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ON ZERO DISTRIBUTIONS OF SOME SELF-RECIPROCAL POLYNOMIALS WITH REAL COEFFICIENTS

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Abstract. If $q(z)$ is a polynomial of degree n with all zeros in the unit circle, then the self-reciprocal polynomial $q(z) + x^n q(1/z)$ has all its zeros on the unit circle. One might naturally ask: where are the zeros of $q(z) + x^n q(1/z)$ located if $q(z)$ has different zero distribution from the unit circle? In this paper, we study this question when

$$q(z) = (z-1)^{n-k}(z-1-c_1)\cdots(z-1-c_k) + (z+1)^{n-k}(z+1+c_1)\cdots(z+1+c_k),$$

where $c_j > 0$ for each j, and $q(z)$ is a 'zeros dragged' polynomial from $(z-1)^n + (z+1)^n$ whose all zeros lie on the imaginary axis.

1. Introduction

It what follows, U denotes the unit circle and n is a positive integer. There is an extensive literature concerning zeros of sums of polynomials. Many papers and books([5], [6], [7]) have been written about these polynomials. An immediate question of sums of polynomials, $A + B = C$, is "given zeros of A and B, what zeros can be given for C?". For example, all (conjugate) zeros of the polynomial

$$\prod_{l=1}^{n}(z-r_l) + \prod_{l=1}^{n}(z+r_l), \tag{1}$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$, lie on the imaginary axis. For the proof and more, see [3]. Perhaps the most basic form of the polynomial (1) is

$$(z+1)^n + (z-1)^n, \tag{2}$$

where, by Fell [2], if all zeros of A and B lie in $[-1, 1]$ with A, B monic and deg $A =$ deg $B = n$, then no zero of C can have modulus exceeding $\cot(\pi/2n)$, the largest zero of (2).

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All polynomials in this paper will be assumed to have real coefficients. A polynomial $P(z)$ of degree n is said to be self-inversive if it satisfies $P(z) = \pm P^*(z)$, where $P^*(z) = z^n P(1/z)$. In particular, if $P(z) = P^*(z)$, $P(z)$ is called self-reciprocal. Many questions about the zeros of a family of self-reciprocal polynomials arise naturally in several areas of mathematics -number theory, coding theory, algebraic curves over finite fields, knot theory, but are also of independent interest. The zeros of a self-reciprocal polynomial either lie on U or occur in pairs conjugate to U. Since the class of self-inversive polynomials of degree n includes polynomials of degree n which have all their zeros on U, it is interesting to mention the condition for a self-reciprocal polynomial having all its zeros on U. For example, in [1], Chen proved a following sufficient and necessary condition for a self-inversive polynomial to have all its zeros on U.

Theorem 1. *A necessary and sufficient condition for all the zeros of* $f_n(z) = \sum_{k=0}^{n} a_k z^k$ *with complex coefficients to lie on* U *is that there is a polynomial* $q_{n-l}(z)$ *with all its zeros in or on* U *such that*

$$f_n(z) = z^l q_{n-l}(z) + e^{i\theta} q^*_{n-l}(z)$$

for some nonnegative integer l *and real* θ.

If $q(z)$ is a polynomial of degree n with all zeros in U, then it follows from Theorem 1 that the self-reciprocal polynomial

$$q(z) + q^*(z)$$

has all its zeros on U. One might naturally ask: where are the zeros of $q(z) + q^*(z)$ located if $q(z)$ has different zero distribution from U? For example, if $q(z)$ is the polynomial (2) whose all zeros are on the imaginary axis, then

$$q(z) + q^*(z) = \begin{cases} 2\left((z+1)^n + (z-1)^n\right) & \text{if } n \text{ is even,} \\ 2(z+1)^n & \text{if } n \text{ is odd.} \end{cases}$$

In this case, for n even, all zeros of $q(z) + q^*(z)$ lie on the imaginary axis, and for n odd, they lie on U.

Suppose we drag the zero -1, 1 of each summand of $q(z)$ in (2) to the outward in the same distance, respectively. More specifically, we consider the polynomial

$$q(z) = (z-1)^{n-k}(z-1-c_1)\cdots(z-1-c_k) + (z+1)^{n-k}(z+1+c_1)\cdots(z+1+c_k),$$

where $c_j > 0$ for each j. Our interests in this paper are zero distributions of $q(z) + q^*(z)$, and we will have some results about these. First, we start to study the polynomial $p_1(z) + p_1^*(z)$, where

$$p_1(z) = (z-1)^{n-1}(z-1-c) + (z+1)^{n-1}(z+1+c)$$

is a 'one zero dragged' polynomial from $(z-1)^n + (z+1)^n$. In fact, this polynomial was studied in [4], and very similar results to Theorem 2 below were given there. But our proof here is different from that in [4], and moreover, we describe in detail what the circle is. The theorem below is interesting in that it does not seem obvious how to construct self-reciprocal polynomials with integer coefficients whose zeros all lie on one circle that is not the unit circle.

Theorem 2. *Let for an odd integer n,*

$$p_1(z) = (z-1)^{n-1}(z-1-c) + (z+1)^{n-1}(z+1+c),$$

where $c_j > 0$, $c \neq 0, -1, -2$ for each j. Then all zeros of the self-reciprocal polynomial

$$\frac{p_1(z) + p_1^*(z)}{z+1}$$

lie on a circle that is not the unit circle. This circle has the center

$$C\left(1 + \frac{2}{|k|^2 - 1}, 0\right)$$

and the radius

$$r = \left|\frac{2|k|}{|k|^2 - 1}\right|,$$

where $k = \left(\frac{c}{c+2}\right)^{\frac{1}{n-1}} > 0$.

In the next theorem, we consider a generalized form of the polynomial $p_1(z)$ in Theorem 2.

Theorem 3. *Let for even integers n and k,*

$$\begin{aligned} p_k(z) = {} & (z-1)^{n-k}(z-1-c_1)\cdots(z-1-c_k) \\ & + (z+1)^{n-k}(z+1+c_1)\cdots(z+1+c_k), \end{aligned}$$

where $c_j > 0$ for each j. If

$$(2 + c_1)\cdots(2 + c_k) > \sum_{\substack{r=2 \\ r\,even}}^{k} \left\{ \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r \le k} c_{i_1}\cdots c_{i_r} \frac{(2+c_1)\cdots(2+c_k)}{(2+c_{i_1})\cdots(2+c_{i_r})} \right\}, \tag{3}$$

then the self-reciprocal polynomial $p_k(z) + p_k^*(z)$ has all its zeros on the imaginary axis.

In the case of $c_1 = c_2 = \cdots = c_k = c$, (3) becomes

$$(2+c)^k > \sum_{\substack{r=2 \\ r\text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c^r (2+c)^{k-r} \right\}.$$

This is equivalent to

$$2(2+c)^k > (2+c)^k + \sum_{\substack{r=2 \\ r\text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c^r (2+c)^{k-r} \right\} = \frac{(2c+2)^k + 2^k}{2},$$

that is,

$$u(c) := 4(c+2)^k - (2c+2)^k - 2^k > 0,$$

which is true since $u(0) > 0$ and for $k = 2$,

$$u'(c) = 4k(c+2)^{k-1} - 2k(2c+2)^{k-1} = 8 > 0.$$

This implies the following Corollary 4 that is the special case of $k = 2$ of Theorem 3.

Corollary 4. *Let for an even integer n and $c > 0$,*

$$p_2(z) = (z-1)^{n-2}(z-1-c)^2 + (z+1)^{n-2}(z+1+c)^2.$$

Then the self-reciprocal polynomial $p_2(z) + p_2^*(z)$ has all its zeros on the imaginary axis.

We recall the polynomial in Theorem 2 was

$$p_1(z) = (z-1)^{n-1}(z-1-c) + (z+1)^{n-1}(z+1+c),$$

where n is an odd integer, and the polynomial in Corollary 4 was

$$p_2(z) = (z-1)^{n-2}(z-1-c)^2 + (z+1)^{n-2}(z+1+c)^2,$$

where n is an even integer. By Theorem 2, the self-reciprocal polynomial $p_1(z)+p_1^*(z)$ has all its zeros other than -1 lies on a circle that is not the unit circle. Corollary 4 is unexpectedly surprising in that the self-reciprocal polynomial $p_2(z) + p_2^*(z)$ with one more dragging has all its zeros on the imaginary axis.

2. PROOFS

In this section, we provide the proofs of our results.

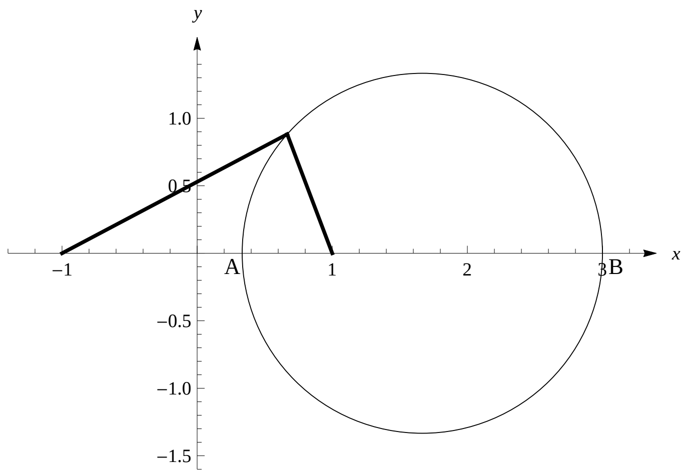

Figure 1. Apollonius circle with $k = 2$

Proof of Theorem 2. With notations of the theorem, the roots of $p_1(z)/(z+1)$ satisfy

$$\left(\frac{z+1}{z-1}\right)^{n-1} = \frac{c}{c+2}.$$

Let $k = \left(\frac{c}{c+2}\right)^{1/(n-1)}$ be a real number. Then $\left|\frac{z+1}{z-1}\right| = |k|$ whose locus is a circle of Apollonius that is the set of points with ratio of distances $|k|$ to two points $(-1, 0)$ and $(1, 0)$ in Figure 1. Let A and B denote the points that the Apollonius circle crosses the real axis.

Then

$$A\left(\frac{|k|-1}{|k|+1}, 0\right), \qquad B\left(\frac{|k|+1}{|k|-1}, 0\right),$$

and the center and the radius of the circle

$$C\left(1 + \frac{2}{|k|^2 - 1}, 0\right), \qquad r = \left|\frac{2|k|}{|k|^2 - 1}\right|,$$

respectively.

For the proof of Theorem 3, we will need the following two theorems.

Theorem 5. (Cohn) *Let* $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$, $(a_n \neq 0)$. *Then all zeros of* P *lie on* $|z| = 1$ *if and only if*
(i) P *is self-inversive,*
(ii) all zeros of P' *lie in* $|z| \leq 1$.

Moreover, if P is self-inversive and

$$\tau = \text{the number of zeros on } |z| = 1 \text{ (counted with multiplicity)},$$

$$\nu = \text{the number of critical points in } |z| \le 1 \text{ (counted with multiplicity)}.$$

Then

$$\tau = 2(\nu + 1) - n.$$

Theorem 6. (Cauchy) *All zeros of* $P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + 2a_2 z + a_1$ *lie in*

$$|z| \le r,$$

where r is the positive root of the equation

$$n|a_n|z^{n-1} - (n-1)|a_{n-1}|z^{n-2} - \cdots - 2|a_2|z - |a_1| = 0.$$

For the proofs of above two theorems, see [7, p. 230] and [6, p. 244].

Proof of Theorem 3. Let n and k be positive even integers with $n > k$, and

$$P_k(z) = p_k(z) + p_k^*(z),$$

where

$$\begin{aligned} p_k(z) = &(z-1)^{n-k}(z-1-c_1)\cdots(z-1-c_k) \\ &+ (z+1)^{n-k}(z+1+c_1)\cdots(z+1+c_k), \end{aligned}$$

where $c_j > 0$ for each j. Then

$$\begin{aligned} p_k^*(z) = &(z-1)^{n-k}\,(1-(1+c_1)z)\cdots(1-(1+c_k)z) \\ &+ (z+1)^{n-k}\,(1+(1+c_1)z)\cdots(1+(1+c_k)z) \end{aligned}$$

and

$$P_k(z) = \{(z+1)^{n-k} + (z-1)^{n-k}\}(A+C) + \{(z+1)^{n-k} - (z-1)^{n-k}\}(B+D),$$

where

$$\begin{aligned} A &= \frac{(z+1+c_1)\cdots(z+1+c_k) + (z-1-c_1)\cdots(z-1-c_k)}{2}, \\ B &= \frac{(z+1+c_1)\cdots(z+1+c_k) - (z-1-c_1)\cdots(z-1-c_k)}{2}, \\ C &= \frac{(1+(1+c_1)z)\cdots(1+(1+c_k)z) + (1-(1+c_1)z)\cdots(1-(1+c_k)z)}{2}, \\ D &= \frac{(1+(1+c_1)z)\cdots(1+(1+c_k)z) - (1-(1+c_1)z)\cdots(1-(1+c_k)z)}{2}. \end{aligned}$$

Then the zeros of $P_k(z)$ satisfy

$$\frac{(z+1)^{n-k}+(z-1)^{n-k}}{(z+1)^{n-k}-(z-1)^{n-k}}=-\frac{B+D}{A+C}.$$

Write

$$l=\frac{(z+1)^{n-k}+(z-1)^{n-k}}{(z+1)^{n-k}-(z-1)^{n-k}}.$$

Then

$$\left(\frac{z+1}{z-1}\right)^{n-k}=\frac{l+1}{l-1}\qquad\text{and}\qquad\frac{z+1}{z-1}=\left(\frac{l+1}{l-1}\right)^{\frac{1}{n-k}}=:L.$$

So

$$z=\frac{L+1}{L-1}\qquad\text{and}\qquad l=\frac{L^{n-k}+1}{L^{n-k}-1}. \tag{4}$$

Since

$$l=\frac{L^{n-k}+1}{L^{n-k}-1}=-\frac{B+D}{A+C},$$

we have

$$(A+B+C+D)L^{n-k}+(A+C-B-D)=0.$$

Let

$$f(L)=\Big\{(A+B+C+D)L^{n-k}+(A+C-B-D)\Big\}(L-1)^k,$$

that is,

$$\begin{aligned}f(L)=\big[&\{(z+1+c_1)\cdots(z+1+c_k)+(1+(1+c_1)z)\cdots(1+(1+c_k)z)\}L^{n-k}\\&+(z-1-c_1)\cdots(z-1-c_k)+(1-(1+c_1)z)\cdots(1-(1+c_k)z)\big](L-1)^k.\end{aligned}$$

By (4), $z=\frac{L+1}{L-1}$ and put this into the right hand side of above equation so that we have

$$\begin{aligned}f(L)=\big\{&((2+c_1)L-c_1)\cdots((2+c_k)L-c_k)\\&+((2+c_1)L+c_1)\cdots((2+c_k)L+c_k)\big\}L^{n-k}\\&+(c_1L-(c_1+2))\cdots(c_kL-(c_k+2))+(c_1L+(c_1+2))\cdots(c_kL+(c_k+2)).\end{aligned}$$

We observe that $L^nf(1/L)=f(L)$ since k is even, that is, $f(L)$ is self reciprocal. We will use Theorem **??** to show that all zeros of f lie on $|L|=1$. First, we may express $f(L)$ by the sum as follows:

(5)

$$f(L)=2E(L^n+1)+2\sum_{\substack{r=2\\ r\text{even}}}^{k}\left\{\sum_{1\le i_1<\cdots<i_r\le k}c_{i_1}\cdots c_{i_r}\frac{E}{(2+c_{i_1})\cdots(2+c_{i_r})}(L^{n-r}+L^r)\right\},$$

where $E = (2 + c_1) \cdots (2 + c_k)$. Then

$$f'(L) = 2nEL^{n-1} + 2 \sum_{\substack{r=2 \\ r \text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c_{i_1} \cdots c_{i_r} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} ((n - r)L^{n-r-1} + rL^{r-1}) \right\}.$$

To use Theorem 5, we let

$$g(L) = 2nEL^{n-1} - 2 \sum_{\substack{r=2 \\ r \text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c_{i_1} \cdots c_{i_r} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} ((n - r)L^{n-r-1} + rL^{r-1}) \right\}.$$

Then

$$g'(L) = 2n(n-1)EL^{n-2} - 2 \sum_{\substack{r=2 \\ r \text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c_{i_1} \cdots c_{i_r} \frac{E}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} ((n - r)(n - r - 1)L^{n-r-2} + r(r - 1)L^{r-2}) \right\}$$

and we have

$$g(0) = 0, \quad g'(0) = -4 \sum_{1 \le i_1 < i_2 \le k} c_{i_1} c_{i_2} \frac{E}{(2 + c_{i_1})(2 + c_{i_2})} < 0. \tag{6}$$

But we observe that

$$g(1) = 2nE - 2 \sum_{\substack{r=2 \\ r \text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c_{i_1} \cdots c_{i_r} \frac{nE}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} \right\} > 0$$

is equivalent to

$$(2 + c_1) \cdots (2 + c_k) > \sum_{\substack{r=2 \\ r \text{even}}}^{k} \left\{ \sum_{1 \le i_1 < \cdots < i_r \le k} c_{i_1} \cdots c_{i_r} \frac{(2 + c_1) \cdots (2 + c_k)}{(2 + c_{i_1}) \cdots (2 + c_{i_r})} \right\}. \tag{7}$$

Hence if the inequlity (7) holds, $g(1) > 0$ and so by (6), $g(L) = 0$ has at least one zero α in the open interval $(0, 1)$. In fact, this zero α is unique in the open interval $(0, 1)$ by Theorem 6. It follows from Theorem 5 that all the zeros of $f(L) = 0$ lie on $|L| \leq \alpha < 1$, where α is the positive zero of the equation $g(L) = 0$. Hence by Theorem 5, all zeros of f lie on $|L| = 1$. But by (4),

$$|L| = \left| \frac{z+1}{z-1} \right| = 1,$$

where z was the zero of $P_k(z)$. One gets that the distances of z from the point -1 equals the distances of z from the point 1. Thus, if z is to the left or to the right of the imaginary axis, one of these distances is bigger. This implies that z lies on the imaginary axis, which completes the proof.

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