

THE STRUCTURE OF PRIORS' SET OF EQUIVALENT MEASURES

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ABSTRACT. The set of priors in the representation of Choquet expectation is expressed as the one of equivalent martingale measures under some conditions. We show that the set of priors, \mathcal{Q}_c in (1.1) is the same set of \mathcal{Q}^θ in (1.3).

1. INTRODUCTION

Kim [6] showed that the set of priors in the representation of Choquet expectation is the one of equivalent martingale measures under some conditions, when the distortion is submodular. That is, if a capacity c is submodular, then we have the representation

$$\int X dc = \max_{Q \in \mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T),$$

where $\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} : Q[A] \leq c(A) \forall A \in \mathcal{F}_T\}$. Here $M_{1,f} := M_{1,f}(\Omega, \mathcal{F})$ is the set of all finitely additive normalized set functions $Q : \mathcal{F} \rightarrow [0, 1]$.

By using g -expectation and related topics [4, 5, 7], Kim [6] showed that \mathcal{Q}_c equals to \mathcal{Q}^θ where \mathcal{Q}^θ is defined as

$$(1.1) \quad \mathcal{Q}^\theta := \left\{ Q^\theta : \theta \in \Theta^g, \frac{dQ^\theta}{dP} \Big|_{\mathcal{F}_t} = \exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right) \right\}$$

where $t \in [0, T]$ and Θ^g is defined as

$$(1.2) \quad \Theta^g = \{(\theta_t)_{t \in [0, T]} : \theta \text{ is } \mathbb{R} - \text{valued, progressively measurable \&} \\ |\theta_t| \leq \nu_t\},$$

for a continuous function ν_t .

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In this paper, we show that for every stopping time $\tau \in [0, T]$ and every $B \in \mathcal{F}_\tau$,

$$(1.3) \quad \mathcal{Q}^\theta = \left\{ Q(\cdot) = \int \left[Q^1(\cdot | \mathcal{F}_\tau) 1_B + Q^2(\cdot | \mathcal{F}_\tau) 1_{B^c} \right] dQ_\tau^3 \mid \{Q^i\}_{i=1}^3 \subset \mathcal{Q}_c^\theta \right\}$$

where Q_τ^3 denotes the restriction of Q^3 to \mathcal{F}_τ (See the paper [2] for details). In fact, Chen and Epstein [2] prove (1.3) by using what to be proved. This paper provides the correct proof in detail.

This paper consists of as follows. Introduction is given in section 1. Some definitions such as conditional expectation, stopping times etc. are stated in section 2. The main theorem is given in section 3.

2. SOME DEFINITIONS SUCH AS CONDITIONAL EXPECTATION, STOPPING TIMES ETC.

Suppose that there is a probability space (Ω, \mathcal{F}, P) , and that P and Q are two probability measures on (Ω, \mathcal{F}, P) .

Definition 2.1. Q is called *absolutely continuous* with respect to P on the σ -field \mathcal{F} , if for all $A \in \mathcal{F}$

$$P(A) = 0 \text{ implies } Q(A) = 0.$$

If Q is absolutely continuous with respect to P and vice versa, then it is called that Q and P are *equivalent*.

Definition 2.2. A family of σ -fields $\{\mathcal{F}_t \mid t \in [0, T]\}$ is a *filtration* if $s, t \in [0, T]$ and $t \leq s$ implies $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$.

Theorem 2.3 (Radon-Nikodym). Q is absolutely continuous with respect to P on \mathcal{F} if and only if there exists an \mathcal{F} -measurable function $\varphi \geq 0$ such that

$$\int f dQ = \int f \varphi dP \text{ for all } \mathcal{F}\text{-measurable function } f \geq 0.$$

The function φ is called the *density* or *Radon-Nikodym derivative* of Q with respect to P , denoted by

$$\frac{dQ}{dP} := \varphi.$$

Definition 2.4. Suppose that X is an integrable random variable on (Ω, \mathcal{F}, P) and that \mathcal{G} is a σ -field in \mathcal{F} . The *conditional expectation* of X given \mathcal{G} , is the random variable $E[X|\mathcal{G}]$ which satisfies these two properties

- (1) $E[X|\mathcal{G}]$ is \mathcal{G} -measurable and integrable

(2) $E[X|\mathcal{G}]$ satisfies the functional equation

$$\int_G E[X|\mathcal{G}]dP = \int_G XdP, \quad G \in \mathcal{G}.$$

Such a random variable $E[X|\mathcal{G}]$ exists. Define a measure ν on \mathcal{G} by $\nu(G) = \int_G XdP$ for $X \geq 0$. Since X is integrable, ν is finite and it is absolutely continuous with respect to P . By the Radon-Nikodym theorem there is a function f , \mathcal{G} -measurable, such that $\nu(G) = \int_G fdP$. This f has properties (1) and (2). If X is not nonnegative, $E[X^+|\mathcal{G}] - E[X^-|\mathcal{G}]$ has the required properties. Generally there are many such random variables $E[X|\mathcal{G}]$. Any one of them is called a *version* of the conditional expectation.

The value $E[X|\mathcal{G}]_\omega$ at ω is to be interpreted as the expected value of X for someone who knows for each $G \in \mathcal{G}$ whether or not it contains the point ω , which itself in general remains unknown. Condition (1) ensures that $E[X|\mathcal{G}]$ can in principle be calculated from this partial information alone. From the condition (2), we have $\int_G (X - E[X|\mathcal{G}])dP = 0$. If the observer, in possession of the partial information contained in \mathcal{G} , is offered the opportunity to bet, paying an entry fee of $E[X|\mathcal{G}]$ and being returned the amount X , and if he/she adopts the strategy of betting if G occurs, this equation says that the game is fair [1].

Lemma 2.1. *If X is integrable and the σ -fields \mathcal{G}_1 and \mathcal{G}_2 satisfy $\mathcal{G}_1 \subset \mathcal{G}_2$, then*

$$E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[X|\mathcal{G}_1].$$

If $\mathcal{G}_1 = \{\emptyset, \Omega\}$ and $\mathcal{G}_2 = \mathcal{G}$, then

$$E[E[X|\mathcal{G}]] = E[X].$$

Let $\tau : \Omega \rightarrow [0, T] \cup \{\infty\}$ be a measurable function.

Definition 2.5. A *stopping time* $\tau = \tau(\omega)$ is a random variable satisfying

$$\{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0.$$

A stopping time is a random time that does not require to investigate the future. In gambling, the gambler can make his/her decision to stop gambling based on all of the information upto that time, but not on what will happen in the future.

Since $\{\tau > t - \frac{1}{n}\} = \{\tau \leq t - \frac{1}{n}\}^c \in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t$ for all positive integers n , we have

$$\{\tau = t\} = \{\tau \leq t\} \cap \left(\bigcap_{n=1}^{\infty} \left\{ \tau > t - \frac{1}{n} \right\} \right) \in \mathcal{F}_t.$$

The stopping time τ has the property that the decision to stop at time t must be based on information available at time t . The condition $\{\tau > t\} = \{\tau \leq t\}^c \in \mathcal{F}_t$ means that one's intention at time t to postpone this decision until later is determined by the information \mathcal{F}_t accessible over the period $[0, t]$, and one can't take into consideration the future.

Definition 2.6. Let $\tau = \tau(\omega)$ be a stopping time. Then the set \mathcal{F}_τ is defined as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for each } t \geq 0\}.$$

The proof of the following Corollary 2.7, Proposition 2.8 and Proposition 2.9 are in the book [3].

Corollary 2.7. *If Q is absolutely continuous with respect to P on \mathcal{F} , then*

$$Q \text{ is equivalent to } P \text{ if and only if } \frac{dQ}{dP} > 0 \quad P - a.s.$$

Proposition 2.8 (Bayes's formula). *Suppose that Q is absolutely continuous with respect to P on \mathcal{F} with density φ and that $\mathcal{G} \subset \mathcal{F}$ is another σ -field. Then for any \mathcal{F} -measurable $f \geq 0$,*

$$E_Q[f|\mathcal{G}] = \frac{1}{E_Q[\varphi|\mathcal{G}]} \cdot E[f\varphi|\mathcal{G}] \quad Q - a.s.$$

Proposition 2.9. *Suppose that Q and P are two probability measures on the measurable space (Ω, \mathcal{F}) and that Q is absolutely continuous with respect to P on \mathcal{F} with density φ . If \mathcal{G} is a σ -field contained in \mathcal{F} , then Q is absolutely continuous with respect to P on \mathcal{G} , and the corresponding density is given by*

$$\left. \frac{dQ}{dP} \right|_{\mathcal{G}} = E[\varphi|\mathcal{G}] \quad P - a.s.$$

3. THE MAIN THEOREM

We specify (1.2) in order to permit the decision-maker to have a nonsingleton set of priors. For each $t \geq 0$, let Θ_t be the function such as

$$\Theta_t : \Omega \rightarrow \mathbb{R}^d.$$

We assume that Θ_t have the following properties.

(1) (Uniform Boundedness) There is a compact subset \mathcal{K} in \mathbb{R}^d such that

$$\Theta_t : \Omega \rightarrow \mathcal{K} \text{ for each } t \geq 0.$$

(2) (Compact-Convex) Θ_t is compact and convex.

- (3) (Measurability) The mapping $(t, \omega) \in [0, s] \times \Omega \mapsto \Theta_t(\omega)$ is $\mathcal{B}([0, s]) \times \mathcal{F}_s$ -measurable for any $0 < s \leq T$.
- (4) (Normalization) $0 \in \Theta_t(\omega) \quad dt \otimes dP - a.e.$

Define Θ , the set of density generators as

$$\Theta = \{\theta_t : \theta_t(\omega) \in \Theta_t(\omega) \quad dt \otimes dP - a.e.\}.$$

Θ is called *stochastically convex* if for any real-valued process (λ_t) with $0 \leq \lambda_t \leq 1$,

$$\theta, \theta' \in \Theta \text{ implies } (\lambda_t \theta_t + (1 - \lambda_t) \theta'_t) \in \Theta.$$

Lemma 3.1. *The set of density generators Θ satisfies the followings.*

- (1) $0 \in \Theta$ and $\sup\{\|\theta\|_{L^\infty([0, T] \times \Omega)} : \theta \in \Theta\} < \infty$.
- (2) Θ is stochastically convex and weakly compact in $L^1([0, T] \times \Omega)$.

For each $\theta \in \Theta$ define z_t^θ as

$$z_t^\theta = \exp\left(-\frac{1}{2} \int_0^t |\theta_s|^2 ds + \int_0^t \theta_s dB_s\right), \quad 0 \leq t \leq T.$$

Then $(z_t^\theta)_{0 \leq t \leq T}$ is a P -martingale since $dz_t^\theta = z_t^\theta \theta_t \cdot dB_t$. Also z_T^θ is a P -density on \mathcal{F}_T since $1 = z_0^\theta = E[z_T^\theta]$. If a probability measure Q^θ on (Ω, \mathcal{F}) is defined as

$$Q^\theta(A) = E[1_A z_T^\theta], \quad A \in \mathcal{F}_T,$$

then Q^θ is equivalent to P .

Note that the set of equivalent martingale measures \mathcal{Q}^θ is defined as

$$\mathcal{Q}^\theta := \left\{ Q^\theta \mid \theta \in \Theta, \quad \frac{dQ^\theta}{dP} = z_T^\theta \text{ or } \frac{dQ^\theta}{dP} \Big|_{\mathcal{F}_t} = z_t^\theta \right\}.$$

Now we prove the main theorem.

Theorem 3.1. *For every deterministic $\tau \in [0, T]$ and every $B \in \mathcal{F}_\tau$, Q^θ is represented by*

$$(3.1) \quad \mathcal{Q}^\theta = \left\{ Q \mid Q(\cdot) := \int [Q^1(\cdot | \mathcal{F}_\tau) 1_B + Q^2(\cdot | \mathcal{F}_\tau) 1_{B^c}] dQ_\tau^3 \mid \{Q^i\}_{i=1}^3 \subset \mathcal{Q}^\theta \right\}$$

where Q^i is a short notation for Q^{θ^i} and Q_τ^3 denotes the restriction of Q^3 to \mathcal{F}_τ .

Proof. It's easy to show that \mathcal{Q}^θ is contained in the right hand side of (3.1). Let $Q^\theta \in \mathcal{Q}^\theta$. Then we have $\frac{dQ^\theta}{dP} \Big|_{\mathcal{F}_t} = z_t^\theta$ and $E[z_T^\theta | \mathcal{F}_\tau] = z_\tau^\theta$ by Proposition 2.9.

If we take $Q^i = Q^\theta$ for $i = 1, 2, 3$, then for $A \in \mathcal{F}_T$

$$\begin{aligned} \int [Q^1(A|\mathcal{F}_\tau)1_B + Q^2(A|\mathcal{F}_\tau)1_{B^c}] dQ_\tau^3 &= \int [Q^\theta(A|\mathcal{F}_\tau)1_B + Q^\theta(A|\mathcal{F}_\tau)1_{B^c}] dQ_\tau^\theta \\ &= \int Q^\theta(A|\mathcal{F}_\tau)z_\tau dP = \int \frac{E[1_A z_T^\theta | \mathcal{F}_\tau]}{E[z_T^\theta | \mathcal{F}_\tau]} z_\tau dP \\ &= E[E[1_A z_T^\theta | \mathcal{F}_\tau]] = E[1_A z_T^\theta] \\ &= Q^\theta(A). \end{aligned}$$

Next we show the reverse inclusion. Let $Q^i \in \mathcal{Q}^\theta$ for $i = 1, 2, 3$. Define θ_t as

$$\theta_t = 1_{\{t \geq \tau\} \times B} \theta_t^1 + 1_{\{t \geq \tau\} \times B^c} \theta_t^2 + 1_{\{t < \tau\}} \theta_t^3.$$

Then $\theta_t \in \Theta$. Moreover, by a direct calculation we can get

$$\begin{aligned} z_t^\theta &= \exp\left(-\frac{1}{2} \int_0^t |\theta_s|^2 ds + \int_0^t \theta_s dB_s\right) \\ &= \begin{cases} z_\tau^3 z_t^1 / z_\tau^1 & \text{if } t \geq \tau \text{ and } \omega \in B \\ z_\tau^3 z_t^2 / z_\tau^2 & \text{if } t \geq \tau \text{ and } \omega \in B^c. \end{cases} \end{aligned}$$

For $A \in \mathcal{F}_T$ define Q^θ as

$$Q^\theta(A) = \int [Q^1(A|\mathcal{F}_\tau)1_B + Q^2(A|\mathcal{F}_\tau)1_{B^c}] dQ_\tau^3.$$

By the Monotone Convergence Theorem for conditional expectation, Q^θ is a probability measure and becomes

$$E_{Q^\theta}[Y] = E_{Q_\tau^3} [E_{Q^1}[Y|\mathcal{F}_\tau]1_B + E_{Q^2}[Y|\mathcal{F}_\tau]1_{B^c}], \quad Y \geq 0.$$

We show that Q^θ is equivalent to P . Let φ_T be the density process of Q^1 with respect to Q_τ^3 . Then we have

$$\varphi_T = \frac{dQ^1}{dQ_\tau^3} = \frac{dQ^1/dP}{dQ_\tau^3/dP} = \frac{z_T^1}{z_\tau^3} \text{ and } \varphi_\tau = E_{Q_\tau^3}[\varphi_T | \mathcal{F}_\tau] = \frac{z_\tau^1}{z_\tau^3}$$

by Proposition 2.9. For $\omega \in B$ and $Y \geq 0$, we have

$$\begin{aligned} E_{Q^\theta}[Y] &= E_{Q_\tau^3} [E_{Q^1}[Y|\mathcal{F}_\tau]] \\ &= E_{Q_\tau^3} \left[\frac{1}{\varphi_\tau} E_{Q_\tau^3}[\varphi_T Y | \mathcal{F}_\tau] \right] = E_{Q_\tau^3} [E_{Q_\tau^3}[\varphi_T / \varphi_\tau Y | \mathcal{F}_\tau]] \\ &= E_{Q_\tau^3} \left[\frac{z_T^1}{z_\tau^1} Y \right] = E \left[\frac{z_\tau^3 z_T^1}{z_\tau^1} Y \right] \\ (3.2) \quad &= E[z_T^\theta Y]. \end{aligned}$$

For $\omega \in B^c$, we can similarly show that

$$\begin{aligned} E_{Q^\theta}[Y] &= E_{Q_\tau^3} [E_{Q^2}[Y|\mathcal{F}_\tau]] = E \left[\frac{z_\tau^3 z_T^2}{z_\tau^2} Y \right] \\ (3.3) \qquad &= E[z_T^\theta Y]. \end{aligned}$$

By taking $Y = 1_A \in \mathcal{F}_T$ in (3.2) and (3.3), we have

$$Q^\theta(A) = E[z_T^\theta 1_A].$$

Therefore, we have shown that $\frac{dQ^\theta}{dP} = z_T^\theta > 0$. Thus Q^θ is equivalent to P . \square

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