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ASYMPTOTIC BEHAVIOR OF A CERTAIN SECOND-ORDER INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this paper we obtain some integral inequalities by using Stieltjes derivatives, and we apply our results to the study of asymptotic behavior of a certain second-order integro-differential equation.

1. Introduction

Differential equations arise in various real world phenomena in mathematical physics, mechanics, engineering, biology and so on. Also integral inequalities are very useful tools in global existence, uniqueness, stability and other properties of the solutions of various nonlinear differential equations, see, e.g., [6, 7].

In this paper, we obtain some integral inequalities of Stieltjes type, and apply the inequalities to the study of asymptotic behavior of a certain second-order integro-differential equation.

The asymptotic behaviors of various second-order nonlinear differential equations have been studied by many authors, see, e.g., [5, 8] and the references cited there.

2. Preliminaries

In this section we state some materials that are needed in this paper.

Let $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ be the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integers, respectively. And for $a \in \mathbf{R}$, we let $\mathbf{R}_a^+ = [a, \infty)$, and let

$G(\mathbf{R}_a^+) = \{f : \mathbf{R}_a^+ \to \mathbf{R} |\ \forall t > a$, both $f(t+)$ and $f(t-)$ exist, and $f(a+)$ exists $\}$.

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For convenience we define

$$\Delta^+ f(t) = f(t+) - f(t),\ \Delta^- f(t) = f(t) - f(t-),\ \Delta f(t) = f(t+) - f(t-).$$

Throughout this paper we use the Kurzweil-Stieltjes integral(sometimes the integral is called as the Perron-Stieltjes integral, see, e.g., [10, 11]), and the Stieltjes derivative. For the integral and derivative, and various notations and results that are used here, see, e.g., [2, 3, 4, 9, 10] and the references cited there.

We use the following results frequently.

Theorem 2.1 ([11, Theorem 2.15])**.** *Assume that $f \in G([a,b])$ and $\alpha \in BV([a,b])$. Then both $f\,\mathrm{d}\alpha$ and $g\,\mathrm{d}f$ are Kurzweil integrable on $[a,b]$.*

Theorem 2.2 ([2, 3])**.** *Assume that $f \in G([a,b])$ and a function $\alpha : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and is not locally constant at $t \in [a,b]$. If f is continuous at t or α is not continuous at t, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(t)} \int_a^t f(s)\,\mathrm{d}\alpha(s) = f(t).$$

Theorem 2.3 ([2, 3])**.** *Assume that $f \in G([a,b])$ and a function $\alpha : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and that if α is constant on some neighborhood of t, then there exists a neighborhood of t such that both f and α are constant there. Suppose that $f'_\alpha(t)$ exists at every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*) \int_a^b f'_\alpha(s)\,\mathrm{d}\alpha(s) = f(b) - f(a).$$

From now on, for a function $f \in G(\mathbf{R}_a^+)$, we define $\mathbf{C}_f$ as the set of continuities of the function f.

Corollary 2.4. *Assume that $f \in G(\mathbf{R}_a^+)$ and a function $\alpha : \mathbf{R}_a^+ \longrightarrow \mathbf{R}$ is nondecreasing, left-continuous, and that if α is constant on some neighborhood of t, then there exists a neighborhood of t such that both f and α are constant there. Suppose that $f'_\alpha(t)$ exists at every $t \in \mathbf{R}_a^+ - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have for every $t \in \mathbf{R}_a^+$,*

$$(K^*) \int_a^t f'_\alpha(s)\,\mathrm{d}\alpha(s) = f(t-) - f(a). \tag{2.1}$$

Proof. Since f, $\alpha \in G(\mathbf{R}_a^+)$, the set of discontinuities of f and α is countable([1, p.17, Corollary 3.2]). Let $t_n \longrightarrow t-$ as $n \to \infty$, where $t_n \in \mathbf{C}_f \cap \mathbf{C}_\alpha$. Then by Theorem 2.3, we have

$$\lim_{n\to\infty} (K^*) \int_a^{t_n} f'_\alpha(s)\,\mathrm{d}\alpha(s) = \lim_{n\to\infty} f(t_n) - f(a) = f(t-) - f(a). \tag{2.2}$$

Since $t_n \in \mathbf{C}_f \cap \mathbf{C}_\alpha$, by the definition of the K*-integral, we have

$$\lim_{n\to\infty} (K^*) \int_a^{t_n} f'_\alpha(s)\,\mathrm{d}\alpha(s) = (K^*) \int_a^{t} f'_\alpha(s)\,\mathrm{d}\alpha(s). \tag{2.3}$$

Thus (2.2) and (2.3) yield (2.1). The proof is complete. □

Theorem 2.5 ([10, p. 45, Theorem 4.32])**.** *If $h : [a,b] \times [a,b] \longrightarrow \mathbf{R}$ is bounded on $[a,b] \times [a,b]$ and $\mathrm{var}_a^b h(s,\cdot) < \infty$ for every $s \in [a,b]$, $\mathrm{var}_a^b h(\cdot,t) < \infty$ for every $t \in [a,b]$, then for any $f, g \in BV([a,b])$ we have*

$$\begin{aligned}\int_a^b \int_a^t h(s,t)\,\mathrm{d}f(s)\,\mathrm{d}g(t) &= \int_a^b \int_s^b h(s,t)\,\mathrm{d}g(t)\,\mathrm{d}f(s)\\ &+ \sum_{t\in(a,b]} \Delta^- f(t)\Delta^- g(t)h(t,t) - \sum_{t\in[a,b)} \Delta^+ f(t)\Delta^+ g(t)h(t,t).\end{aligned}$$

Remark 2.6. In the above theorem, the sums $\sum_{t\in(a,b]}$ and $\sum_{t\in[a,b)}$ are actually countable sums, since $f, g \in BV([a,b])$ implies that the sets of discontinuities of f and g are countable, respectively.

Corollary 2.7. *If $h \in G([a,b])$, and $f, g \in BV([a,b])$ are left-continuous on $[a,b]$, then for every $t \in [a,b]$, we have*

$$\int_a^t \int_a^s h(v)\,\mathrm{d}f(v)\,\mathrm{d}g(s) = \int_a^t [g(t) - g(v+)]h(v)\,\mathrm{d}f(v).$$

Proof. Since $h \in G(]a,b])$, there is a sequence $\{h_n\}$ of step functions that is uniformly convergent to h on $[a,b]$ ([1, p. 16, Theorem 3.1]). Since f is left-continuous on $[a,b]$, $\Delta^+ f(t) = \Delta f(t)$, and so we have

$$\sum_{v\in[a,t)} \Delta^+ f(v)\Delta^+ g(v)h_n(v) = \sum_{v\in[a,t)} \Delta^+ g(v)h_n(v)\Delta f(v) = \int_a^t \Delta^+ g(v)h_n(v)\,\mathrm{d}f(v),$$

(see the bottom of [10, p.47]). Hence by Theorem 2.5, we get

$$\int_a^t \int_a^s h_n(v)\,\mathrm{d}f(v)\,\mathrm{d}g(s) = \int_a^t \int_v^t h_n(v)\,\mathrm{d}g(s)\,\mathrm{d}f(v) - \sum_{v\in[a,t)} \Delta^+ f(v)\Delta^+ g(v) h_n(v)$$

$$= \int_a^t [g(t)-g(v)]h_n(v)\,\mathrm{d}f(v) - \int_a^t \Delta^+ g(v) h_n(v)\,\mathrm{d}f(v)$$

$$= \int_a^t [g(t)-g(v)-\Delta^+ g(v)]h_n(v)\,\mathrm{d}f(v) = \int_a^t [g(t)-g(v+)]h_n(v)\,\mathrm{d}f(v).$$

Thus by [9, Corollary 1.32] we have

$$\int_a^t \int_a^s h(v)\,\mathrm{d}f(v)\,\mathrm{d}g(s) = \lim_{n\to\infty} \int_a^t \int_a^s h_n(v)\,\mathrm{d}f(v)\,\mathrm{d}g(s)$$

$$= \lim_{n\to\infty} \int_a^t [g(t)-g(v+)]h_n(v)\,\mathrm{d}f(v) = \int_a^t [g(t)-g(v+)]h(v)\,\mathrm{d}f(v).$$

The proof is complete. □

The following result is a L'Hôpital's rule for Stieltjes derivatives.

Theorem 2.8. *Let functions $f, g, \alpha \in G(\mathbf{R}_a^+)$ be left-continuous and α be nondecreasing on $\mathbf{R}_a^+$, If both $f'_\alpha(t)$ and $g'_\alpha(t)$ exist for all $t \in \mathbf{R}_a^+$ with $\lim_{t\to\infty} g(t) = \infty$, and $g'_\alpha(t) > 0$. Then, $\lim_{t\to\infty} \frac{f'_\alpha(t)}{g'_\alpha(t)} = L \in \mathbf{R}$ implies $\lim_{t\to\infty} \frac{f(t)}{g(t)} = L$.*

Proof. Since $\lim_{t\to\infty} \frac{f'_\alpha(t)}{g'_\alpha(t)} = L$, there is $N(\varepsilon) > 0$ such that

$$\left| \frac{f'_\alpha(t)}{g'_\alpha(t)} - L \right| \le \varepsilon,$$

for all $t > N(\varepsilon)$. This implies that

$$-\varepsilon g'_\alpha(t) \le f'_\alpha(t) - L g'_\alpha(t) \le \varepsilon g'_\alpha(t).$$

Integrating the above inequalities from s to t, where $s \in \mathbf{C}_f \cap \mathbf{C}_g$, by Theorem 2.3, we have

$$-\varepsilon[g(t)-g(s)] \le [f(t)-f(s)] - L[g(t)-g(s)] \le \varepsilon[g(t)-g(s)].$$

So we get

$$(L-\varepsilon)[g(t)-g(s)] \le f(t)-f(s) \le (L+\varepsilon)[g(t)-g(s)].$$

This implies

$$(L-\varepsilon)\left(1-\frac{g(s)}{g(t)}\right) \le \frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \le (L+\varepsilon)\left(1-\frac{g(s)}{g(t)}\right).$$

Letting $t \to \infty$, we get

$$L-\varepsilon \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le L+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = L.$$

The proof is complete. □

From now on we define

$$\int_a^\infty f(s)\,\mathrm{d}\alpha(s) = \lim_{t\to\infty} \int_a^t f(s)\,\mathrm{d}\alpha(s)$$

Theorem 2.9. *If* $\int_a^\infty |f(s)|\,\mathrm{d}\alpha(s)$ *exists and* α *is nondecreasing, then* $\int_a^\infty f(s)\,\mathrm{d}\alpha(s)$ *is also convergent.*

Proof. The proof of this theorem is not difficult and essentially identical to that of the Riemann integral. So we omit it. □

Throughout this paper, unless otherwise specified, we always assume the following hypotheses:

(H1) All one variable functions belong to $G(\mathbf{R}_1^+)$.

(H1) A function $k : \mathbf{R}_1^+ \times \mathbf{R}_1^+ \longrightarrow \mathbf{R}^+$ satisfies that, for every $S, T \ge 1$, k is bounded on $[1, S] \times [1, T]$, and for fixed $s, t \in \mathbf{R}_1^+$, $k(\cdot, s), k(t, \cdot) \in G(\mathbf{R}_1^+)$.

(H3) Functions m_i's : $\mathbf{R}_1^+ \longrightarrow \mathbf{R}$, $(i \in \mathbf{N})$ are all strictly increasing on $\mathbf{R}_1^+$, and a function $w : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is nondecreasing, continuous and positive on $(0, \infty)$. And for $t \in (0, \infty)$ we define

$$E(t) = \int_1^t \frac{\mathrm{d}s}{w(s)},$$

and E^{-1} represents the inverse of the function E, and $\mathrm{Dom}(E^{-1})$ represents the domain of the function E^{-1}.

3. Some Integral Inequalities

Throughout this section all functions are nonnegative on their domains respectively.

In order to obtain some integral inequalities we need the following result.

Lemma 3.1 ([4])**.** *Let $\alpha : \mathbf{R}_1^+ \longrightarrow \mathbf{R}$ be nondecreasing and left-continuous on $\mathbf{R}_1^+$. Assume that a positive left-continuous function z is nondecreasing on $\mathbf{R}_1^+$.*

If z is continuous at t and $z'_\alpha(t)$ exists, then we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(t)}E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}\alpha(t)}\int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} = \frac{z'_\alpha(t)}{w(z(t))}.$$

If $t \notin \mathbf{C}_\alpha$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(t)}E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}\alpha(t)}\int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} \le \frac{z'_\alpha(t)}{w(z(t))}.$$

The following result is a Gronwall-Bellman type integral inequality.

Theorem 3.2. *Assume that a function c is nondecreasing on $\mathbf{R}_1^+$ and that a nondecreasing function α is left-continuous on $\mathbf{R}_1^+$. If a nonnegative function u satisfies*

$$u(t) \le c(t) + \int_1^t w(u(s))\,\mathrm{d}\alpha(s), \tag{3.1}$$

then for $1 \le t \le T$ we have

$$u(t) \le E^{-1}\Big[E(c(t)) + \alpha(t) - \alpha(1)\Big], \tag{3.2}$$

where the number T is chosen so that, for all $t \in [1, T]$, $E(c(t)) + \alpha(t) - \alpha(1) \in$ $\mathrm{Dom}(E^{-1})$.

Proof. From (3.1), we have, for $1 \le s \le t \le T$, where t is fixed,

$$u(s) \le c(s) + \int_1^s w(u(\sigma))\,\mathrm{d}\alpha(\sigma) \le c(t) + \int_1^s w(u(\sigma))\,\mathrm{d}\alpha(\sigma) \equiv z(s).$$

Then by Theorem 2.2 we have, except for a countable subset of $\mathbf{C}_\alpha$,

$$z'_\alpha(s) = w(u(s)) \le w(z(s)).$$

This implies

$$\frac{z'_\alpha(s)}{w(z(s))} \le 1.$$

Thus by Lemma 3.1 and Theorem 2.3 we get

$$(E\circ z)(s) - (E\circ z)(1) = \int_1^s (E\circ z)'_\alpha(\sigma)\,\mathrm{d}\alpha(\sigma)$$

$$\le \int_1^s \frac{z'_\alpha(\sigma)}{w(z(\sigma))}\,\mathrm{d}\alpha(\sigma) \le \int_1^s \mathrm{d}\alpha(\sigma) = \alpha(s) - \alpha(1).$$

This yields for every $s \in [0,t]$

$$u(s) \le z(s) \le E^{-1}\Big[E(z(1)) + \alpha(s) - \alpha(1)\Big] = E^{-1}\Big[E(c(t)) + \alpha(s) - \alpha(1)\Big].$$

Since the above inequality is true for $s = t$ we have

$$u(t) \le E^{-1}\Big[E(c(t)) + \alpha(t) - \alpha(1)\Big].$$

This inequality yields (3.2). Since the number $t \in [1,T]$ was arbitrary, the proof is complete. $\square$

Theorem 3.3. *Assume that a function c is nondecreasing on $\mathbf{R}_1^+$ and a nondecreasing function m_1 is left-continuous on $\mathbf{R}_1^+$. If a nonnegative function u satisfies*

$$u(t) \le c(t) + \int_1^t \left[a(s)w(u(s)) + \int_1^s k(s,\sigma)w(u(\sigma))\,\mathrm{d}m_2(\sigma)\right]\mathrm{d}m_1(s), \tag{3.3}$$

then for $1 \le t \le T$ we have

$$u(t) \le E^{-1}[\gamma(t)],$$

where

$$\gamma(t) = E(c(t)) + \int_1^t \left[a(s) + \int_1^s k(s,\sigma)\,\mathrm{d}m_2(\sigma)\right]\mathrm{d}m_1(s),$$

and the number T is chosen so that

$$\gamma(t) \in \mathrm{Dom}(E^{-1}),$$

for every $t \in [1,T]$.

Proof. Assume that $1 \le t \le t^* \le T$, where t^* is fixed. Then since the function c is nondecreasing, for all $t \in [1, t^*]$, we have

$$u(t) \le c(t^*) + \int_1^t \left[a(s)w(u(s)) + \int_1^s k(s,\,\sigma)w(u(\,\sigma))\,\mathrm{d}m_2(\,\sigma) \right] \mathrm{d}m_1(s) \equiv z(t). \tag{3.4}$$

Then $u(t) \le z(t)$. Now by Theorem 2.2, except for a countable subset of $\mathbf{C}_{m_1}$, we have

$$\begin{aligned} z'_{m_1}(t) &\le a(t)w(u(t)) + \int_1^t k(t,\,\sigma)w(u(\,\sigma))\,\mathrm{d}m_2(\,\sigma) \\ &\le \left[a(t) + \int_1^t k(t,\,\sigma)\,\mathrm{d}m_2(\,\sigma) \right] w(z(t)). \end{aligned}$$

So by Lemma 3.1, we have

$$(E \circ z)'_{m_1}(t) \le \frac{z'_{m_1}(t)}{w(z(t))} \le a(t) + \int_1^t k(t,\,\sigma)\,\mathrm{d}m_2(\,\sigma).$$

Thus by Theorem 2.3 we get

$$\begin{aligned} &E(z(t)) - E(z(1)) \\ &= \int_1^t (E \circ z)'_{m_1}\,\mathrm{d}m_1 \le \int_1^t \left[a(s) + \int_1^s k(s,\,\sigma)\,\mathrm{d}m_2(\,\sigma) \right] \mathrm{d}m_1(s). \end{aligned}$$

This yields

$$u(t) \le z(t) \le E^{-1} \left[E(c(t^*)) + \int_1^t \left[a(s) + \int_1^s k(s,\,\sigma)\,\mathrm{d}m_2(\,\sigma) \right] \mathrm{d}m_1(s) \right].$$

Since the above inequality is true for $t = t^*$, we get

$$u(t^*) \le E^{-1} \left[E(c(t^*)) + \int_1^{t^*} \left[a(s) + \int_1^s k(s,\,\sigma)\,\mathrm{d}m_2(\,\sigma) \right] \mathrm{d}m_1(s) \right].$$

Since $t^* \in [0, T]$ was arbitrary, the proof is complete. □

Similarly we can obtain the following result.

Theorem 3.4. *Let $s, t, u, v, S, T, U, V \ge 1$ be arbitrarily fixed numbers. Assume that a function $k : (\mathbf{R}_1^+)^4 \longrightarrow \mathbf{R}^+$ satisfies that, k is bounded on $[1, S] \times [1, T] \times$*

$[1,U]\times[1,V]$, and $k(\cdot,t,u,v), k(s,\cdot,u,v), k(s,t,\cdot,v), k(s,t,u,\cdot) \in G(\mathbf{R}_1^+)$. And assume that a function $c : \mathbf{R}_1^+ \times \mathbf{R}_1^+ \longrightarrow \mathbf{R}^+$ satisfies that both functions $c(\cdot.t), c(s,\cdot)$ are nondecreasing on $\mathbf{R}_1^+$, and functions $a, x : \mathbf{R}_1^+ \times \mathbf{R}_1^+ \longrightarrow \mathbf{R}^+$ satisfy that a and x are bounded on $[1,S]\times[1,T]$, and $a(\cdot,t), a(s,\cdot), x(\cdot,t), x(s,\cdot) \in G(\mathbf{R}_1^+)$. Let a nondecreasing function m_1 be left-continuous on $\mathbf{R}_1^+$.

If the function x satisfies

$$\begin{aligned} x(s,t) &\le c(s,t) \\ &+ \int_1^s\int_1^t \left[a(u,v)w(x(u,v)) + \int_1^u\int_1^v k(u,v,\sigma,\tau)w(x(\sigma,\tau))\,\mathrm{d}m_4(\tau)\,\mathrm{d}m_3(\sigma)\right] \\ &\qquad\qquad \times\ \mathrm{d}m_2(v)\,\mathrm{d}m_1(u), \end{aligned}$$

then for $(s,t)\in[1,S]\times[1,T]$ we have

$$x(s,t) \le E^{-1}[\gamma(s,t)], \tag{3.5}$$

where

$$\begin{aligned} \gamma(s,t) &= E(c(s,t)) \\ &+ \int_1^s\int_1^t \left(a(u,v) + \int_1^u\int_1^v k(u,v,\sigma,\tau)\,\mathrm{d}m_4(\tau)\,\mathrm{d}m_3(\sigma)\right)\mathrm{d}m_2(v)\,\mathrm{d}m_1(u), \end{aligned}$$

and the numbers S, T are chosen so that

$$\gamma(s,t) \in \mathrm{Dom}(E^{-1}),$$

for all $(s,t)\in[1,S]\times[1,T]$.

Proof. Assume that $1\le s\le s^*\le S$. Then, since, for every fixed t, a function $c(\cdot,t)$ is nondecreasing, we have

$$\begin{aligned} x(s,t) &\le c(s^*,t) \\ &+ \int_1^s\int_1^t \left[a(u,v)w(x(u,v)) + \int_1^u\int_1^v k(u,v,\sigma,\tau)w(x(\sigma,\tau))\,\mathrm{d}m_4(\tau)\,\mathrm{d}m_3(\sigma)\right] \\ &\qquad\qquad \times\ \mathrm{d}m_2(v)\,\mathrm{d}m_1(u). \end{aligned} \tag{3.6}$$

Define a function $y(s,t)$ by the right-hand side of (3.6), where t is fixed. Then for every $1\le s\le s^*$, $x(s,t)\le y(s,t)$. Let $z(s)=y(s,t)$, where t is fixed. Then we have, except for a countable subset of $\mathbf{C}_{m_1}$,

$$\begin{aligned} &z'_{m_1}(s) \\ &= \int_1^t \left[a(s,v)w(x(s,v)) + \int_1^s \int_1^v k(s,v,\,\sigma,\,\tau)w(x(\,\sigma,\,\tau))\,\mathrm{d}m_4(\,\tau)\,\mathrm{d}m_3(\,\sigma) \right] \mathrm{d}m_2(v) \\ &\le \int_1^t \left[a(s,v) + \int_1^s \int_1^v k(s,v,\,\sigma,\,\tau)\,\mathrm{d}m_4(\,\tau)\,\mathrm{d}m_3(\,\sigma) \right] \mathrm{d}m_2(v) \cdot w(z(s)). \end{aligned}$$

By Lemma 3.1, this implies

$$\begin{aligned} (E \circ z)'_{m_1}(s) &\le \frac{z'_{m_1}(s)}{w \circ z(s)} \\ &\le \int_1^t \left[a(s,v) + \int_1^s \int_1^v k(s,v,\,\sigma,\,\tau)\,\mathrm{d}m_4(\,\tau)\,\mathrm{d}m_3(\,\sigma) \right] \mathrm{d}m_2(v). \end{aligned}$$

Thus by Theorem 2.3 we have

$$\begin{aligned} x(s,t) \le & z(s) \le E^{-1} \Bigg[E \circ z(1) \\ & + \int_1^s \int_1^t \left(a(u,v) + \int_1^u \int_1^v k(u,v,\,\sigma,\,\tau)\,\mathrm{d}m_4(\,\tau)\,\mathrm{d}m_3(\,\sigma) \right) \mathrm{d}m_2(v)\,\mathrm{d}m_1(u) \Bigg]. \end{aligned}$$

Since the above inequality is true for $s = s^*$, and $(s^*, t) \in [1, S] \times [1, T]$ was arbitrary, the inequality (3.5) is valid. The proof is complete. $\square$

4. Some Applications

There are many applications of the inequalities obtained in the previous section. Here we shall give some examples that are sufficient to show the usefulness of our results.

From now on we assume the following conditions:

(C1) A function α is nondecreasing on $\mathbf{R}_1^+$, and nondecreasing functions m_1, m_2 are all left-continuous on $\mathbf{R}_1^+$.

(C2) A function $x \in G(\mathbf{R}_1^+)$ is left-continuous, and, for all $t \in \mathbf{R}_1^+$, both $x'_{m_1}(t)$ and $x''_m(t) \equiv (x'_{m_1})'_{m_2}(t)$ exist, and $x'_{m_1} \in G(\mathbf{R}_1^+)$, and a set $\mathbf{C}$ satisfies that $\mathbf{R}_1^+ - \mathbf{C}$ is a countable set, and x'_{m_1} is continuous at every $t \in \mathbf{C}$.

(C3) A function $I : \mathbf{R}_1^+ \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfies that $I(\cdot, x(\cdot))$ is Borel-measurable, and for every $s \notin \mathbf{C}$

$$x'_{m_1}(s) = I(s, x(s)),$$

and, for some bounded nonnegative function $\varphi \in G(\mathbf{R}_1^+)$, $|I(s,x)| \leq \varphi(s) \leq L$, for some $L \geq 1$ and for all $(s,x) \in \mathbf{R}_1^+ \times \mathbf{R}$.

(C4) Functions $F : \mathbf{R}_1^+ \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}, G : \mathbf{R}_1^+ \times \mathbf{R}_1^+ \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfy that for every $u, v \in G(\mathbf{R}_1^+)$, and, for every $t \in \mathbf{R}_1^+$, both functions $F(\cdot, u(\cdot), v(\cdot))$ and $G(t, \cdot, v(\cdot))$ are Borel-measurable, and for some nonnegative functions $a, b \in G(\mathbf{R}_1^+)$, and for every $(t,\, \sigma,\, \tau) \in \mathbf{R}_1^+ \times \mathbf{R} \times \mathbf{R}$,

$$|F(t,\, \sigma,\, \tau)| \leq a(t)\; w\!\left(\frac{|\,\sigma|}{m_1(t)}\right) + b(t) w(|\,\tau|),$$

and for every $(t,\, \sigma,\, \tau) \in \mathbf{R}_1^+ \times \mathbf{R}_1^+, \times \mathbf{R}$,

$$|G(t,\, \sigma,\, \tau)| \leq k(t,\, \sigma)\; w\!\left(\frac{|\,\tau|}{m_1(t)}\right).$$

Now we consider the following second-order integro-differential equation:

$$\begin{cases} x''_m(t) = F[t, x(t), x'_{m_1}(t-)] + \int\limits_1^t G(t,\, \sigma, x(\,\sigma))\, \mathrm{d}\alpha(\,\sigma), \\ x(1) = c_1,\; x'_{m_1}(1) = c_2,\;\; t \in [1, \infty). \end{cases} \tag{4.1}$$

Lemma 4.1. *Assume that there is a number $K \geq 0$ such that $K \in \mathrm{Dom}(E^{-1})$, and for all $t \in \mathbf{R}_1^+$,*

$$\begin{aligned} \gamma(t) =& E(|c_1| + |c_2| + L) \\ &+ \int\limits_1^t \left[[a(s) + b(s)] + \int\limits_1^s k(s,\, \sigma)\, \mathrm{d}\alpha(\,\sigma) \right] \mathrm{d}m_2(s) \leq K. \end{aligned}$$

And suppose that $m_1(1) \geq 1$.

If x is a solution of the equation (4.1), then for some $\rho \in \mathbf{R}$, we have

$$\lim_{t \to \infty} \int\limits_1^t \left[F[s, x(s), x'_{m_1}(s-)] + \int\limits_1^s G(s,\, \sigma, x(\,\sigma))\, \mathrm{d}\alpha(\,\sigma) \right] \mathrm{d}m_2(s) = \rho. \tag{4.2}$$

Proof. Integrating (4.1) from 1 to t, by Corollary 2.4, we get

$$x'_{m_1}(t-) = c_2 + \int\limits_1^t \left[F[s, x(s), x'_{m_1}(s-)] + \int\limits_1^s G(s,\, \sigma, x(\,\sigma))\, \mathrm{d}\alpha(\,\sigma) \right] \mathrm{d}m_2(s). \tag{4.3}$$

So by hypotheses we have

$$|x'_{m_1}(t-)| \le |c_2| + \int_1^t \left[a(s)w\left(\frac{|x(s)|}{m_1(s)}\right) + b(s)w(|x'_{m_1}(s-)|) + \int_1^s k(s,\,\sigma)w\left(\frac{|x(\sigma)|}{m_1(\sigma)}\right)\,\mathrm{d}\alpha(\sigma)\right]\mathrm{d}m_2(s). \tag{4.4}$$

Now let

$$P(s) = \begin{cases} 1, & s \notin \mathbf{C} \\ 0, & s \in \mathbf{C}, \end{cases} \qquad Q(s) = \begin{cases} 1, & s \in \mathbf{C} \\ 0, & s \notin \mathbf{C} \end{cases}.$$

Then we get

$$x'_{m_1}(t) = P(t)I(t, x(t)) + Q(t)\left(c_2 + \int_1^t \left[F[s, x(s), x'_{m_1}(s-)] + \int_1^s G(s,\,\sigma, x(\sigma))\,\mathrm{d}\alpha(\sigma)\right]\mathrm{d}m_2(s)\right).$$

Hence by Theorem 2.3 we have

$$x(t) = c_1 + \int_1^t P(s)I(s, x(s))\,\mathrm{d}m_1(s) + \int_1^t Q(s)\left(c_2 + \int_1^s \left[F[\sigma, x(\sigma), x'_{m_1}(\sigma-)] + \int_1^\sigma G(\sigma, v, x(v))\,\mathrm{d}\alpha(v)\right]\mathrm{d}m_2(\sigma)\right)\mathrm{d}m_1(s). \tag{4.5}$$

So using $m_1(t) \ge 1, \varphi(t) \le L$ and Corollary 2.7, we have

$$|x(t)| \le |c_1| + \int_1^t |P(s)I(s, x(s))|\,\mathrm{d}m_1(s) + \int_1^t |Q(s)|\left(|c_2| + \int_1^s \left[|F(\sigma, x(\sigma), x'_{m_1}(\sigma-))| + \int_1^\sigma |G(\sigma, v, x(v))|\,\mathrm{d}\alpha(v)\right]\mathrm{d}m_2(\sigma)\right)\mathrm{d}m_1(s)$$

$$
\begin{aligned}
&\le |c_1| + L \int_1^t \mathrm{d}m_1(s) \\
&\quad + \int_1^t \Bigg(|c_2| + \int_1^s \Bigg[|F[\sigma, x(\sigma), x'_{m_1}(\sigma-)]| \\
&\qquad\qquad + \int_1^\sigma |G(\sigma, v, x(v))| \,\mathrm{d}\alpha(v) \Bigg] \mathrm{d}m_2(\sigma) \Bigg) \mathrm{d}m_1(s) \\
&\le |c_1| + L[m_1(t) - m_1(1)] + |c_2|[m_1(t) - m_1(1)] \\
&\quad + \int_1^t \int_1^s \Bigg[|F[\sigma, x(\sigma), x'_{m_1}(\sigma-)]| + \int_1^\sigma |G(\sigma, v, x(v))| \,\mathrm{d}\alpha(v) \Bigg] \mathrm{d}m_2(\sigma) \,\mathrm{d}m_1(s) \\
&\le |c_1| m_1(t) + L m_1(t) + |c_2| m_1(t) \\
&\quad + \int_1^t [m_1(t) - m_1(\sigma+)] \Bigg[|F[\sigma, x(\sigma), x'_{m_1}(\sigma-)]| + \int_1^\sigma |G(\sigma, v, x(v))| \,\mathrm{d}\alpha(v) \Bigg] \\
&\qquad\qquad \times \,\mathrm{d}m_2(\sigma) \\
&\le (|c_1| + |c_2| + L) m_1(t) + m_1(t) \int_1^t \Bigg[a(s) w\left(\frac{|x(s)|}{m_1(s)} \right) + b(s) w(|x'_{m_1}(s-)|) \\
&\qquad\qquad + \int_1^s k(s, \sigma) w\left(\frac{|x(\sigma)|}{m_1(\sigma)} \right) \mathrm{d}\alpha(\sigma) \Bigg] \mathrm{d}m_2(s).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{|x(t)|}{m_1(t)} \le |c_1| + |c_2| + L + \int_1^t \Bigg[a(s) w\left(\frac{|x(s)|}{m_1(s)} \right) + b(s) w(|x'_{m_1}(s-)|) \\
+ \int_1^s k(s, \sigma) w\left(\frac{|x(\sigma)|}{m_1(\sigma)} \right) \mathrm{d}\alpha(\sigma) \Bigg] \mathrm{d}m_2(s) \equiv z(t).
\end{aligned}
\tag{4.6}
$$

So (4.4) and (4.6) yield

$$
|x'_{m_1}(t-)| \le z(t), \;\; \frac{|x(t)|}{m_1(t)} \le z(t). \tag{4.7}
$$

This implies

$$\begin{aligned} z(t) &\le |c_1| + |c_2| + L \\ &\quad + \int_1^t \left[a(s)w(z(s)) + b(s)w(z(s)) + \int_1^s k(s,\,\sigma)w(z(\sigma))\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s), \\ &\le |c_1| + |c_2| + L \\ &\quad + \int_1^t \left[[a(s)+b(s)]w(z(s)) + \int_1^s k(s,\,\sigma)w(z(\sigma))\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s). \end{aligned}$$

Then by Theorem 3.3, we have

$$z(t) \le E^{-1}[\gamma(t)] \le E^{-1}(K) \equiv M < \infty.$$

Thus by (4.7) and the conditions that we supposed, we get

$$\begin{aligned} &\int_1^t \left| \left[F[s, x(s), x'_{m_1}(s-)] + \int_1^s G(s,\,\sigma, x(\sigma))\,\mathrm{d}\alpha(\sigma) \right] \right| \mathrm{d}m_2(s) \\ &\le \int_1^t \left[|F[s, x(s), x'_{m_1}(s-)]| + \int_1^s |G(s,\,\sigma, x(\sigma))|\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s) \\ &\le \int_1^t \left[[a(s)+b(s)]w(z(s)) + \int_1^s k(s,\,\sigma)w(z(\sigma))\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s) \\ &\le \int_1^t \left[[a(s)+b(s)]w(M) + \int_1^s k(s,\,\sigma)w(M)\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s) \\ &\le w(M) \int_1^t \left[a(s)+b(s) + \int_1^s k(s,\,\sigma)\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s) \le w(M)\cdot K < \infty. \end{aligned} \tag{4.8}$$

Then by Theorem 2.9 this implies, for some $\rho \in \mathbf{R}$,

$$\lim_{t\to\infty} \int_1^t \left[F[s, x(s), x'_{m_1}(s-)] + \int_1^s G(s,\,\sigma, x(\sigma))\,\mathrm{d}\alpha(\sigma) \right] \mathrm{d}m_2(s) = \rho.$$

The proof is complete. ☐

Now we apply the above result to the following result.

Theorem 4.2. *In Lemma 4.1, if x'_{m_1} is continuous on $\mathbf{R}_1^+$, then for some $\lambda \in \mathbf{R}$ we have*

$$\lim_{t\to\infty} \frac{x(t)}{m_1(t)} = \lambda.$$

Proof. Since x'_{m_1} is continuous on $\mathbf{R}_1^+$, we take $\mathbf{C} = \mathbf{R}_1^+$. Then for all $t \in \mathbf{R}_1^+$ by (4.3) we have

$$x'_{m_1}(t) = c_2 + \int_1^t \left[F[s, x(s), x'_{m_1}(s)] + \int_1^s G(s,\, \sigma, x(\,\sigma))\,\mathrm{d}\alpha(\,\sigma) \right] \mathrm{d}m_2(s).$$

Thus by Theorem 2.8 and Lemma 4.1 we get

$$\begin{aligned}
&\lim_{t\to\infty} \frac{x(t)}{m_1(t)} = \lim_{t\to\infty} \frac{x'_{m_1}(t)}{(m_1)'_{m_1}(t)} = \lim_{t\to\infty} x'_{m_1}(t) \\
&= c_2 + \lim_{t\to\infty} \int_1^t \left[F[s, x(s), x'_{m_1}(s)] + \int_1^s G(s,\, \sigma, x(\,\sigma))\,\mathrm{d}\alpha(\,\sigma) \right] \mathrm{d}m_2(s) \\
&= c_2 + \rho \equiv \lambda.
\end{aligned}$$

The proof is complete. $\square$

Using Lemma 4.1, in the following result, we can obtain a more detailed result on the asymptotic behavior of the equation (4.1).

Theorem 4.3. *Under the same conditions as in Lemma 4.1, suppose that*

$$\int_1^\infty \int_s^\infty \left[a(\,\sigma) + b(\,\sigma) + \int_1^\sigma k(\,\sigma,\, \tau)\,\mathrm{d}\alpha(\,\tau) \right] \mathrm{d}m_2(\,\sigma)\,\mathrm{d}m_1(s) < \infty,$$

and that $\int_1^\infty \varphi(s)\,\mathrm{d}m_1(s) < \infty$. If x is a solution of the equation (4.1), then for some $a, b \in \mathbf{R}$ we have

$$\lim_{t\to\infty} \left[x(t) - \left(a \int_1^t Q(s)\,\mathrm{d}m_1(s) + b \right) \right] = 0.$$

Proof. Let

$$H(t) = F[t, x(t), x'_{m_1}(t-)] + \int_1^t G(t,\, \sigma, x(\,\sigma))\,\mathrm{d}\alpha(\,\sigma).$$

Then using (4.5) we get for all $t \geq 1$

$$x(t) = x(1) + \int_1^t P(s)I(s,x(s))\,\mathrm{d}m_1(s) + x'_{m_1}(1)\int_1^t Q(s)\,\mathrm{d}m_1(s) + \int_1^t\int_1^s Q(s)H(\sigma)\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s). \tag{4.9}$$

From Lemma 4.1 we have $\int_1^\infty H(\sigma)\,\mathrm{d}m_2(\sigma) < \infty$. So we get

$$\begin{aligned}
&\int_1^t\int_1^s Q(s)H(\sigma)\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s) \\
&= \int_1^t Q(s)\left[\int_1^\infty H(\sigma)\,\mathrm{d}m_2(\sigma) - \int_s^\infty H(\sigma)\,\mathrm{d}m_2(\sigma)\right]\mathrm{d}m_1(s) \\
&= \int_1^t Q(s)\,\mathrm{d}m_1(s)\int_1^\infty H(\sigma)\,\mathrm{d}m_2(\sigma) - \int_1^t\int_s^\infty Q(s)H(\sigma)\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s).
\end{aligned} \tag{4.10}$$

Considering hypotheses and the inequality (4.8), we conclude that

$$\begin{aligned}
&\int_1^\infty\int_s^\infty |Q(s)||H(\sigma)|\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s) \\
&\leq w(M)\int_1^\infty\int_s^\infty\left[a(\sigma) + b(\sigma) + \int_1^\sigma k(\sigma,\tau)\,\mathrm{d}\alpha(\tau)\right]\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s) < \infty.
\end{aligned}$$

This means that for some $\mu \in \mathbf{R}$

$$\lim_{t\to\infty}\int_1^t\int_s^\infty Q(s)H(\sigma)\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s) = \mu. \tag{4.11}$$

And by hypotheses

$$\int_1^t |P(s)|\cdot|I(s,x(s))|\,\mathrm{d}m_1(s) \leq \int_1^\infty \varphi(s)\,\mathrm{d}m_1(s) < \infty.$$

So for some $\nu \in \mathbf{R}$

$$\lim_{t\to\infty}\int_1^t P(s)\cdot I(s,x(s))\,\mathrm{d}m_1(s) = \nu. \tag{4.12}$$

Hence, using (4.9) $\sim$ (4.12), we get for some $b \in \mathbf{R}$

$$\begin{aligned} x(t) - \left[x'_{m_1}(1) + \int_1^\infty H(\sigma)\,\mathrm{d}m_2(\sigma) \right] \cdot \int_1^t Q(s)\,\mathrm{d}m_1(s) \\ = x(1) + \int_1^t P(s) I(s, x(s))\,\mathrm{d}m_1(s) - \int_1^t \int_s^\infty Q(s) H(\sigma)\,\mathrm{d}m_2(\sigma)\,\mathrm{d}m_1(s) \\ \longrightarrow b \text{ as } \; t \longrightarrow \infty. \end{aligned}$$

Thus we conclude that for some numbers $a, b \in \mathbf{R}$

$$x(t) - \left[a \int_1^t Q(s)\,\mathrm{d}m_1(s) + b \right] \longrightarrow 0 \text{ as } \; t \longrightarrow \infty.$$

The proof is complete. $\square$

Now we let $\mathbf{N}^* = \{2, 3, 4, \cdots\}$, and we define a function ϕ as

$$\phi(t) = \begin{cases} t, & \text{if } t \in [1, 2] \\ t + k - 1, & \text{if } t \in (k, k+1], \quad k \in \mathbf{N}^*. \end{cases} \tag{4.13}$$

For the function ϕ we have the following result.

Lemma 4.4 ([3])**.** *We have*

$$\int_1^t f(s)\,\mathrm{d}\phi(s) = \int_1^t f(s)\,\mathrm{d}s + \sum_{1<k<t} f(k).$$

Corollary 4.5. *Assume that $m_1 = \phi = m_2$, and that the conditions in Theorem 4.3 are satisfied. And suppose that, for every $t \neq k (\forall k \in \mathbf{N}^*)$, both $x'(t)$ and $x''(t)$ exist. If x is a solution of the impulsive integro-differential equation*

$$\begin{cases} x''(t) = F(t, x(t), x'(t)) + \int_1^t G(t, \sigma, x(\sigma))\,\mathrm{d}\sigma, \; t \neq k, \\ \Delta x'(k) = F(k, x(k), x'(k-)) + \int_1^k G(k, \sigma, x(\sigma))\,\mathrm{d}\sigma, \\ \Delta x(k) = I(k, x(k)), \; \forall k \in \mathbf{N}^*, \end{cases} \tag{4.14}$$

then for some numbers $a, b \in \mathbf{R}$

$$x(t) - [a \cdot t + b] \longrightarrow 0 \text{ as } \; t \longrightarrow \infty.$$

Proof. Since $m_1 = \phi = m_2$, we see that

$$\Delta x(k) = x'_{m_1}(k), \quad \Delta x'(k) = x''_m(k).$$

This implies that the equation (4.14) becomes the equation (4.1). And if we take $\mathbf{R}_1^+ - \mathbf{C} = \mathbf{N}^*$, then we have

$$Q(t) = \begin{cases} 1, & t \notin \mathbf{N}^* \\ 0, & t \in \mathbf{N}^*. \end{cases}$$

And by Lemma 4.4 we have

$$\int_1^t Q(s)\, \mathrm{d}\phi(s) = \int_1^t Q(s)\, \mathrm{d}s = \int_1^t \mathrm{d}s = t - 1.$$

By Theorem 4.3 this implies that, for some $a, b \in \mathbf{R}$,

$$x(t) - [a(t-1) + b] = x(t) - [a \cdot t + b - a] \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

The proof is complete. □

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