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A NUMERICAL METHOD OF FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we propose a numerical method to solve fuzzy differential equations. Numerical experiments show that when the step size is small, the new method has significantly good approximate solutions of fuzzy differential equation. Graphical representation of fuzzy solutions in three-dimension is also provided as a reference of visual convergence of the solution sequence.

1. Introduction

Fuzzy set initially presented by Zadeh [15] has been developed into fuzzy mathematics including fuzzy logic, fuzzy probabilities, fuzzy information, and so on. Fuzzy-valued mapping was developed by Puri and Ralescu [10] and then a theory for fuzzy differential equations (FDEs) has been developed by Kaleva [8].

There are many works done by several authors for solving FDEs based on the fuzzy set [15]. For examples, hybrid predictor-corrector method [14], variational iteration method [1], a partial averaging scheme with maxima [9], Laplace decomposition method [6], Milne's predictor-corrector method [3], variational iteration method [5], and fuzzy Laplace transform method [2]. On the other hand, slightly different fuzzy number so called a linear fuzzy real number was discussed in [7, 11, 12, 13]. However, there is no literature so far dealing with FDEs in algorithmic point of view over linear fuzzy real numbers. In this paper, we present an algorithm to solve FDEs on linear fuzzy real numbers.

The paper is organized as follows. In Section 2, we provide some preliminary definitions on linear fuzzy real numbers. In Section 3, numerical algorithm and experiments are presented to solve fuzzy differential equations. Lastly, we will make concluding remarks in Section 4.

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2. Preliminaries

In this section, we discuss some important definitions and properties of linear fuzzy real numbers [7, 11, 12, 13]. When we consider the set of all real numbers R, one way to associate a fuzzy number with a fuzzy subset of real numbers is as a function $\mu : R \to [0, 1]$, where the value $\mu(x)$ is to represent a degree of belonging to the subset of R.

Definition 2.1 (Linear fuzzy real number)**.** Let R be the set of all real numbers and $\mu : R \to [0, 1]$ be a function defined by

$$\mu(x) = \begin{cases} 0, & \text{if } x < a \text{ or } x > c, \\ \frac{x-a}{b-a}, & \text{if } a \le x < b, \\ 1, & \text{if } x = b, \\ \frac{c-x}{c-b}, & \text{if } b < x \le c. \end{cases}$$

Then $\mu(a, b, c)$ is called a *linear fuzzy real number* with associated triple of real numbers (a, b, c) where $a \le b \le c$ shown in Figure 1.

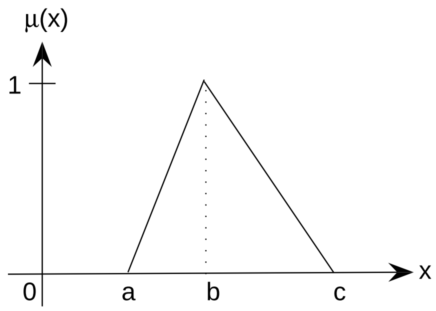


Figure 1. Linear fuzzy real number $\mu(a, b, c)$

Let LFR be the set of all linear fuzzy real numbers. Then we note that any real number $t \in R$ can be written as a linear fuzzy real number $r(t) \in LFR$, where $r(t) = \mu(t, t, t)$, and hence $R \subseteq LFR$. As a linear fuzzy real number, we consider $r(t)$ to represent the real number t itself. Operations on LFR [7, 11, 12, 13], sequence, and differentiability are defined as the followings.

Definition 2.2 (Operations)**.** For given two linear fuzzy real numbers $\mu_1 = \mu(a_1, b_1, c_1)$ and $\mu_2 = \mu(a_2, b_2, c_2)$, we define addition, subtraction, multiplication, and division by

(1) $\mu_1 + \mu_2 = \mu(a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(2) $\mu_1 - \mu_2 = \mu(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(3) $\mu_1 \cdot \mu_2 = \mu(\min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\}, b_1b_2, \max\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\})$
(4) $\frac{\mu_1}{\mu_2} = \mu_1 \cdot \frac{1}{\mu_2}$ where $\frac{1}{\mu_2} = \mu(\min\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \text{median}\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \max\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\})$.

Since a real number t can be considered as a linear fuzzy real number $r(t) = \mu(t, t, t)$, we have $t \cdot \mu(a, b, c) = \mu(t \cdot a, t \cdot b, t \cdot c)$ for $t > 0$.

Definition 2.3 (Sequence)**.** Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of LFR where $X_n = \mu(a_n, b_n, c_n)$. The LFR-sequence $\{X_n\}$ has the limit $X^* = \mu(a^*, b^*, c^*)$ and we write $\lim_{n\to\infty} X_n = X^*$, if the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have the limit a^*, b^*, and c^*, respectively. If $\lim_{n\to\infty} X_n$ exists, we say the LFR-sequence $\{X_n\}$ is *convergent*. Otherwise, we say the sequence is *divergent*.

Definition 2.4 (Differentiability)**.** Let $X : [a, b] \to LFR$ be a mapping, $t_0 \in [a, b]$, and $L \in LFR$. We say that $X(t)$ has the limit L as t approaches t_0 and we write $\lim_{t\to t_0} X(t) = L$, if we can make the values of $X(t)$ arbitrarily close to L by taking t to be sufficiently close to t_0 but not equal to t_0. We say that $X(t)$ is *differentiable* at t_0 if there exists $X'(t_0) \in LFR$ such that

$$X'(t_0) = \lim_{h\to 0} \frac{X(t_0 + h) - X(t_0)}{h}.$$

3. Solving Fuzzy Differential Equations

In this section, we consider the following fuzzy initial value problem:

$$\begin{cases} X'(t) = F(t, X(t)), t \in [a, b] \\ X(t_0) = X_0 \in LFR, \end{cases} \tag{3.1}$$

where $F : [a, b] \times LFR \to LFR$. In this case, $X'(t) = F(t, X(t))$ is called a LFR-valued fuzzy differential equation and $X(t_0) = X_0 \in LFR$ is called an initial condition at t_0. Associated crisp initial value problem is

$$\begin{cases} x'(t) = f(t, x(t)), t \in [a, b] \\ x(t_0) = x_0 \in R, \end{cases} \tag{3.2}$$

where $x'(t)$ is the crisp derivative of a function $x : [a, b] \to R$.

Solving the fuzzy initial value problem (3.1) over LFR is possible with a modification of classical Euler's method of the crisp initial value problem (3.2) over real numbers. Let N be a fixed positive integer and $h = \frac{b-a}{N}$. Take $t_n = a + nh$, for $n = 0, 1, \cdots, N$. Then by Taylor's theorem [4], for any $t_n \in [a, b]$, there exists a number $\xi_n \in [t_n, t_{n+1}]$ with

$$x(t_{n+1}) = x(t_n + h) = x(t_n) + hx'(t_n) + \frac{1}{2}h^2 x''(\xi_n),$$

and hence crisp Euler's method is $x_{n+1} = x_n + hf(t_n, x_n)$ for $n = 0, 1, \cdots, N-1$.

Thus, the modified Euler's method to the fuzzy initial value problem (3.1) over LFR is

$$X_{n+1} = X_n + hF(t_n, X_n) \text{ for each } n = 0, 1, \cdots, N-1. \tag{3.3}$$

Now we provide the algorithm of the new scheme using (3.3), referred to as LFR Euler's algorithm, to solve the fuzzy initial value problem (3.1) over LFR.

Algorithm: (LFR Euler's algorithm)

INPUT: fuzzy mapping F, interval $[a, b]$, integer N, initial condition X_0
OUTPUT: approx. $X_n = \mu(a_n, b_n, c_n)$ to $X(t_n)$ at the $(N+1)$ values of t
Step 1: Set $h = (b-a)/N$ and $t_0 = a$.
Step 2: For $n = 1, 2, \cdots, N$ do Steps 3, 4.
Step 3: Set $t_n = a + nh$.
Step 4: Set $X_n = X_{n-1} + hF(t_{n-1}, X_{n-1})$.
Step 5: OUTPUT(all t_n and X_n) and STOP.

Example 3.1. Consider the following fuzzy initial value problem:

$$\begin{cases} X'(t) = t - X, t \in [0, 2] \\ X(0) = \mu(0.5, 1, 1.5) \end{cases}$$

Let $F(t, X) = t - X$ and initial condition $X_0 = \mu(0.5, 1, 1.5) \in LFR$. If we choose $N = 10$ so that the step size $h = 0.2$, then we can generate a sequence of approximate solutions $\{X_n\}_{n=0}^{10}$, where $X_n = \mu(a_n, b_n, c_n)$, using Equation (3.3). For example,

$$X_1 = X_0 + hF(t_0, X_0) = X_0 + h(t_0 - X_0) = \mu(0.4, 0.8, 1.2).$$

Entire terms of the sequence of approximate solutions $\{X_n\}$ are listed in Table 1 along with the exact solutions $x(t_n)$ at t_n for $n = 0, 1, \cdots, 10$, since the deterministic associated problem: $x'(t) = t - x$, $t \in [0, 2]$, $x(0) = 1$ has the exact solution $x(t) = t - 1 + 2e^{-t}$. Table 2 shows the results using $N = 100$ in the algorithm. We can see in Table 2 that X_{100} in Table 2 is more accurate than X_{10} in Table 1 at the same last time level $t = 2$.

Table 1. Approximate solutions by LFR Euler and exact solutions ($N = 10$, $h = 0.2$)

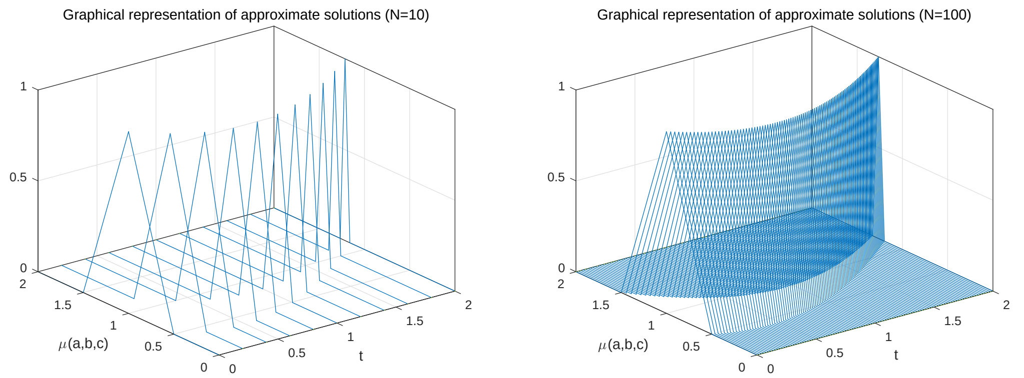
t_n	$X_n = \mu(a_n, b_n, c_n)$	$x(t_n)$
$t_0 = 0$	$X_0 = \mu(0.5, 1, 1.5)$	$x(t_0) = 1.0$
$t_1 = 0.2$	$X_1 = \mu(0.4000, 0.8000, 1.2000)$	$x(t_1) = 0.8375$
$t_2 = 0.4$	$X_2 = \mu(0.3600, 0.6800, 1.0000)$	$x(t_2) = 0.7406$
$t_3 = 0.6$	$X_3 = \mu(0.3680, 0.6240, 0.8800)$	$x(t_3) = 0.6976$
$t_4 = 0.8$	$X_4 = \mu(0.4144, 0.6192, 0.8240)$	$x(t_4) = 0.6987$
$t_5 = 1.0$	$X_5 = \mu(0.4915, 0.6554, 0.8192)$	$x(t_5) = 0.7358$
$t_6 = 1.2$	$X_6 = \mu(0.5932, 0.7243, 0.8554)$	$x(t_6) = 0.8024$
$t_7 = 1.4$	$X_7 = \mu(0.7146, 0.8194, 0.9243)$	$x(t_7) = 0.8932$
$t_8 = 1.6$	$X_8 = \mu(0.8517, 0.9355, 1.0194)$	$x(t_8) = 1.0038$
$t_9 = 1.8$	$X_9 = \mu(1.0013, 1.0684, 1.1355)$	$x(t_9) = 1.1306$
$t_{10} = 2.0$	$X_{10} = \mu(1.1611, \mathbf{1.2147}, 1.2684)$	$x(t_{10}) = \mathbf{1.2707}$

Table 2. Approximate solutions by LFR Euler and exact solutions ($N = 100$, $h = 0.02$)

t_n	$X_n = \mu(a_n, b_n, c_n)$	$x(t_n)$
$t_0 = 0$	$X_0 = \mu(0.5, 1, 1.5)$	$x(t_0) = 1.0$
$t_{10} = 0.2$	$X_{10} = \mu(0.4256, 0.8341, 1.2427)$	$x(t_{10}) = 0.8375$
$t_{20} = 0.4$	$X_{20} = \mu(0.4014, 0.7352, 1.0690)$	$x(t_{20}) = 0.7406$
$t_{30} = 0.6$	$X_{30} = \mu(0.4182, 0.6910, 0.9637)$	$x(t_{30}) = 0.6976$
$t_{40} = 0.8$	$X_{40} = \mu(0.4686, 0.6914, 0.9143)$	$x(t_{40}) = 0.6987$
$t_{50} = 1.0$	$X_{50} = \mu(0.5463, 0.7283, 0.9104)$	$x(t_{50}) = 0.7358$
$t_{60} = 1.2$	$X_{60} = \mu(0.6463, 0.7951, 0.9439)$	$x(t_{60}) = 0.8024$
$t_{70} = 1.4$	$X_{70} = \mu(0.7647, 0.8862, 1.0078)$	$x(t_{70}) = 0.8932$
$t_{80} = 1.6$	$X_{80} = \mu(0.8980, 0.9973, 1.0966)$	$x(t_{80}) = 1.0038$
$t_{90} = 1.8$	$X_{90} = \mu(1.0435, 1.1246, 1.2058)$	$x(t_{90}) = 1.1306$
$t_{100} = 2.0$	$X_{100} = \mu(1.1989, \mathbf{1.2652}, 1.3315)$	$x(t_{100}) = \mathbf{1.2707}$

Tables 1, 2 show the comparison between the approximate values at t_n and the actual values. We can see that the accuracy of approximate solutions is better when a smaller step size h is used.

We also provide three-dimensional graphical representation of approximate solutions in Figure 2 at $N = 10$ and $N = 100$, in which we are able to see the convergence of the solution sequence.



(a) Solution by LFR Euler (N=10) (b) Solution by LFR Euler (N=100)

Figure 2. Graphical representation of approximate solutions by LFR Euler's method

4. Conclusion

A differential equation over linear fuzzy real numbers is called a fuzzy differential equation. In this paper, a numerical method has been introduced to solve fuzzy differential equations with a modification of crisp Euler's method. The numerical experiments show that the approximate solutions of fuzzy differential equations have very good accuracy when the step size is small. We have also presented graphical representation of those solutions in three-dimension as a reference of visual convergence of the solution sequence.

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