J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2016.23.4.377 ISSN(Online) 2287-6081
Volume 23, Number 4 (November 2016), Pages 377–383

THE MAXIMAL PRIOR SET IN THE REPRESENTATION OF COHERENT RISK MEASURE

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Abstract. The set of priors in the representation of coherent risk measure is expressed in terms of quantile function and increasing concave function. We show that the set of prior, $\mathcal{Q}_c$ in (1.2) is equal to the set of $\mathcal{Q}_m$ in (1.6), as maximal representing set $\mathcal{Q}_{max}$ defined in (1.7).

1. Introduction

Kim [4] showed that the set of priors in the representation of Choquet expectation [2] is the one of equivalent martingale measures under some conditions, when the distortion is submodular. That is, if a capacity c is submodular, then the coherent risk measure is represented as

$$\rho(X) := \int X\, dc = \max_{Q \in \mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T), \tag{1.1}$$

where $\mathcal{Q}_c$ is defined as

$$\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} \,:\, Q[A] \le c(A) \quad \forall A \in \mathcal{F}_T\} \tag{1.2}$$

that is equal to the maximal set $\mathcal{Q}_{max}$ representing ρ. Here $M_{1,f} := M_{1,f}(\Omega, \mathcal{F})$ is the set of all finitely additive normalized set functions $Q : \mathcal{F} \to [0,1]$.

By using g-expectation [6] and related topics [1, 8], Kim [4] showed that Q_c equals to $\mathcal{Q}^\theta$ where $\mathcal{Q}^\theta$ and Θ^g are respectively defined as

$$\mathcal{Q}^\theta := \left\{ Q^\theta : \theta \in \Theta^g, \frac{dQ^\theta}{dP}\Big|_{\mathcal{F}_t} = exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds \right) \right\} \tag{1.3}$$

and

$$\Theta^g = \{(\theta_t)_{t\in[0,T]} : \quad \theta \text{ is } \mathbb{R}-\text{valued, progressively measurable } \& \; |\theta_t| \le \nu_t\}, \tag{1.4}$$

Received by the editors September 11, 2016. Accepted October 04, 2016
2010 *Mathematics Subject Classification.* 60G42, 60G44, 60H10.
Key words and phrases. set of priors, coherent risk measure, Choquet expectation, quantile, minimal penalty function.

for a continuous function ν_t for $t \in [0, T]$.

We consider the Banach spaces $L^p(\Omega, \mathcal{F}, P)$ for $1 \le p < \infty$. Let $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and define

$$\mathcal{M}_1^q(P) := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, \frac{dQ}{dP} \in L^q \right\}.$$

It is well-known in the literature [3, 5, 7] that the coherent risk measures ρ_m defined as

$$\rho_m(X) := \int_{(0,1]} AV@R_\lambda(X)\, m(d\lambda)$$

for $m \in \mathcal{M} \subset \mathcal{M}_1^2((0, 1])$, can be expressed as Choquet expectation and consequently

$$\rho_m(X) = \max_{Q \in \mathcal{Q}_m} E_Q[-X], \tag{1.5}$$

where the set $\mathcal{Q}_m$ is defined as

(1.6)
$$\mathcal{Q}_m := \left\{ Q \in \mathcal{M}_1^2(P) \,\middle|\, \varphi := \frac{dQ}{dP} \text{ satisfies } \int_t^1 q_\varphi(s) ds \le \psi(1-t) \text{ for } t \in (0, 1) \right\}.$$

In this paper, we show that $\mathcal{Q}_{c_\psi} = \mathcal{Q}_{max} = \mathcal{Q}_m$ as maximal representing set,

$$\mathcal{Q}_{max} = \{ Q \in \mathcal{M}_1^2(P) \,|\, \alpha_{min}(Q) = 0 \}. \tag{1.7}$$

The minimal penalty function α_{min} is defined as

$$\alpha_{min}(Q) := \sup_{X \in \mathcal{A}_\rho} E_Q[-X] \quad \text{ for } Q \in \mathcal{M}_1^2(P),$$

where $\mathcal{A}_\rho$ is the acceptance set of ρ on a measurable set $\mathcal{X}$ defined as

$$\mathcal{A}_\rho := \{ X \in \mathcal{X} \,|\, \rho(X) \le 0 \}.$$

This paper consists of as follows. Introduction is given in section 1. Some primary definitions such as coherent risk measure, quantile function etc. are stated in section 2. The main theorem is given in section 3.

2. Some Primary Definitions

We give some definitions such as coherent risk measure, λ-quantile, Choquet expectation, etc..

Definition 2.1. For $\lambda \in (0, 1)$, a λ-*quantile* of a random variable X on $(\Omega, \mathcal{F}, P)$ is defined as any real number q satisfying

$$P[X \leq q] \geq \lambda \text{ and } P[X < q] \leq \lambda,$$

and the set of all λ-quantiles of X is an interval $[q_X^-(\lambda), q_X^+(\lambda)]$, where

$$q_X^-(\lambda) = \sup\{x \,|\, P[X < x] < t\} = \inf\{x \,|\, P[X \leq x] \geq t\}$$

and

$$q_X^+(\lambda) = \inf\{x \,|\, P[X \leq x] > t\} = \sup\{x \,|\, P[X < x] \leq t\}.$$

Definition 2.2. Let $\lambda \in (0,1)$ and $X \in \mathcal{X}$. The *Value at Risk* at level λ is defined as

$$V@R_\lambda(X) := -q_X^+ = q_{-X}^-(1-\lambda) = \inf\{\sigma \,|\, P[X + \sigma < 0] \leq \lambda\}.$$

Definition 2.3. The *Average Value at Risk* at level $\lambda \in (0,1]$ of a position $X \in \mathcal{X}$ is defined as

$$AV@R_\lambda(X) := \frac{1}{\lambda} \int_0^\lambda V@R_s(X)\, ds.$$

Definition 2.4. A *coherent risk measure* $\rho : L^2 \to \mathbb{R} \cup \{\infty\}$ is a mapping satifying the following properties for X, $Y \in L^2$

(1) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ (subadditivity),
(2) $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$ (positive homogeneity),
(3) $\rho(X) \geq \rho(Y)$ if $X \leq Y$ (monotonicity),
(4) $\rho(Y + m) = \rho(Y) - m$ for $m \in \mathbb{R}$ (translation invariance).

Definition 2.5. A mapping $\rho : L^2 \to \mathbb{R} \cup \{\infty\}$ is called a *monetary measure* of risk if it satisfies the conditions (3) (montonicity) and (4) (translation invariance) in Definition 2.4.

Definition 2.6. A monetary measure of risk ρ on $\mathcal{X} = L^2(\Omega, \mathcal{F}, P)$ is called *law-invariant* if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under P.

Definition 2.7. A coherent risk measure ρ on $\mathcal{X} = L^2$ is called that ρ *is continuous from above* if it satisfies

$$X_n \searrow X \Longrightarrow \rho(X_n) \nearrow \rho(X),$$

and that ρ *is continuous from below* if it satisfies

$$X_n \nearrow X \Longrightarrow \rho(X_n) \searrow \rho(X).$$

Assume that given probability space $(\Omega, \mathcal{F}, P)$ is atomless throughout this paper.

Theorem 2.1 ([3]). *A coherent risk measure ρ is continuous from above and law-invariant if and only if*

$$\rho(X) := \sup_{m \in \mathcal{M}} \int_{(0,1]} AV@R_\lambda(X)\, m(d\lambda)$$

for some set $\mathcal{M} \subset \mathcal{M}_1^2((0,1])$.

Define the coherent risk measures ρ_m as

$$\rho_m(X) := \int_{(0,1]} AV@R_\lambda(X)\, m(d\lambda)$$

for $m \in \mathcal{M} \subset \mathcal{M}_1^2((0,1])$.

Since $AV@R_\lambda$ is coherent, continuous from below and law-invariant, so are any mixture ρ_m for a probability measure m on $(0, 1]$.

It is shown in Theorem 3.3 that ρ_m can be identified with the Choquet integral of the loss $-X$ with respect to the set function $c_\psi(A) := \psi(P[A])$, where ψ is the concave function defined in Lemma 3.2.

Definition 2.8. A set function $c : \mathcal{F} \to [0, 1]$ is called *monotone* if

$$c(A) \le c(B) \quad \text{for} \quad A \subset B$$

and *normalized* if

$$c(\emptyset) = 0 \quad \text{and} \quad c(\Omega) = 1.$$

A monotone set function is called *submodular* if

$$c(A \cup B) + c(A \cap B) \le c(A) + c(B).$$

Definition 2.9. Let $c : \mathcal{F} \to [0, 1]$ be any set function which is normalized and monotone. Then *Choquet integral* of a bounded measurable function X on $(\Omega, \mathcal{F})$ with respect to c is defined as

$$\int X\, dc := \int_{-\infty}^{0} (c(X > x) - 1) dx + \int_{0}^{\infty} c(X > x) dx.$$

Definition 2.10. Let $\psi : [0, 1] \to [0, 1]$ be an increasing function such that $\psi(0) = 0$ and $\psi(1) = 1$. The set function

$$c_\psi(A) = \psi(P[A]), \quad A \in \mathcal{F},$$

is called the *distortion* of the probability measure P with respect to the distiortion function ψ.

3. The Main Theorem

In this section, the main theorem is stated and proven. Proposition 3.1, Lemma 3.2, Theorem 3.3 and Theorem 3.4 are quoted in the book [3].

Proposition 3.1. *Let c_ψ be the distortion of P with respect to the distiortion function ψ. If ψ is concave, then c_ψ is submodular.*

Note that a concave function ψ take a right-continuous right-hand derivative ψ'_+.

Lemma 3.2. *The identity*

$$\psi'_+(t) = \int_{(t,1]} s^{-1}\, m(ds), \quad 0 < t < 1,$$

defines a one-to-one correspondence between probability measures m on $[0,1]$ and increasing concave functions $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$. Moreover, we have $\psi(0+) = m(\{0\})$.

Theorem 3.3. *Let m be a probability measure on $[0,1]$ and ψ be the concave function defined in Lemma 3.2. Then for $X \in \mathcal{X}$,*

$$\begin{aligned} \rho_m(-X) &= \psi(0+)AV@R_0(-X) + \int_0^1 q_X(t)\psi'(1-t)\, dt \\ &= \int_{[0,1]} X\, dc_\psi. \end{aligned}$$

Theorem 3.4. *Let ρ be a coherent risk measure and suppose that ρ is continuous from above. Then ρ is law-invariant if and only if its minimal penalty function $\alpha_{min}(Q)$ depends only on the law of $\varphi_Q := \frac{dQ}{dP}$ under P when $Q \in \mathcal{M}_1^q(P)$. In this case, ρ has the representation*

$$\rho(X) = \sup_{Q\in\mathcal{M}_1^q(P)} \left(\int_0^1 q_{-X}(t) q_{\varphi_Q}(t)\, dt - \alpha_{min}(Q) \right),$$

and the minimal penalty function satisfies

$$\alpha_{min}(Q) = \sup_{X\in\mathcal{A}_\rho} \int_0^1 q_{-X}(t) q_{\varphi_Q}(t)\, dt = 0. \tag{3.1}$$

Theorem 3.5. *Let m be a probability measure on $[0,1]$. Let ψ be the corresponding concave function defined in Lemma 3.2. Then ρ_m can be represented*

$$\rho_m(X) = \max_{Q\in\mathcal{Q}_m} E_Q[-X],$$

where the set $\mathcal{Q}_m$ is defined as

$$\mathcal{Q}_m := \left\{ Q \in \mathcal{M}_1^2(P) \,\Big|\, \varphi := \frac{dQ}{dP} \text{ satisfies } \int_t^1 q_\varphi(s)ds \le \psi(1-t) \text{ for } t \in (0,1) \right\}.$$

Here $\mathcal{Q}_m$ is the maximal subset of $\mathcal{M}_1^2(P)$ that represents ρ_m.

Proof. The risk measure ρ_m is clearly coherent and continuous from above. The ρ_m can be represented by taking the supremum of expectations over the set $\mathcal{Q}_{max} = \{Q \in \mathcal{M}_1^2(P) \,|\, \alpha_{min}(Q) = 0\}$. The equation (3.1) and Theorem 3.3 imply that a measure $Q \in \mathcal{M}_1^2(P)$ with density $\varphi = dQ/dP$ belongs to $\mathcal{Q}_{max}$ if and onlt if

$$\begin{aligned} \int_0^1 q_X(s) q_\varphi(s)\, ds &\le \rho_m(-X) \\ &= \psi(0+) AV@R_0(-X) + \int_0^1 q_X(s)\psi'(1-s)\, ds \end{aligned} \tag{3.2}$$

for all $X \in L^2$. Set $X \equiv t$ which is constant random variable. Then we have $q_X = I_{[t,1]}$ a.e., $AV@R_0(-X) := V@R_0(-X) := \operatorname{ess\,sup}(t) \le 1$ and the inequalty (3.2) becomes

$$\int_t^1 q_\varphi(s)\, ds \le \psi(0+) + \int_t^1 \psi'(1-s)\, ds = \psi(1-t)$$

for all $t \in (0,1)$. Therefore, $\mathcal{Q}_{max} \subset \mathcal{Q}_m$. Now let's prove the reverse inclusion $\mathcal{Q}_m \subset \mathcal{Q}_{max}$. Let $Q \in \mathcal{Q}_m$ be fixed. We show that the density $\varphi = dQ/dP$ satisfies the equation (3.2) for any given $X \in L^2$. Let ν be the positive finite measure on $[0,1]$ such that $q_X^+(s) = \nu([0,s])$. Fubini's theorem and the definition of $\mathcal{Q}_m$ imply that

$$\begin{aligned} \int_0^1 q_X(s) q_\varphi(s)\, ds &= \int_{[0,1]} \int_t^1 q_\varphi\, ds\, \nu(dt) \\ &\le \int_{[0,1]} \psi(1-t)\, \nu(dt) \\ &= \psi(0+)\nu([0,1]) + \int_0^1 \psi'(1-s) \int_{[0,s]} \nu(dt)\, ds, \end{aligned}$$

which coincides with the right-hand side of (3.2). Note that $\nu([0,1]) = q_X^+(1) = \sup\{x \in \mathbb{R} \,|\, F_X(x) \le 1\} = \operatorname{ess\,sup} X := AV@R_0(-X)$ and $\int_{[0,s]} \nu(dt) = \nu([0,s]) = q_X^+(s) = q_X(s)$ a.e.. □

The following is the main theorem.

Theorem 3.6. *We have*

$$\mathcal{Q}_m = \mathcal{Q}_{c_\psi} = \mathcal{Q}_{max}.$$

Moreover, we get

$$\rho_m(-X) = \max_{Q \in \mathcal{Q}_m} E_Q[X] = \int_{[0,1]} X \, dc_\psi = \max_{Q \in \mathcal{Q}_{c_\psi}} E_Q[X].$$

Proof. The equation (1.1), Theorem 3.3 and Theorem 3.5 give the proof. □

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