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COMPARISON OF SMALLEST EIGENVALUES FOR RIGHT FOCAL ATICI-ELOE FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. The theory of u_0-positive operators is applied to obtain smallest eigenvalue comparison results for right focal boundary value problems of Atici-Eloe fractional difference equations.

1. Introduction

Fractional calculus, the continuous case, has a long history and there is a renewed interest in both the study of fractional calculus and fractional differential equations. In fact, rapid progress is being made in the study of boundary value problems for fractional differential equations. We refer the reader to Kilbas *et al.* [12], F. Mainardi [15], Miller and Ross [17], Podlubny [18] and Samko *et al.* [19], etc., for accounts of the historical development; we refer the reader to [5, 6, 9, 16, 20, 21, 22] for samples of recent studies in boundary value problems for fractional differential equations. Also, discrete fractional calculus has generated a lot of interest in recent years. While, there has been little progress made in developing the theory of fractional difference equations or, moreover, the general theory of fractional calculus on an arbitrary time scale. Recently, though, there have appeared a number of papers on the discrete fractional calculus, which has helped to build up some of the basic theory of this area. For example, Atici and Şengül [4] shows that fractional difference equations may provide for useful biological models. Jia, Erbe and Peterson[10] give the monotonicity results for nabla and delta fractional differences. In addition to these works, one can consult [1, 2, 23] and the references therein to see more progress that has been made in the discrete fractional calculus. This paper, can be considered a contribution to this new, emerging area of mathematics.

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In this paper, for $b \in \mathbb{N}$, we are concerned with the existence of smallest positive eigenvalues and their comparisons for the νth order Atici-Eloe fractional difference equations,

$$\Delta^{\nu} y(t) + \lambda_1 p(t+\nu-1) y(t+\nu-1) = 0, \quad t \in \{1, 2, \ldots, b+1\}, \tag{1}$$

$$\Delta^{\nu} y(t) + \lambda_2 q(t+\nu-1) y(t+\nu-1) = 0, \quad t \in \{1, 2, \ldots, b+1\}, \tag{2}$$

with each satisfying the right focal boundary conditions

$$y(\nu-2) = 0, \quad \Delta y(b+\nu) = 0, \tag{3}$$

where Δ^{ν} is the νth Atici-Eloe fractional difference with $1 < \nu \leq 2$ a real number, and $p(t), q(t) : \{\nu, \nu+1, \ldots, b+\nu\} \to (0, +\infty)$.

The purpose of this article is to establish comparison results for smallest positive eigenvalues of the right focal boundary value problems for these fractional difference equations.

In Section 2, we give some preliminary definitions and theorems from the theory of cones in Banach spaces that are employed to establish existence of the smallest positive eigenvalues. In Section 3, we first provide some sign properties of the Green's function for $-\Delta^{\nu} y = 0$ under the boundary conditions (3), which will play the role of the kernel of suitable positive operators to which the u_0-positive operator theory can be applied. Also, a suitable cone in a Banach space is defined, and then applications of the preliminary results are made for the comparison of the smallest positive eigenvalues.

2. PRELIMINARIES

We will state some definitions from fractional difference equations along with some definitions and theorems from cone theory on which the paper's main results depend.

Definition 2.1 ([1, 7]). Let $n - 1 < \nu \leq n$ be a real number and $t \in \{a+\nu, a+\nu+1, \ldots\}$, $n \geq 2$ is a integer. The νth Atici-Eloe fractional sum of the function u is defined by

$$\Delta^{-\nu} u(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} u(s),$$

where $t^{(\nu)} = (\Gamma(t+1))/(\Gamma(t+1-\nu))$ is the falling function and Γ is the Gamma function. Also, the νth Atici-Eloe fractional difference of the function u is defined by

$$\Delta^{\nu}u(t) = \Delta^{n-(n-\nu)}u(t) = \Delta^{n}(\Delta^{-(n-\nu)}u(t)),$$

where Δ is the forward difference defined as $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^{i}u(t) = \Delta(\Delta^{i-1}u(t))$, $i = 2, 3, \ldots$.

Remark 2.2. It is easy to check that $x^{(\nu)}$ is an increasing function for $x \in (\nu, \nu + 1, \ldots)$.

Remark 2.3. We note that for u defined on $\{a, a+1, \ldots\}$, then $\Delta^{-\nu}u$ is defined on $\{a+\nu, a+\nu+1, \ldots\}$.

Let $(\mathcal{B}, ||\cdot||)$ be a real Banach space. $\mathcal{P} \subset \mathcal{B}$ is a *cone* provided (i) $\alpha u + \beta v \in \mathcal{P}$, for all $\alpha, \beta \geq 0$ and for all $u, v \in \mathcal{P}$, and (ii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. A cone $\mathcal{P}$ is *solid* if the interior, $\mathcal{P}^{\circ}$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is *reproducing* if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.4. Krasnosel'skii [13] showed that every solid cone is reproducing.

A cone $\mathcal{P}$ in a real Banach space $\mathcal{B}$ induces a partial order on $\mathcal{B}$; namely, for $u, v \in \mathcal{B}$, $u \preceq v$ with respect to $\mathcal{P}$, if $u - v \in \mathcal{P}$. A bounded linear operator $L : \mathcal{B} \to \mathcal{B}$ is said to be *positive* with respect to the cone $\mathcal{P}$ if $L : \mathcal{P} \to \mathcal{P}$. $L : \mathcal{B} \to \mathcal{B}$ is u_0-*positive with respect to* $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \preceq Lu \preceq k_2 u_0$ with respect to $\mathcal{P}$.

The following three theorems are fundamental to our results and are attributed to Krasnosel'skii [13]. The proof of the second theorem can be found in Krasnosel'skii's book [13], and the proof of the third theorem is provided by Keener and Travis [11] as an extension of Krasnosel'skii's results. In each of the following theorems, assume that $\mathcal{B}$ is a Banach space and $\mathcal{P}$ is a reproducing cone, and that $M, N : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive operators with respect to $\mathcal{P}$.

Theorem 2.5. *Let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P} \setminus \{0\} \to \mathcal{P}^{\circ}$, then M is u_0-positive.*

Theorem 2.6. *Let $M : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive linear operator. Then M has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.7. *Let M, N be bounded, linear operators on Banach space, and assume that at least one of the operators is u_0-positive with respect to $\mathcal{P}$. If $M \preceq N$ with respect to $\mathcal{P}$, and if there exist nonzero u_1, u_2 and positive real numbers λ_1 and λ_2 such that $\lambda_1 u_1 \preceq M u_1$ and $N u_2 \preceq \lambda_2 u_2$, then $\lambda_1 \le \lambda_2$. Moreover, if $\lambda_1 = \lambda_2$, then u_1 is a scalar multiple of u_2.*

3. Comparison of Smallest Eigenvalues

Atici and Eloe [3] derived the expression for the Green's function, $G(t,s)$, for $-\Delta^\nu y = 0$, $t \in \{1, 2, \ldots, b+1\}$, and satisfying (3), which, for us, will play the role of the kernel for appropriate compact linear operators. By direct computation, they obtained, for $1 < \nu \le 2$,

$$G(t,s) = \frac{(b+2)^{(2-\nu)}}{\Gamma(\nu)} \cdot \begin{cases} t^{(\nu-1)}(\nu+b-s-1)^{(\nu-2)} - (t-s-1)^{(\nu-1)}(\nu+b)^{(\nu-2)}, & (t,s) \in T_1, \\ t^{(\nu-1)}(\nu+b-s-1)^{(\nu-2)}, & (t,s) \in T_2. \end{cases}$$

where

$$T_1 := \{(t,s) \in \{\nu-1, \nu, \ldots, \nu+b+1\} \times \{0, 1, \ldots, b+1\} : \ 0 \le s < t-\nu+1 \le b+2\},$$

$$T_2 := \{(t,s) \in \{\nu-1, \nu, \ldots, \nu+b+1\} \times \{0, 1, \ldots, b+1\} : \ 0 \le t-\nu+1 \le s \le b+2\}.$$

Some properties of $G(t,s)$ that are useful for our main results are stated in the next theorem.

Theorem 3.1. *The following properties hold:*

(a) $G(t,s) > 0$ *for* $t \in \{\nu-1, \nu, \ldots, \nu+b+1\}$, $s \in \{0, 1, \ldots, b+1\}$.
(b) $G(\nu-2, s) = 0$ *and* $\Delta G(\nu+b, s) = 0$ *for* $s \in \{0, 1, \ldots, b+1\}$.
(c) $\Delta_t G(t,s) \le 0$ *for* $(t,s) \in T_1$ *and* $\Delta_t G(t,s) \ge 0$ *for* $(t,s) \in T_2$.
(d) $\max_{t \in \{\nu-1, \nu, \ldots, \nu+b+1\}} G(t,s) = G(s+\nu-1, s)$ *for* $s \in \{0, 1, \ldots, b+1\}$.

Proof. For the proof of (a), see Goodrich [[8], Proposition 4.2]. The statement (b) is easy to be verified. Now, we want to show (c). For $(t,s) \in T_2$, from Remark 2.4, it is easy to see $\Delta_t G(t,s) \ge 0$. For $(t,s) \in T_1$,

$$\Delta_t G(t,s) = \frac{(b+2)^{(2-\nu)}}{\Gamma(\nu-1)} [t^{(\nu-2)}(\nu+b-s-1)^{\nu-2} - t^{(\nu-2)}(\nu+b-s-1)^{\nu-2}]$$

$$\begin{aligned}
&= \frac{(b+2)^{(2-\nu)}\Gamma(t-s)\Gamma(\nu+b-s)}{\Gamma(\nu-1)\Gamma(t-\nu-3)\Gamma(b+3)}[(t-s)(t-s+1)\cdots t \\
&\quad \cdot (b-s+2)(b-s+3)\cdots(b+2) \\
&\quad -(t-\nu-s+2)(t-\nu-s+3)\cdots(t-\nu+2) \\
&\quad \cdot(\nu+b-s)(\nu+b-s+1)\cdots(\nu+b)].
\end{aligned}$$

Note that

$$\begin{aligned}
V(t,s) &= (t-s)(t-s+1)\cdots t\cdot(b-s+2)(b-s+3)\cdots(b+2) \\
&\quad -(t-\nu-s+2)(t-\nu-s+3)\cdots(t-\nu+2) \\
&\quad \cdot(\nu+b-s)(\nu+b-s+1)\cdots(\nu+b).
\end{aligned}$$

Since $0\le t-\nu+1\le s\le b+1$ and $t-\nu-b<0$, we have

$$V(t,0)=t(b+2)-(t-\nu+2)(\nu+b)]=(2-\nu)(t-\nu-b)\le 0.$$

$$\begin{aligned}
V(t,1) &= t(t-1)(b+2)(b+1)-(t-\nu+2)(t-\nu+1)(\nu+b-1)(\nu+b) \\
&= [t(b+2)\cdot(t-1)(b+1)-(t-\nu+2)(\nu+b)\cdot(t-1)(b+2)] \\
&\quad +[(t-\nu+2)(\nu+b)\cdot(t-1)(b+2)-(t-\nu+2)(\nu+b) \\
&\quad \cdot(t-\nu+1)(\nu+b-1)] \\
&= V(t,0)(t-1)(b+1)+(t-\nu+2)(\nu+b)(2-\nu)(t-b-\nu) \\
&< 0.
\end{aligned}$$

$$\begin{aligned}
V(t,2) &= t(t-1)(t-2)(b+2)(b+1)b \\
&\quad -(t-\nu+2)(t-\nu+1)(t-\nu)(\nu+b)(\nu+b-1)(\nu+b-2) \\
&= [t(t-1)(b+2)(b+1)\cdot(t-2)b \\
&\qquad -(t-\nu+2)(t-\nu+1)(\nu+b)(\nu+b-1)\cdot(t-2)b] \\
&\quad +[(t-\nu+2)(t-\nu+1)(\nu+b)(\nu+b-1)\cdot(t-2)b \\
&\qquad -(t-\nu+2)(t-\nu+1)(\nu+b)(\nu+b-1)\cdot(t-\nu)(\nu+b-2)] \\
&= V(t,1)(t-2)b \\
&\quad +(t-\nu+2)(t-\nu+1)(\nu+b)(\nu+b-1)(2-\nu)(t-b-\nu) \\
&< 0.
\end{aligned}$$

By induction, we can obtain $V(t,s)<0$ for $s\in\{0,1,2,\dots,t-\nu\}$. Then, we have $\Delta_t G(t,s)\le 0$ for $(t,s)\in T_1$. □

Next, we consider the Banach space $(\mathcal{B}, ||\cdot||)$ defined by

$$\mathcal{B} := \{y : \{\nu-2, \nu-1, \ldots, b+\nu+1\} \to \mathbb{R} \mid y(\nu-2) = 0,\ \Delta y(b+\nu) = 0\}$$

with the norm

$$||y|| := \max_{t\in\{\nu-2,\nu-1,\ldots,b+\nu+1\}} |y(t)|.$$

Also, we define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{y \in \mathcal{B} \mid y(t) \geq 0 \text{ on } \{\nu-2, \nu-1, \ldots, b+\nu+1\}\}.$$

The cone $\mathcal{P}$ is a reproducing cone since, if $y \in \mathcal{B}$, then

$$y_1(t) = \max\{0,\ y(t)\}, \quad y_2(t) = \max\{0,\ -y(t)\},$$

belong to $\mathcal{P}$ and $y = y_1 - y_2$.

Remark 3.2. The cone $\mathcal{P}$ has nonempty interior and

$$\Omega := \{y \in \mathcal{B} \mid y(t) > 0 \ \textit{on}\ \{\nu-1, \nu, \ldots, b+\nu\}\} \subset \mathcal{P}^\circ.$$

Hence, $\mathcal{P}$ is a solid cone.

Now, for the eigenvalue problems (1)-(3) and (2)-(3), we consider in an equivalent manner eigenvalues of the linear operators, $M, N :\ \mathcal{B} \to \mathcal{B}$ defined by

$$(4)\quad My(t) := \sum_{s=0}^{b+1} G(t,s)p(s+\nu-1)y(s+\nu-1), \quad t \in \{\nu-2, \nu-1, \ldots, b+\nu+1\},$$

$$(5)\quad Ny(t) := \sum_{s=0}^{b+1} G(t,s)q(s+\nu-1)y(s+\nu-1), \quad t \in \{\nu-2, \nu-1, \ldots, b+\nu+1\}.$$

Lemma 3.3. *The operators M and N are positive with respect to $\mathcal{P}$. In addition, $M, N : \mathcal{P}\setminus\{0\} \to \mathcal{P}^\circ$.*

Proof. We establish the statement for the operator M. First, the positivity of M with respect to $\mathcal{P}$ is an easy consequence of the sign condition on the function p and the properties of Green's function $G(t,s)$. Now, we shall show that $M : \mathcal{P}\setminus\{0\} \to \mathcal{P}^\circ$.

For $y \in \mathcal{P}\setminus\{0\}$, there exists $t_0 \in \{1, 2, \ldots, b+1\}$ such that $y(t_0+\nu-1) > 0$. So, for $t \in \{\nu-1, \nu, \ldots, b+\nu+1\}$, we have

$$\begin{aligned} My(t) &= \sum_{s=1}^{b+1} G(t,s)p(s+\nu-1)y(s+\nu-1) \\ &\geq G(t,t_0)p(t_0+\nu-1)y(t_0+\nu-1) \\ &> 0. \end{aligned}$$

Since $G(\nu-2,s)=0$ and $\Delta_t G(b+\nu,s)=0$, $My(\nu-2)=0$ and $\Delta My(b+\nu)=0$. Hence, $My \in \Omega \subset \mathcal{P}^\circ$. $\square$

Remark 3.4. According to Theorem *2.5*, M and N are u_0-positive with respect to $\mathcal{P}$.

Remark 3.5. (i) The fact that M and N are compact is clear since $\{\nu-2,\nu-1,\dots,b+\nu+1\}$ is a discrete set.
(ii) $\lambda_1 \neq 0$ for all eigenvalues of problem (1)-(3). So if $y(t)$ is an eigenvector corresponding to an eigenvalue λ_1 of (1)-(3), then

$$y(t)=\sum_{s=1}^{b+1} G(t,s)\lambda_1 p(s+\nu-1)y(s+\nu-1), \quad t\in\{\nu-2,\nu-1,\dots,b+\nu+1\};$$

that is, $\frac{1}{\lambda_1}y = My$. So, eigenvalues of (1)-(3) are reciprocals of the eigenvalues of (4), and conversely.

With this in mind, the following results straightforward.

Theorem 3.6. *M has an essentially unique eigenvector $y \in \mathcal{P}\setminus\{0\}$, and the corresponding eigenvalue Λ is positive, simple, and larger than the absolute value of any other eigenvalue.*

Proof. Theorem 2.6 establishes such an eigenvalue Λ and corresponding eigenvector y in $\mathcal{P}$. By the proof of Lemma 3.3, $My \in \mathcal{P}^\circ$. In fact, $My(t)>0$, for any $t \in \{\nu-1,\dots,b+\nu+1\}$. Since, $\Lambda y = My$, $y=\frac{1}{\Lambda}My$. So, $y(t)>0$ for $t\in\{\nu-1,\dots,b+\nu+1\}$. Hence, $y(t)\in\mathcal{P}^\circ$. $\square$

Theorem 3.7. *Let $p(t)\le q(t)$ on $\{\nu,\nu+1,\dots,b+\nu\}$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem* 3.6 *associated with M and N, respectively, with the essentially unique eigenvectors y_1 and y_2 in $\mathcal{P}^\circ$. Then $\Lambda_1\le\Lambda_2$. Furthermore, $\Lambda_1=\Lambda_2$ if and only if $p(t)=q(t)$ on $\{\nu,\nu+1,\dots,b+\nu\}$.*

Proof. Let $p(t)\le q(t)$ on $\{\nu,\nu+1,\dots,b+\nu\}$. Then for any $y\in\mathcal{P}$ and $t\in\{\nu,\nu+1,\dots,b+\nu\}$,

$$(Ny-My)(t)=\sum_{s=1}^{b+1} G(t,s)(q(s+\nu-1)-p(s+\nu-1))y(s+\nu-1)\ge 0.$$

So $Ny-My\in\mathcal{P}$ for all $y\in\mathcal{P}$; that is, $M\preceq N$ with respect to $\mathcal{P}$. Then by Theorem 2.7, $\Lambda_1\le\Lambda_2$.

For the final statement of the theorem, suppose $p(t) \neq q(t)$. So, for some $t_0 \in \{\nu, \nu+1, \ldots, b+\nu\}$, $p(t_0) < q(t_0)$. Then, from Theorem 3.6, there exists y_1 such that $(N-M)y_1 \in \mathcal{P}^\circ$, and so there exists $\varepsilon > 0$ such that $(N-M)y_1 - \varepsilon y_1 \in \mathcal{P}$. Hence, $\Lambda_1 y_1 + \varepsilon y_1 = My_1 + \varepsilon y_1 \leq Ny_1$, which implies $(\Lambda_1 + \varepsilon)y_1 \leq Ny_1$. Since $N \preceq N$ and $Ny_2 = \Lambda_2 y_2$, by Theorem 2.7, $\Lambda_1 + \varepsilon \leq \Lambda_2$, and $\Lambda_1 < \Lambda_2$. By contrapositive, if $\Lambda_1 = \Lambda_2$, then $p(t) = q(t)$ for $t \in \{\nu, \nu+1, \ldots, b+\nu\}$. □

Recall from Remark 3.5 that eigenvalues of M are reciprocals of eigenvalues of the BVP (1)-(3), and that eigenvalues of N are reciprocals of eigenvalues of the BVP (2)-(3) in Introduction. The following theorem is an immediate consequence of Theorems 3.6 and 3.7.

Theorem 3.8. *Assume the hypotheses of Theorem* 3.7. *Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (1)-(3) *and* (2)-(3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenvectors corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in \{\nu, \nu+1, \ldots, b+\nu\}$.*

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