J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2017.24.3.171 ISSN(Online) 2287-6081
Volume 24, Number 3 (August 2017), Pages 171–178

STABILITY OF C^*-TERNARY QUADRATIC 3-JORDAN HOMOMORPHISMS

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Abstract. In this paper, we define C^*-ternary quadratic 3-Jordan homomorphisms associated with the quadratic mapping $f(x+y)+f(x-y)=2f(x)+2f(y)$, and prove the Hyers-Ulam stability of C^*-ternary quadratic 3-Jordan homomorphisms.

1. Introduction and Preliminaries

As it is extensively discussed in [17], the full description of a physical system S implies the knowledge of three basic ingredients: the set of the observables, the set of the states and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally the set of the observables were considered to be a C^*-algebra [9].

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called quadratic functional equation. In addition, every solution of the above equation is said to be a quadratic mapping. Czerwik [5] proved the Cauchy-Rassias stability of the quadratic functional equation. Since then, the stability problems of various functional equation

Received by the editors July 12, 2017. Accepted August 22, 2017.
2010 *Mathematics Subject Classification.* Primary 39B52, 39B82, 46B99, 17A40.
Key words and phrases. Hyers-Ulam stability, C^*-ternary algebra, quadratic functional equation, C^*-ternary quadratic 3-Jordan homomorphism.
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have been extensively investigated by a number of authors (for instances, [3, 7, 18, 19, 20, 21]).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists (see [12]). As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [6] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc ([1, 23]). The comments on physical applications of ternary structures can be found in ([4, 8, 11, 13, 14, 15]).

A ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u]v\Big] = \Big[[x, y, z], u, v\Big]$ and satisfies $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$. A C^*-ternary algebra is a complex Banach space A equipped with a ternary product which is associative and $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and $\|[x, x, x]\| = \|x\|^3$ (see [24]). If a C^*-ternary algebra $(A, [\cdot\,\cdot\,\cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $xoy := [x, e, y]$, $\quad x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if (A, o) is a unital C^*-algebra, then $[x, y, z] := xoy^*oz$ makes A into a C^*-ternary algebra.

Throughout this paper, let A and B be Banach ternary algebras.

A quadratic mapping $Q : A \to B$ is called a *C^*-ternary quadratic homomorphism* if

$$Q([x, x, x]) = [Q(x), Q(x), Q(x)]$$

for all $x \in A$.

Definition 1.1. Let A and B be C^*-ternary algebras. A quadratic mapping $Q : A \to B$ is called a *C^*-ternary quadratic 3-Jordan homomorphism* if it satisfies

$$Q([[x, x, x], [y, y, y], [z, z, z]]) = [Q([x, x, x]), Q([y, y, y]), Q([z, z, z])]$$

for all $x, y, z \in A$.

In this paper, we prove the Hyers-Ulam stability of C^*-ternary quadratic 3-Jordan homomorphisms in C^*-ternary algebras.

2. Stability of C^*-ternary Quadratic 3-Jordan Homomorphisms

In this section, we prove the Hyers-Ulam stability of C^*-ternary quadratic 3-Jordan homomorphisms for the quadratic functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y).$$

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that*

$$\sum_{i=0}^{\infty} 4^{9i}\varphi\Big(\frac{x}{2^i},\frac{y}{2^i},\frac{z}{2^i}\Big) < \infty,$$

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varphi(x,y,0) \tag{2.1}$$

$$\Big\|f([[x,x,x],[y,y,y],[z,z,z]])-[f([x,x,x]),f([y,y,y]),f([z,z,z])]\Big\| \le \varphi(x,y,z) \tag{2.2}$$

for all $x,y,z\in A$. Then there exists a unique C^*-ternary quadratic 3-Jordan homomorphism $Q : A \to B$ such that

$$\|f(x)-Q(x)\| \le \widetilde{\varphi}\Big(\frac{x}{2},\frac{x}{2},0\Big) \tag{2.3}$$

for all $x\in A$, where

$$\widetilde{\varphi}(x,y,z) := \sum_{i=0}^{\infty} 4^{i}\varphi\Big(\frac{x}{2^i},\frac{y}{2^i},\frac{z}{2^i}\Big)$$

for all $x,y,z\in A$.

Proof. It follows from (2.1) that $f(0)=0$.

Letting $y=x$ in (2.1), we get

$$\|f(2x)-4f(x)\| \le \varphi(x,x,0) \tag{2.4}$$

for all $x\in A$. So

$$\|f(x)-4f\Big(\frac{x}{2}\Big)\| \le \varphi\Big(\frac{x}{2},\frac{x}{2},0\Big)$$

for all $x\in A$. Hence

$$\begin{aligned}\Big\|4^l f\Big(\frac{x}{2^l}\Big)-4^m f\Big(\frac{x}{2^m}\Big)\Big\| &\le \sum_{i=1}^{m-1}\Big\|4^i f\Big(\frac{x}{2^i}\Big)-4^{i+1} f\Big(\frac{x}{2^{i+1}}\Big)\Big\| \le \sum_{i=0}^{m-1}4^i\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},0\Big)\\ &\le \sum_{i=0}^{m-1}4^{9i}\varphi\Big(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},0\Big)\end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) = \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&= \lim_{n\to\infty} 4^n \Big\| f\Big(\frac{x+y}{2^n}\Big) + f\Big(\frac{x-y}{2^n}\Big) - 2f\Big(\frac{x}{2^n}\Big) - 2f\Big(\frac{y}{2^n}\Big)\Big\| \\
&\le \lim_{n\to\infty} 4^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0\Big) \le \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0\Big) = 0
\end{aligned}$$

and so

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in A$.

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned}
&\Big\| Q([[x,x,x],[y,y,y],[z,z,z]]) - [Q([x,x,x]), Q([y,y,y]), Q([z,z,z])]\Big\| \\
&= \lim_{n\to\infty} 4^{9n} \Big\| f\Big([[\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}], [\frac{y}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}], [\frac{z}{2^n}, \frac{z}{2^n}, \frac{z}{2^n}]]\Big) \\
&\quad - [f\Big([\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}]\Big), f\Big([\frac{y}{2^n}, \frac{y}{2^n}, \frac{y}{2^n}]\Big), f\Big([\frac{z}{2^n}, \frac{z}{2^n}, \frac{z}{2^n}]\Big)]\Big\| \\
&\le \lim_{n\to\infty} 4^{9n} \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) = 0
\end{aligned}$$

and so

$$Q([[x,x,x],[y,y,y],[z,z,z]]) = [Q([x,x,x]), Q([y,y,y]), Q([z,z,z])]$$

for all $x, y, z \in A$.

Now, let $T : A \to B$ be another quadratic mapping satisfying (2.3). Then we have

$$\begin{aligned}
\|Q(x) - T(x)\| &= 4^n \Big\| Q\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\Big\| \\
&\le 4^n \Big(\Big\| Q\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big)\Big\| + \Big\| T\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x}{2^n}\Big)\Big\|\Big) \\
&\le 2 \cdot 4^n \varphi\Big(\frac{x}{2^n}, \frac{x}{2^n}, 0\Big),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $Q(x) = T(x)$ for all $x \in A$. This proves the uniqueness of Q. Thus the quadratic mapping

$Q : A \to B$ is a unique C^*-ternary quadratic 3-Jordan homomorphism satisfying (2.3). $\square$

Corollary 2.2. *Let r, θ be nonnegative real numbers with $r > 18$ and let $f : A \to B$ be a mapping satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^r + \|y\|^r), \tag{2.6}$$

$$\begin{aligned} &\Big\|f([[x,x,x],[y,y,y],[z,z,z]]) - [f([x,x,x]), f([y,y,y]), f([z,z,z])]\Big\| \\ &\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{2.7}$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary quadratic 3-Jordan homomorphism $Q : A \to B$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in A$.

Proof. Defining $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.1, we get the desired result. $\square$

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying (2.1) and (2.2) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{i=0}^{\infty} \frac{1}{4^i}\varphi(2^i x, 2^i y, 2^i z) < \infty$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary quadratic 3-Jordan homomorphisms $Q : A \to B$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4}\widetilde{\varphi}(x,x,0) \tag{2.8}$$

for all $x \in A$

Proof. It follows from (2.4) that

$$\Big\|f(x) - \frac{1}{4}f(2x)\Big\| \leq \frac{1}{4}\varphi(x,x,0)$$

for all $x \in A$

(2.9)

$$\Big\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\Big\| \leq \sum_{j=l}^{m-1}\Big\|\frac{1}{4^j}f(2^j x) - \frac{1}{4^{j+1}}f(2^{j+1}x)\Big\| \leq \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^j x, 2^j x, 0)$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.9) that the sequence $\{(\frac{1}{4^n})f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B

is complete, the sequence $\{(\frac{1}{4^n})f(2^nx)\}$ converges. So one can define the mapping $Q : A \to B$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.8).

It follows from (2.2) and the continuity of the ternary product that

$$\begin{aligned}
&\Big\|Q([[x,x,x],[y,y,y],[z,z,z]]) - [Q([x,x,x]),Q([y,y,y]),Q([z,z,z])]\Big\| \\
&= \lim_{n\to\infty} \frac{1}{4^{9n}} \Big\|f([[2^nx,2^nx,2^nx],[2^ny,2^ny,2^ny],[2^nz,2^nz,2^nz]]) \\
&\quad - [f([2^nx,2^nx,2^nx]),f([2^ny,2^ny,2^ny]),f([2^nz,2^nz,2^nz])]\Big\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{9n}} \varphi\Big(2^nx,2^ny,2^nz\Big) \le \lim_{n\to\infty} \frac{1}{4^n} \varphi\Big(2^nx,2^ny,2^nz\Big) = 0
\end{aligned}$$

and so

$$Q([[x,x,x],[y,y,y],[z,z,z]]) = [Q([x,x,x]),Q([y,y,y]),Q([z,z,z])]$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.1 □

Corollary 2.4. *Let r, θ be nonnegative real numbers with $r < 2$ and let $f : A \to B$ be a mapping satisfying (2.6) and (2.7). Then there exists a unique C^*-ternary quadratic 3-Jordan homomorphism $Q : A \to B$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r} \|x\|^r$$

for all $x \in A$.

Proof. Defining $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.3, we get the desired result. □

ACKNOWLEDGMENTS

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937), and the second author was supported by Hanshin University Research Grant.

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