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ON GENERALIZED Z-RECURRENT MANIFOLDS

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Abstract. The object of the present paper is to study generalized Z-recurrent manifolds. Some geometric properties of generalized Z-recurrent manifolds have been studied under certain curvature conditions. Finally, we give an example of a generalized Z-recurrent manifold.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan[3], who, in particular, obtained a classification of those spaces.
Let $(M^n, g), (n = dimM)$ be a Riemannian manifold, that is, a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [21]. The class of Riemannian locally symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds by Walker [30], conformally recurrent manifolds by Adati and Miyazawa[1], conformally symmetric Ricci-recurrent spaces by Roter[27], pseudo symmetric manifolds by Chaki[6] etc. The notion of recurrent manifolds have been generalized by various authors such as Ricci-recurrent manifolds

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by Patterson [25], 2-recurrent manifolds by Lichnerowicz [15], projective 2-recurrent manifolds by Ghosh [14] and others.

A tensor field T of type (0,q) is said to be recurrent [27] if the relation

$$(\nabla_X T)(Y_1, Y_2, ..., Y_q)T(Z_1, Z_2, ..., Z_q)$$
$$- T(Y_1, Y_2, ..., Y_q)(\nabla_X T)(Z_1, Z_2, ..., Z_q) = 0,$$

holds on (M^n, g). From the definition it follows that if at a point $x \in M$, $T(x) \neq 0$, then on some neighbourhood of x, there exists a unique 1-form A satisfying

$$(\nabla_X T)(Y_1, Y_2, ..., Y_q) = A(X)T(Y_1, Y_2, ..., Y_q).$$

In 1952, Patterson [25] introduced Ricci-recurrent manifolds. According to Patterson, a manifold (M^n, g) of dimension n, is called Ricci-recurrent if the Ricci tensor S is satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z),$$

for some 1-form A. He denoted such a manifold by R_n. Ricci-recurrent manifolds have been studied by several authors ([4], [26], [27], [31]) and many others. In a paper De, Guha and Kamilya [12] introduced the notion of generalized Ricci recurrent manifolds which is defined as follows:

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called generalized Ricci recurrent if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \tag{1.1}$$

where A and B are two non-zero 1-forms. Such a manifold is denoted by GR_n. If the associated 1-form B becomes zero, then the manifold GR_n reduces to a Ricci-recurrent manifold R_n. This justifies the name generalized Ricci-recurrent manifold and the symbol GR_n for it.

Generalized recurrent and generalized Ricci recurrent manifolds have been studied by several authors such as Özgür ([22], [23], [24]), Mallick, De and De [16], Arslan et al [2] and many others. Also Mantica and Suh [17] studied quasi-conformally recurrent Riemannian manifolds. In [11] De and Gazi proved that a generalized concircularly recurrent manifold with constant scalar curvature is a GR_n.

On the other hand, quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. A non-flat Riemannian manifold

$(M^n, g)(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where $a, b \in \mathbb{R}$ and η is a non-zero 1-form such that

$$g(X,\xi) = \eta(X),$$

for all vector fields X, η is the 1-form metrically equivalent to the vector field ξ.

In a recent paper [18] Mantica and Molinari introduced weakly-Z-symmetric manifolds which is denoted by $(WZS)_n$. It was a generalization of the notion of weakly Ricci symmetric manifolds [29], pseudo Ricci symmetric manifolds [7], pseudo projective Ricci symmetric manifolds [8]. A (0,2) symmetric tensor is a generalized Z tensor if

$$Z(X,Y) = S(X,Y) + \phi g(X,Y), \tag{1.2}$$

where ϕ is an arbitrary scalar function. The scalar Z is obtained by contracting (1.2) over X and Y as follows:

$$Z = r + n\phi. \tag{1.3}$$

Also pseudo-Z-symmetric and recurrent Z forms on Riemannian manifolds have been studied in [19] and [20] respectively.

Motivated by the above studies in the present paper we study a type of non-flat connected Riemannian manifold which is called generalized Z-recurrent manifolds.

A manifold is called generalized Z-recurrent and denoted by GZ_n, if the generalized Z tensor is non-zero and satisfies the condition (1.1), that is,

$$(\nabla_X Z)(Y,W) = A(X)Z(Y,W) + B(X)g(Y,W), \tag{1.4}$$

where Z is the generalized Z tensor. The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}r$, where r is the scalar curvature. Hereafter we refer to the generalized Z tensor simply as the Z tensor. Also we assume that

$$g(X,P) = A(X) \text{ and } g(X,Q) = B(X), \text{ for all } X. \tag{1.5}$$

Then P, Q are called the basic vector fields of the manifolds corresponding to the associated 1-forms A and B respectively.

The paper is organized as follows:

After preliminaries in Section 2, we obtain a necessary and sufficient condition for the scalar curvature of a GZ_n to be a constant. Next we study Ricci-recurrent

GZ_n. In Section 5, we study conformally flat $GZ_n (n > 3)$. Section 6 is devoted to study a GZ_n satisfying $C.S = 0$. Section 7 deals with decomposable GZ_n. Finally, we give an example of GZ_n.

2. PRELIMINARIES

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively. Also let L denotes the symmetric tensor of type (1,1) corresponding to the Ricci tensor S, that is ,

$$g(LX, Y) = S(X, Y), \tag{2.1}$$

for any vector fields X, Y. Let $\bar{A}$ and $\bar{B}$ are two 1-forms defined by

$$A(LX) = \bar{A}(X), \ \ B(LX) = \bar{B}(X). \tag{2.2}$$

Then $\bar{A}$ and $\bar{B}$ are called auxiliary 1-forms corresponding to the 1-forms A and B respectively. We also have

$$Z(X, Y) = Z(Y, X). \tag{2.3}$$

We obtain from (1.4)

$$\begin{aligned}(\nabla_X Z)(Y, W) - (\nabla_W Z)(X, Y) = {} & A(X)Z(Y, W) + B(X)g(Y, W) \\ & - A(W)Z(X, Y) - B(W)g(X, Y).\end{aligned} \tag{2.4}$$

A conformally flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be a manifold of quasi constant curvature [9] if its curvature $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}\tilde{R}(X, Y, U, W) = {} & p[g(Y, U)g(X, W) - g(X, U)g(Y, W)] \\ & + q[g(X, W)H(Y)H(U) + g(Y, U)H(X)H(W)] \\ & - g(X, U)H(Y)H(W) - g(Y, W)H(X)H(U)],\end{aligned} \tag{2.5}$$

where $\tilde{R}(X, Y, U, W) = g(R(X, Y)U, W)$ and R is the curvature tensor of type (1,3), p and q are scalar functions of which $q \neq 0$ and H is a non-zero 1-form defined by $g(X, \mu) = H(X)$ for all X, μ being a unit vector field. In such a case p and q are called associated scalars, H is called the associated 1-form and μ is called the generator of the manifold. In 1956, Chern [10] studied a type of Riemannian manifold whose curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\tilde{R}(X, Y, U, W) = F(Y, U)F(X, W) - F(X, U)F(Y, W), \tag{2.6}$$

where F is a symmetric tensor of type (0,2). Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor F and was denoted by $\psi(F_n)$. Such a manifold is important for the following reasons: Firstly for possesing some remarkable properties relating to curvature and characteristic classes and secondly, for containing a manifold of quasi-constant curvature as a subclass.

3. Necessary and Sufficient Condition for the Scalar Curvature of a GZ_n to be a Constant

Using (1.2) in (1.4) we get

$$(\nabla_X S)(Y,W) = A(X)S(Y,W) + B(X)g(Y,W) + \phi A(X)g(Y,W) - (X\phi)g(Y,W). \tag{3.1}$$

Now contracting (3.1) over Y and W we obtain

$$dr(X) = (r + n\phi)A(X) + nB(X) - n(X\phi). \tag{3.2}$$

If possible, let the scalar curvature r is constant. Then from (3.2) we get

$$(X\phi) = [\frac{r + n\phi}{n}]A(X) + B(X). \tag{3.3}$$

Thus we have the following:

Theorem 3.1. *In a GZ_n the scalar curvature r is constant if and only if (3.3) holds.*

Again if possible, let $\phi = -\frac{r}{n}$, then from (3.3) we obtain

$$(X\phi) = B(X), \tag{3.4}$$

which implies

$$g(grad\phi, X) = g(X, Q). \tag{3.5}$$

Hence $grad\ \ \phi = Q$. Thus we have the following:

Theorem 3.2. *In a GZ_n with constant scalar curvature, the associated vector field $Q = grad\phi$, provided $\phi = -\frac{r}{n}$.*

Now, if possible let $\phi = constant$. Then from (3.3) we get

$$k_1 A(X) + B(X) = 0, \tag{3.6}$$

where $k_1 = \frac{r+n\phi}{n} = constant(\neq 0)$. Hence we have the following:

Theorem 3.3. *If in a GZ_n with constant scalar curvature, $\phi = constant$, then the 1-form A is closed if and only if the 1-form B is also closed.*

4. Ricci-recurrent Generalized Z-recurrent Manifolds

In this section we assume that GZ_n is Ricci-recurrent, then we have

$$(\nabla_X S)(Y,W) = E(X)S(Y,W), \tag{4.1}$$

where $E(X)$ is a non-zero 1-form. If the manifold is also GZ_n then using (1.2) and (4.1) in (1.4) we get

$$\begin{aligned} &E(X)S(Y,W) + (X\phi)g(Y,W) \\ &= A(X)S(Y,W) + \phi A(X)g(Y,W) + B(X)g(Y,W), \end{aligned} \tag{4.2}$$

which implies

$$S(Y,W) = \frac{[\phi A(X) + B(X) - (X\phi)]}{E(X) - A(X)} g(Y,W), \tag{4.3}$$

provided

$$E(X) \neq A(X) \tag{4.4}$$

which can be written as

$$S(Y,W) = \lambda g(Y,W), \tag{4.5}$$

where $\lambda = \frac{\phi A(X)+B(X)-(X\phi)}{E(X)-A(X)}$. Hence we have the following:

Theorem 4.1. *A Ricci-recurrent GZ_n is an Einstein manifold, provided (4.4) holds.*

5. Conformally Flat $GZ_n(n > 3)$

Suppose (M^n, g) is a Riemannian manifold of dimension $n > 3$ and X is any vector field on M. Then the divergence of the vector field X, denoted by $divX$ and is defined as $divX = \Sigma_{i=1}^n g(\nabla_{e_i} X, e_i)$, where $\{e_i\}$ is an orthonormal basis of the tangent space T_pM at any point $p \in M$. Again, if K is a tensor field of type (1,3), then its divergence $divK$ is a tensor field of type (0,3) defined as $(divK)(X_1, ..., X_3) = \Sigma_{i=1}^n g((\nabla_{e_i} K)(X_1, ..., X_3), e_i)$.

In this section we assume that the manifold $GZ_n(n > 3)$ is conformally flat. Then $divC = 0$, where C denotes the Weyl's conformal curvature tensor and $'div'$ denotes divergence. Hence we have [13]

$$(\nabla_X S)(Y,W) - (\nabla_W S)(X,Y) = \frac{1}{2(n-1)}[g(Y,W)dr(X) - g(X,Y)dr(W)]. \tag{5.1}$$

Now using (1.2) in (2.4) we obtain

$$\begin{aligned}(\nabla_X S)(Y,W) + (X\phi)g(Y,W) - (\nabla_W S)(X,Y) - (W\phi)g(X,Y)\\ = A(X)[S(Y,W) + \phi g(Y,W)] + B(X)g(Y,W)\\ - A(W)[S(X,Y) + \phi g(X,Y)] - B(W)g(X,Y).\end{aligned} \tag{5.2}$$

Contracting (5.2) over Y and W we get

$$\begin{aligned}dr(X) = 2[r + (n-1)\phi]A(X) - 2\bar{A}(X)\\ + 2(n-1)B(X) - 2(n-1)(X\phi).\end{aligned} \tag{5.3}$$

Using (5.2) and (5.3) in (5.1), yields

$$\begin{aligned}A(X)[S(Y,W) + \phi g(Y,W)] - A(W)[S(X,Y) + \phi g(X,Y)]\\ = \frac{1}{2(n-1)}[g(Y,W)\{2(r + (n-1)\phi)A(X) - 2\bar{A}(X)\}\\ - g(X,Y)\{2(r + (n-1)\phi)A(W) - 2\bar{A}(W)\}].\end{aligned} \tag{5.4}$$

Now putting $Y = P$ in (5.4) we obtain

$$A(X)\bar{A}(W) = \bar{A}(X)A(W). \tag{5.5}$$

Again putting $X = P$ in (5.5) we get

$$\bar{A}(W) = \frac{\bar{A}(P)A(W)}{A(P)},$$

which can be written as

$$\bar{A}(W) = sA(W), \quad for\ \ all\ \ W, \tag{5.6}$$

where

$$s = \frac{\bar{A}(P)}{A(P)}. \tag{5.7}$$

Since $A \neq 0$, putting $X = P$ in (5.4) we obtain

$$\begin{aligned}A(P)[S(Y,W) + \phi g(Y,W)] - A(W)[\bar{A}(Y) + \phi A(Y)]\\ = \frac{1}{2(n-1)}[g(Y,W)\{2(r + (n-1)\phi)A(P) - 2\bar{A}(P)\}\\ - A(Y)\{2(r + (n-1)\phi)A(W) - 2\bar{A}(W)\}].\end{aligned} \tag{5.8}$$

Using (5.6) in (5.8) we get

$$S(Y,W) = \frac{(r-s)}{(n-1)}g(Y,W) + \frac{(ns-r)}{(n-1)}\frac{A(Y)A(W)}{A(P)}, \tag{5.9}$$

which can be written as

$$S(Y,W) = ag(Y,W) + bT(Y)T(W), \tag{5.10}$$

where

$$a = \frac{(r-s)}{(n-1)} \text{ and } b = \frac{(ns-r)}{(n-1)} \text{ are scalars}$$
$$\text{and } T(X) = \frac{A(X)}{\sqrt{A(P)}}. \tag{5.11}$$

A Riemannian manifold is said to be a quasi Einstein manifold if its Ricci tensor is of the form (5.10). Hence we have the following theorem:

Theorem 5.1. *A conformally flat $GZ_n(n > 3)$ is a quasi Einstein manifold.*

Now from (5.9) it follows that

$$S(Y,W) = \frac{(r-s)}{(n-1)}g(Y,W) + \frac{(ns-r)}{(n-1)}\frac{A(Y)A(W)}{A(P)}. \tag{5.12}$$

Putting $W = P$ in (5.12) we get

$$S(Y,P) = \bar{A}(Y) = sA(Y) = sg(Y,P). \tag{5.13}$$

Thus we have the following:

Corollary 5.1. *The vector field P corresponding to the 1-form A is an eigen vector of the Ricci tensor S corresponding to the eigen value s.*

Let us suppose that the associated vector field P corresponding to the 1-form A is a unit vector field. Therefore from (5.11) it follows that

$$T(X) = A(X), \tag{5.14}$$

since $A(P) = 1$. In a conformally flat Riemannian manifold the curvature tensor $\tilde{R}$ of type (0,4) satisfies the following condition

$$\begin{aligned}\tilde{R}(X,Y,U,W) = {} & \frac{1}{(n-2)}[S(Y,U)g(X,W) - S(X,U)g(Y,W) \\ & + S(X,W)g(Y,U) - S(Y,W)g(X,U)] \\ & - \frac{r}{(n-1)(n-2)}[g(Y,U)g(X,W) - g(X,U)g(Y,W)],\end{aligned} \tag{5.15}$$

where $\tilde{R}(X,Y,U,W) = g(R(X,Y)U,W)$ and R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature. Now using (5.10) and (5.14) in (5.15) we get

$$\begin{aligned}\tilde{R}(X,Y,U,W) = &\left[\frac{2a}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][g(Y,U)g(X,W) - g(X,U)g(Y,W)]\\ &+ \frac{b}{n-2}[g(X,W)A(Y)A(U) + g(Y,U)A(X)A(W)\\ &- g(X,U)A(Y)A(W) - g(Y,W)A(X)A(U)],\end{aligned} \tag{5.16}$$

which can be written as

$$\begin{aligned}\tilde{R}(X,Y,U,W) = &\,p[g(Y,U)g(X,W) - g(X,U)g(Y,W)]\\ &+ q[g(X,W)A(Y)A(U) + g(Y,U)A(X)A(W)\\ &- g(X,U)A(Y)A(W) - g(Y,W)A(X)A(U)],\end{aligned} \tag{5.17}$$

where $p = \frac{r-2s}{(n-1)(n-2)}$, $q = \frac{ns-r}{(n-1)(n-2)}$. This implies that the manifold is of quasi-constant curvature. Thus we can state the following theorem:

Theorem 5.2. *A conformally flat $GZ_n(n > 3)$ is a manifold of quasi-constant curvature provided the vector field metrically equivalent to the 1-form A is a unit vector field.*

Now, we suppose that in a manifold of quasi-constant curvature

$$F(X,Y) = \sqrt{p}g(X,Y) + \frac{q}{\sqrt{p}}H(X)H(Y). \tag{5.18}$$

It is obvious that

$$F(X,Y) = F(Y,X). \tag{5.19}$$

Thus F is a symmetric tensor of type (0,2). Now (2.6) can be written as

$$\tilde{R}(X,Y,U,W) = F(Y,U)F(X,W) - F(X,U)F(Y,W).$$

Thus a manifold of quasi-constant curvature is a $\psi(F)_n$. Hence $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass. So we have the following:

Proposition 5.1. *A manifold of quasi-constant curvature is a $\psi(F)_n$.*

From this Proposition 5.1 and Theorem 5.2 we can conclude that

Corollary 5.2. *A conformally flat $GZ_n(n > 3)$ is a $\psi(F)_n$.*

6. GZ_n SATISFYING THE CURVATURE CONDITION $C.S = 0$

In this section we consider a GZ_n satisfying

$$C.S = 0, \tag{6.1}$$

where $C(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for all tangent vectors X, Y and S is the Ricci tensor. that is,

$$(C(X, Y).S)(U, V) = -S(C(X, Y)U, V) - S(U, C(X, Y)V), \tag{6.2}$$

From (6.1) and (6.2) it follows that

$$S(C(X, Y)U, V) + S(U, C(X, Y)V) = 0. \tag{6.3}$$

Using (1.2) in (6.3) we get

$$\begin{aligned} &Z(C(X, Y)U, V) - \phi g(C(X, Y)U, V) \\ &+ Z(U, C(X, Y)V) - \phi g(U, C(X, Y)V) = 0. \end{aligned}$$

This can be written as

$$\begin{aligned} &Z(C(X, Y)U, V) + Z(U, C(X, Y)V) \\ &- \phi\tilde{C}(X, Y, U, V) - \phi\tilde{C}(X, Y, V, U) = 0, \end{aligned} \tag{6.4}$$

where $\tilde{C}(X, Y, U, V) = g(C(X, Y)U, V)$ and C is the Weyl conformal curvature tensor of type (1,3). Since $\tilde{C}$ is skew-symmetric, (6.4) yields

$$Z(C(X, Y)U, V) + Z(U, C(X, Y)V) = 0, \tag{6.5}$$

which implies that

$$(C(X, Y).Z)(U, V) = 0. \tag{6.6}$$

Thus

$$C.Z = 0. \tag{6.7}$$

holds, where $C(X, Y)$ is a derivation operating on the Z tensor. Again let us suppose that $C.Z = 0$ holds in GZ_n. Then using (1.2) in (6.6) and using skew-symmetric properties of $\tilde{C}$ we get after some simple calculations that $C.S = 0$. Hence we have the following:

Theorem 6.1. *A GZ_n satisfies the curvature condition $C.S = 0$ if and only if $C.Z = 0$.*

7. Decomposable GZ_n

A Riemannian manifold (M^n, g) is said to be decomposable or a product manifold [28] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{7.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^*; $a, b, c, ...$ run from 1 to p and $\alpha, \beta, \gamma, ...$ run from p+1 to n.

The two parts of (7.1) are the metrics of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Let M^n be a decomposable manifold such that $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ are components of this manifold.

Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by 'star' is assumed to be from M_2.

Let $\bar{X}, \bar{Y}, \bar{W}, \bar{U}, \bar{V} \in \chi(M_1)$ and $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, the following relations hold [32]:

$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); S(X^*, Y^*) = S^*(X^*, Y^*)$,

$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{W}) = (\bar{\nabla}_{\bar{X}} \bar{S})(\bar{Y}, \bar{W}); (\nabla_{X^*} S)(Y^*, Z^*) = (\nabla^*_{X^*} S^*)(Y^*, Z^*)$, where the meaning of $\bar{X}$, $\bar{Y}$ and $\bar{W}$ is different on each side, similarly for X^*, Y^* and Z^*, and $r = \bar{r} + r^*$, where $r, \bar{r}$ and r^* are scalar curvatures of M, M_1 and M_2 respectively.

Let us consider a Riemannian manifold (M^n, g), which is a decomposable GZ_n. Then $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Now using (1.2) in (1.4), we get

$$\begin{aligned}(\nabla_X S)(Y, W) &= A(X)S(Y, W) + B(X)g(Y, W) \\ &\quad + \phi A(X)g(Y, W) - (X\phi)g(Y, W).\end{aligned} \tag{7.2}$$

Thus from (7.2) we obtain

$$\begin{aligned}(\nabla_{\bar{X}} S)(\bar{Y}, \bar{W}) &= A(\bar{X})S(\bar{Y}, \bar{W}) + B(\bar{X})g(\bar{Y}, \bar{W}) \\ &\quad + [\phi A(\bar{X}) - (\bar{X}\phi)]g(\bar{Y}, \bar{W}).\end{aligned} \tag{7.3}$$

This can be written as

$$(\nabla_{X^*}S)(Y^*,W^*) = A(X^*)S(Y^*,W^*) + B(X^*)g(Y^*,W^*) + [\phi A(X^*) - (X^*\phi)]g(Y^*,W^*). \tag{7.4}$$

From (7.3) it follows that M_1 is GR_n if

$$\phi A(\bar{X}) = (\bar{X}\phi). \tag{7.5}$$

The converse is also true. Similarly we can conclude from (7.5) that M_2 is GR_n if and only if

$$\phi A(X^*) = (X^*\phi). \tag{7.6}$$

Thus we have the following theorem:

Theorem 7.1. *Let M^n be a Riemannian manifold of dimension n such that $M = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2)$. If M^n be a GZ_n, then M_1(respectively M_2) is a generalized Ricci recurrent manifold of dimension p, that is, GR_p (respectively M_2 is generalized Ricci recurrent manifold of dimension $n-p$, that is, GR_{n-p})if and only if (7.5) holds in M_1(respectively (7.6) holds in M_2).*

8. Example of GZ_n

This section deals with an example of GZ_n.

Example 8.1. We define a Riemannian metric on the 4-dimensional real number space $\mathbb{R}^4 - \{x^1 = 0,\ \ x^3 = 0,\ \ x^1 = x^3\}$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = x^1[(dx^1)^2 + (dx^2)^2] + x^3[(dx^3)^2 + (dx^4)^2], \tag{8.1}$$

where $i, j = 1, 2, ..., 4$.

Then the non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are respectively:

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \frac{1}{2x^1},\ \ \Gamma^3_{33} = -\Gamma^3_{44} = \Gamma^4_{34} = \frac{1}{2x^3},$$

$$R_{1221} = -\frac{1}{2x^1},\quad R_{3443} = -\frac{1}{2x^3},\ \ R_{11} = R_{22} = -\frac{1}{2(x^1)^2},$$

$$R_{33} = R_{44} = -\frac{1}{2(x^3)^2}$$

and the components which can be obtained from these by the symmetric properties. The non-zero covariant derivative of R_{ij} are:

$$R_{11,1} = R_{22,1} = \frac{3}{2(x^1)^3}, \quad R_{33,3} = R_{44,3} = \frac{3}{2(x^3)^3},$$

and the components which can be obtained from these by the symmetric properties, where ',' denotes the covariant derivative with respect to the metric tensor. Using the above relations, it can be easily shown that the scalar curvature of the manifold is $-\frac{(x^1)^3+(x^3)^3}{(x^1x^3)^3}$. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold M^4 whose scalar curvature is non-zero and non-constant.

Let us choose an arbitrary scalar function ϕ as $\phi = \frac{1}{x^1}$. Hence the non-vanishing components of the Z tensor and their covariant derivatives are respectively:

$$Z_{11} = Z_{22} = -\frac{1}{2(x^1)^2} + 1, \quad Z_{33} = Z_{44} = -\frac{1}{2(x^3)^2} + \frac{x^3}{x^1},$$

$$Z_{11,1} = Z_{22,1} = \frac{3}{2(x^1)^3} - \frac{1}{x^1}, \quad Z_{33,1} = Z_{44,1} = -\frac{x^3}{(x^1)^2}, \quad Z_{33,3} = Z_{44,3} = \frac{3}{2(x^3)^3}.$$

We shall now show that $\mathbb{R}^4$ is an GZ_n. Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{3(x^3)^3}{x^1\{(x^1)^3-(x^3)^3\}} & \text{for i=1} \\ -\frac{3(x^1)^3}{x^3\{(x^1)^3-(x^3)^3\}} & \text{for i=3} \\ 0 & \text{otherwise,} \end{cases} \tag{8.2}$$

$$B_i(x) = \begin{cases} \frac{3x^1-2(x^1)^3-4(x^3)^3}{2(x^1)^2\{(x^1)^3-(x^3)^3\}} & \text{for i=1} \\ -\frac{3\{2(x^1)^2-1\}}{2x^3\{(x^1)^3-(x^3)^3\}} & \text{for i=3} \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equations

$$Z_{11,1} = A_1 Z_{11} + B_1 g_{11}, \tag{8.4}$$

$$Z_{22,1} = A_1 Z_{22} + B_1 g_{22}, \tag{8.5}$$

$$Z_{33,1} = A_1 Z_{33} + B_1 g_{33}, \tag{8.6}$$

$$Z_{33,3} = A_3 Z_{33} + B_3 g_{33}, \tag{8.7}$$

$$Z_{44,1} = A_1 Z_{44} + B_1 g_{44}, \tag{8.8}$$

$$Z_{44,3} = A_3 Z_{44} + B_3 g_{44}, \tag{8.9}$$

since, for the other cases (1.4) holds trivially. By (8.2) and (8.3) we get the following relation for the right hand side(R.H.S.) and the left hand side(L.H.S.) of (8.4)

$$\begin{aligned}
\text{R.H.S. of (8.4)} &= A_1 Z_{11} + B_1 g_{11} \\
&= \frac{3(x^3)^3}{x^1\{(x^1)^3 - (x^3)^3\}}\{-\frac{1}{2(x^1)^2} + 1\} + \frac{\{3x^1 - 2(x^1)^3 - 4(x^3)^3\}}{2(x^1)^2\{(x^1)^3 - (x^3)^3\}}x^1 \\
&= \frac{\{3 - 2(x^1)^2\}}{2(x^1)^3} \\
&= Z_{11,1} \\
&= \text{L.H.S. of } \quad (8.4).
\end{aligned}$$

By similar argument it can be shown that the relations from (8.5) to (8.9) are true. So, $\mathbb{R}^4$ is an GZ_n whose scalar curvature is non-zero and non-constant.

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