\alpha \in \upsilon(X, \mathcal{F}) \mid F \in \alpha\}$$

as a base for the closed sets, coincides with the subspace topology on $\upsilon(X, \mathcal{F})$ of $\omega(X, \mathcal{F})$, and $\upsilon(X, \mathcal{F})$ is in fact a realcompactification of X, called a *Wallman realcompactification* of X [13]. The poof of the following Lemma 2.2 can be found in [2, Theorem 2, Corollary 2.1] and [13, Theorem 2.2].

Lemma 2.2. *Let X be a space and $\mathcal{F} \in \mathcal{L}(X)$. Then we have the following:*

(1) $Z(\omega(X, \mathcal{F}))_X = \mathcal{F}$,
(2) $\upsilon(X, \widehat{\mathcal{F}}) = \upsilon(X, \mathcal{F})$, *and*
(3) $\omega(X, \widehat{\mathcal{F}}) = \beta(\upsilon(X, \mathcal{F}))$, *where* $\widehat{\mathcal{F}} = Z(\upsilon(X, \mathcal{F}))_X$.

Let X be a space and $\mathcal{T}_\delta$ the topology on X generated by the family of all G_δ-sets in X, and let $\mathrm{cl}_{(X,\delta)}(A)$ be the closure of A in $(X, \mathcal{T}_\delta)$ for any $A \subseteq X$. Then $x \in \mathrm{cl}_{(X,\delta)}(A)$ if and only if $Z \cap A \neq \emptyset$ for any zero-set Z in X with $x \in Z$. The closure $\mathrm{cl}_{(X,\delta)}(A)$ is also called the Q-closure of A in X [2, 7].

Proposition 2.3. *Let X be a space and (Y, j) a realcompactification of X. The following statements are equivalent.*

(1) *Y is a Wallman realcompactification of X.*
(2) $\upsilon(X, Z(Y)_X) = Y$.
(3) $\mathrm{cl}_{(\beta Y,\delta)}(X) = Y$.
(4) *if Z is a zero-set in Y with $Z \cap X = \emptyset$, then $Z = \emptyset$.*

Proof.

(1) $\Rightarrow$ (2) Since Y is a Wallman realcompactification of X, there is an $\mathcal{F} \in \mathcal{L}(X)$ such that $Y = \upsilon(X, \mathcal{F})$. By Lemma 2.2, $Y = \upsilon(X, Z(Y)_X)$.

(2) $\Rightarrow$ (3) Let $y \in \beta Y - Y$. Since Y is realcompact, there is a zero-set Z in βY such that $y \in Z$ and $Z \cap Y = \emptyset$ [4, Remark 8.8]. Hence $Z \cap X = \emptyset$. So, we have $y \notin \mathrm{cl}_{(\beta Y,\delta)}(X)$, showing $\mathrm{cl}_{(\beta Y,\delta)}(X) \subseteq Y$. On the other hand, let $t \in Y - \mathrm{cl}_{(\beta Y,\delta)}(X)$. There is a zero-set A in βY such that $t \in A$ and $A \cap X = \emptyset$. If $A \cap Y \neq \emptyset$, then

there is an $\alpha \in Y \cap A = \upsilon(X, \mathcal{F}) \cap A$. Since $A \in Z(\beta Y)$, there is a continuous map $f : \beta Y \longrightarrow [0, 1]$ such that $f^{-1}(0) = A$. For each $n \in \mathbb{N}$, let

$$Z_n = f^{-1}([0, \frac{1}{n}]).$$

It then follows that $Z_{n+1} \subseteq \mathrm{int}_{\beta Y}(Z_n)$ and $Z_n \cap X \in Z(Y)_X$ with

$$A = \bigcap\{Z_n \mid n \in \mathbb{N}\}.$$

Suppose that $Z_n \cap X \notin \alpha$ for some $n \in \mathbb{N}$. Since α is a $Z(Y)_X$-ultrafilter, there is a $B \in Z(\beta Y)$ such that $B \cap X \in \alpha$ and $Z_n \cap X \cap B = \emptyset$. Hence

$$\mathrm{int}_{\beta Y}(Z_n) \cap \mathrm{cl}_{\beta Y}(X \cap B) = \emptyset,$$

showing $A \cap \mathrm{cl}_{\beta Y}(B \cap X) = \emptyset$ as $A \subseteq \mathrm{int}_{\beta Y}(Z_n)$. Since $\alpha \in \mathrm{cl}_Y(B \cap X)$, we obtain $\alpha \in A \cap \mathrm{cl}_{\beta Y}(B \cap X)$, which is a contradiction. Thus $Z_n \cap X \in \alpha$ for all $n \in \mathbb{N}$. Now, as α has the countable intersection property, $A \cap X \in \alpha$, which is a contraction. Hence $A \cap Y = \emptyset$. Thus we must conclude $t \notin Y - \mathrm{cl}_{(\beta Y,\delta)}(X)$. Therefore $Y = \mathrm{cl}_{(\beta Y,\delta)}(X)$ as desired.

(3) $\Rightarrow$ (4) It is trivial.

(4) $\Rightarrow$ (1) Let $\mathcal{F} = Z(Y)_X$. Since $Z(\beta Y)_X = \mathcal{F}$, there is a continuous map $k : \omega(X, \mathcal{F}) \longrightarrow \beta Y$ with $k \circ \omega_{\mathcal{F}} = \beta_Y \circ j$. Take any zero-sets A and B in $\omega(X, \mathcal{F})$ such that $A \cap B \cap X = \emptyset$. By Lemma 2.2, we have $Z(\omega(X, \mathcal{F}))_X = \mathcal{F}$. Thus, there are zero-sets C and D of Y such that $A \cap X = C \cap X$ and $B \cap X = D \cap X$. So, $C \cap D \cap X = \emptyset$. By hypothesis (4), necessarily $C \cap D = \emptyset$. Hence $\mathrm{cl}_{\beta Y}(C) \cap \mathrm{cl}_{\beta Y}(D) = \emptyset$ [4], showing

$$\mathrm{cl}_{\beta Y}(A \cap X) \cap \mathrm{cl}_{\beta Y}(B \cap X) = \emptyset.$$

By Urysohn's extension theorem, βY and $\omega(X, \mathcal{F})$ are homeomorphic. Now, as $\mathrm{cl}_{(\omega(X,\mathcal{F}),\delta)}(X) = \upsilon(X, \mathcal{F})$ [7, Theorem 5.3], we have

$$Y = \mathrm{cl}_{(\beta Y,\delta)}(X) = \mathrm{cl}_{(\omega(X,\mathcal{F}),\delta)}(X) = \upsilon(X, \mathcal{F}),$$

showing that Y is a Wallman realcompactification of X. $\square$

Recall that a continuous map $f : X \longrightarrow Y$ is called *a covering map* if f is onto, perfect, closed, and irreducible [12, Chapter 8].

Proposition 2.4. *Let X be a space and $f : Y \to \beta X$ a covering map such that Y is a Wallman compactification of $f^{-1}(X)$. Then $v(f^{-1}(X), Z(Y)_{f^{-1}(X)}) \subseteq f^{-1}(\upsilon X)$.*

Proof. Since Y is a Wallman compactification of $f^{-1}(X)$,

$$\beta(\upsilon(F^{-1}(X), Z(Y)_{f^{-1}(X)})) = \omega(f^{-1}(X), Z(Y)_{f^{-1}(X)})$$

because $\upsilon(F^{-1}(X), Z(Y)_{f^{-1}(X)})$ is a wallman realcompactification of $f^{-1}(X)$(see the proof of (4) $\Rightarrow$ (1) in Proposition 2.3). By Lemma 2.2, we have

$$Y = \omega(f^{-1}(X), Z(Y)_{f^{-1}(X)}) = \beta(\upsilon(f^{-1}(X), Z(Y)_{f^{-1}(X)})) = \omega(f^{-1}(X), \widehat{\mathcal{F}}),$$

where $\mathcal{F} = Z(Y)_{f^{-1}(X)}$. Let $S = \upsilon(f^{-1}(X), Z(Y)_{f^{-1}(X)})$ and $t \in Y - f^{-1}(\upsilon X)$. Since $f(t) \notin \upsilon X$, there is a zero-set Z in βX such that $f(t) \in Z$ and $Z \cap \upsilon X = \emptyset$. Thus $f^{-1}(Z) \cap f^{-1}(X) = \emptyset$, showing $f^{-1}(Z) \cap S \cap f^{-1}(X) = \emptyset$. Now, as $f^{-1}(Z) \cap S$ is a zero-set in S, by Proposition 2.3, $f^{-1}(Z) \cap S = \emptyset$. Therefore $t \notin S$ as desired. □

Corollary 2.5. *Let X be a space and $f : Y \longrightarrow \beta X$ a covering map such that Y is a Wallman compactification of $f^{-1}(X)$. Then $\upsilon(f^{-1}(X), Z(Y)_{f^{-1}(X)}) = f^{-1}(\upsilon X)$ if and only if $f^{-1}(\upsilon X)$ is a Wallman realcompactification of $f^{-1}(X)$.*

Proof. ($\Rightarrow$) It is trivial.

($\Leftarrow$) Let $S = \upsilon(f^{-1}(X), Z(Y)_{f^{-1}(X)})$. By Proposition 2.4, we have $S \subseteq f^{-1}(\upsilon X)$. Suppose that there is a $t \in f^{-1}(\upsilon X) - S$. Then there is a zero-set Z in βS such that $t \in Z$ and $Z \cap S = \emptyset$. Since $Z \cap f^{-1}(\upsilon X)$ is a non-empty zero-set in $f^{-1}(\upsilon X)$ and $f^{-1}(\upsilon X)$ is a Wallman realcompactification of $f^{-1}(X)$, by Proposition 2.3, $Z \cap f^{-1}(X) \neq \emptyset$. This is a contradiction. □

3. Projective Objects in the Category of Realcompact Spaces and $z^{\#}$-irreducible Maps

In this last section, we prove the main Theorem 3.2 about the quasi-F compactifications of spaces.

First, we recall from [4] that a subspace X of a space Y is called *C^*-embedded in Y* if for any real-valued continuous map $f : X \longrightarrow \mathbb{R}$, there is a continuous map $g : Y \longrightarrow \mathbb{R}$ such that $g|_X = f$. Also, a space X is called a *quasi-F space* if every dense cozero-set in X is C^*-embedded in X, or, equivalently, for any zero-sets A, B in X, $\mathrm{cl}_X(\mathrm{int}_X(A \cap B)) = \mathrm{cl}_X(\mathrm{int}_X(A)) \cap \mathrm{cl}_X(\mathrm{int}_X(B))$ [10, Lemma 2.10]. We further recall the following definitions [10].

Definition 3.1. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : X \longrightarrow Y$ is a covering map,
(2) *a quasi-F cover of* X if (Y, f) is a cover of X and Y is quasi-F space, and

(3) *a minimal quasi-F cover of X* if (Y, f) is a quasi-F cover of X, and for any quasi-F cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ such that $f \circ h = g$.

Theorem 3.2. *Let Y be a compactification of a space X. If Y is a quasi-F space, then Y is a Wallman compactification of X. In fact, $Y = w(X, Z(Y)_X)$.*

Proof. Consider the Wallman compactification $K = w(X, \mathcal{G})$ of X associated with $\mathcal{G} = Z(Y)_X$. By the basic fact mentioned in Section 2, there is a continuous map $f : K \to Y$ such that $f \circ \omega_{\mathcal{G}} = j$, where $j : X \to Y$ is a dense embedding. Take any disjoint closed sets A, B in K. Since K is a compact space, there are disjoint zero-sets C, D in K such that $A \subseteq \mathrm{int}_K(C)$ and $B \subseteq \mathrm{int}_K(D)$. Since $Z(K)_X = Z(w(X, \mathcal{G}))_X = \mathcal{G}$, certainly $C \cap X$ and $D \cap X$ belong to $\mathcal{G}$. Also, since $\mathcal{G} = Z(Y)_X$, there are $E, F \in Z(Y)$ such that $C \cap X = E \cap X$ and $D \cap X = F \cap X$. Clearly, $\mathrm{int}_Y(E) \cap X \subseteq \mathrm{int}_X(E \cap X)$. Take any $x \in \mathrm{int}_X(E \cap X)$. Then there is an open neighborhood U of x in Y such that $U \cap X \subseteq E \cap X$. Since X is dense in Y and U is open in Y, $\mathrm{cl}_Y(U) = \mathrm{cl}_Y(U \cap X) \subseteq E$ and $x \in \mathrm{int}_Y(E)$. Hence $\mathrm{int}_Y(E) \cap X = \mathrm{int}_X(E \cap X)$. Similarly, $\mathrm{int}_K(C) \cap X = \mathrm{int}_X(C \cap X)$. Thus

$$\mathrm{int}_Y(E) \cap X = \mathrm{int}_K(C) \cap X \quad \text{and} \quad \mathrm{int}_Y(F) \cap X = \mathrm{int}_K(D) \cap X.$$

Since $C \cap D = \emptyset$, $\mathrm{int}_Y(E) \cap X \cap \mathrm{int}_Y(F) = \emptyset$, we have $\mathrm{int}_Y(E) \cap \mathrm{int}_Y(F) = \emptyset$. Since Y is a quasi-F space, $\mathrm{cl}_Y(\mathrm{int}_Y(E)) \cap \mathrm{cl}_Y(\mathrm{int}_Y(F)) = \emptyset$. Further notice that

$$A \cap X \subseteq \mathrm{int}_K(C) \cap X = \mathrm{int}_Y(E) \cap X \subseteq \mathrm{int}_Y(E)$$

and $B \cap X \subseteq \mathrm{int}_Y(F)$. Thus we have $\mathrm{cl}_Y(A \cap X) \cap \mathrm{cl}_Y(B \cap X) = \emptyset$. Now, by the Urysohn's extension theorem, there is a continuous map $g : Y \to K$ with $g \circ j = w_{\mathcal{G}}$. Composing with f, we obtain $f \circ g \circ j = f \circ w_{\mathcal{G}} = 1_Y \circ j$. Since $j : X \longrightarrow Y$ is a dense embedding, $f \circ g = 1_Y$. Hence f is a homeomorphism. □

It is known that every space X has the minimal quasi-F cover (QFX, Φ_X). For the detailed accounts for quasi-F covers of X, see [3, 9, 12].

Corollary 3.3. *If K is a compactification of X, then QFK is a Wallman compactification of $\Phi_K^{-1}(X)$.*

In the following, for any space X, let $(QF\beta X, \Phi_\beta)$ denote the minimal quasi-F cover of βX, and let $S = \upsilon(\Phi_\beta^{-1}(X), Z(QF\beta X)_{\Phi_\beta^{-1}(X)})$. By Corollary 3.3, we have the following Proposition 3.4.

Proposition 3.4. *Let X be a space. Then we have the following.*

(1) $QF\beta X = \beta S$.

(2) *S is a quasi-F space.*

(3) *$(\Phi_\beta^{-1}(\upsilon X), \Phi_\upsilon)$ is the minimal quasi-F cover of υX.*

(4) *$\beta QF\upsilon X = QF\beta X$, where $\Phi\upsilon : \Phi_\beta^{-1}(\upsilon X) \longrightarrow \upsilon X$ is the restriction and corestriction of Φ_β with respect to $\Phi_\beta^{-1}(\upsilon X)$ and υX, respectively.*

Proof. (1) By Lemma 2.2 and Corollary 3.3, we have $QF\beta X = \beta S$.

(2) Since $QF\beta X = \beta S$, certainly S is a quasi-F space [3, Theorem 5.1].

(3) By Theorem 3.2, the space $QF\beta X$ is a Wallman compactification of $\Phi_\beta^{-1}(X)$. Also, by Proposition 2.4, we have $S \subseteq \Phi_\beta^{-1}(\upsilon X)$. Since $\beta S = QF\beta X$ and $\Phi_\beta^{-1}(\upsilon X) \subseteq QF\beta X$, it follows that $\beta S = \beta\Phi_\beta^{-1}(\upsilon X)$ and $\Phi_\beta^{-1}(\upsilon X)$ is a quasi-F space. Hence $(\Phi_\beta^{-1}(\upsilon X), \Phi_\upsilon)$ is the minimal quasi-F cover of υX [11].

(4) The fact that S is C^*-embedded in $QF\beta X$ implies that $\Phi_\beta^{-1}(\upsilon X)$ is C^*-embedded in $QF\beta X$. Since $S \subseteq \Phi_\beta^{-1}(\upsilon X) \subseteq QF\beta X$, by (1), we obtain $QF\beta X = \beta QF\upsilon X$. □

As noted in the introduction, we now turn to the conditions on the spaces X under which $QF\beta X = \beta QFX$. We recall that a space X is called *weakly Lindelöf*, if for any open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{V \mid V \in \mathcal{V}\}$ is dense in X. Henriksen, Vermeer, and Woods in [10] showed that $QF\beta X = \beta QFX$ for any weakly Lindelöff space X.

By Proposition 3.4, we obtain the following Corollary 3.5. We note in passing, however, that there is no direct relationship between realcompact spaces and weakly Lindelöf spaces.

Corollary 3.5. *For any realcompact space X, $QF\beta X = \beta QFX$.*

Let **C** be a topological subcategory of the category **Top** of topological spaces and continuous maps [10, Section 4]. An object X in **C** is called *a projective object* in **C** if for any morphism $f : X \longrightarrow Y$ in **C** and any onto morphism $g : Z \longrightarrow Y$ in **C**, there is a morphism $h : X \longrightarrow Z$ in **C** such that $g \circ h = f$. A pair (Y, f) is called *a projective cover of an object* X in **C** if Y is a projective object in **C** and $f : Y \longrightarrow X$ is a morphism in **C** such that f is an onto, closed, irreducible map.

Gleason in [5] showed that the projective objects in the category of all compact spaces and their continuous maps are exactly the extremally disconnected spaces and that each compact space has a unique projective cover, namely its absolute.

Further recall from [10] that a covering map $f : Y \longrightarrow X$ is called *$z^{\#}$-irreducible* if

$$\{f(A) \mid A \in Z(Y)^{\#}\} = Z(X)^{\#}.$$

Henriksen, Vermeer, and Woods in [10] showed that the quasi-F spaces are the projective objects in the category $\mathbf{Tych}_{\#}$ of all spaces and their $z^{\#}$-irreducible maps [10, Theorem 4.3] and that a space X has a projective cover in $\mathbf{Tych}_{\#}$ if and only if $QF\beta X = \beta QFX$ [10, Theorem 4.5].

Let $\mathbf{Rcomp}_{\#}$ be the category of all realcompact spaces and their $z^{\#}$-irreducible maps. Now, using the fact that if $\beta QFX = QF\beta X$, then the covering map $\Phi : QFX \longrightarrow X$ is a $z^{\#}$-irreducible map [10, Theorem 3.5], we obtain the following.

Corollary 3.6. *A realcompact space X is a projective object in* $\mathbf{Rcomp}_{\#}$ *if and only if X is a quasi-F space.*

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