J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2018.25.2.73 ISSN(Online) 2287-6081
Volume 25, Number 2 (May 2018), Pages 73–94

EMPLOYING α-ψ-CONTRACTION TO PROVE COUPLED COINCIDENCE POINT THEOREM FOR GENERALIZED COMPATIBLE PAIR OF MAPPINGS ON PARTIALLY ORDERED METRIC SPACES

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Abstract. We introduce some new type of admissible mappings and prove a coupled coincidence point theorem by using newly defined concepts for generalized compatible pair of mappings satisfying $\alpha - \psi$ contraction on partially ordered metric spaces. We also prove the uniqueness of a coupled fixed point for such mappings in this setup. Furthermore, we give an example and an application to integral equations to demonstrate the applicability of the obtained results. Our results generalize some recent results in the literature.

1. Introduction and Preliminaries

For simplicity, if $x \in X$, we denote $T(x)$ by Tx. In [9], Guo and Lakshmikantham introduced the following notion of coupled fixed point for single-valued mappings:

Definition 1. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of F if

$$F(x,\ y) = x \text{ and } F(y,\ x) = y.$$

In [3], Bhaskar and Lakshmikantham gave the notion of mixed monotone property and proved some coupled fixed point theorems for a mapping having mixed monotone property in partially ordered metric spaces.

Bhaskar and Lakshmikantham [3] introduced the following:

Definition 2. Let $(X, \preceq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

Received by the editors November 26, 2016. Accepted March 09, 2018.
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. coupled coincidence point, $\alpha - \psi$ contraction, generalized compatibility, increasing mapping, mixed monotone mapping, commuting mapping.
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$$(u,\ v) \preceq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u,\ v),\ (x,\ y) \in X \times X.$$

Definition 3. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

Lakshmikantham and Ciric [13] extended the notion of mixed monotone property to mixed $g-$monotone property and generalized the results of Bhaskar and Lakshmikantham [3].

In [13], Lakshmikantham and Ciric introduced the following:

Definition 4. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings F and g if

$$F(x,\ y) = gx \text{ and } F(y,\ x) = gy.$$

Definition 5. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. An element $(x, y) \in X \times X$ is called a *common coupled fixed point* of the mappings F and g if

$$x = F(x,\ y) = gx \text{ and } y = F(y,\ x) = gy.$$

Definition 6. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *commutative* if

$$gF(x,\ y) = F(gx,\ gy), \text{ for all } (x,\ y) \in X \times X.$$

Definition 7. Let $(X, \leq)$ be a partially ordered set. Suppose $F : X \times X \to X$ and $g : X \to X$ are given mappings. Then F has the *mixed $g-$monotone property* if for all $x, y \in X$, we have

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y)$$

and

$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

In [5], Choudhury and Kundu gave the notion of compatibility in the context of coupled coincidence point and used this notion to improve the results of Lakshmikantham and Ciric [13].

Definition 8 ([5])**.** The mappings $F : X \times\ X \to X$ and $g : X \to X$ are said to be *compatible* if

$$\begin{aligned}\lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0,\\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0,\end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y,\ \text{for some } x,\ y \in X.\end{aligned}$$

In [11], Hussain et al. introduced a new concept of generalized compatibility of a pair of mappings defined on a product space and proved some coupled coincidence point results and also proved some coupled fixed point results without mixed monotone property.

In [11], Hussain et al. introduced the following:

Definition 9. Suppose that $F,\, G : X \times X \to X$ are two mappings. F is said to be $G-$*increasing* with respect to $\preceq$ if for all $x,\, y,\, u,\, v \in X$, with $G(x,\, y) \leq G(u,\, v)$ we have $F(x,\, y) \leq F(u,\, v)$.

Example 10. Let $X = (0,\, +\infty)$ be endowed with the natural ordering of real numbers $\leq$. Define mappings $F,\, G : X \times X \to X$ by $F(x,\, y) = \ln(x + y)$ and $G(x,\, y) = x + y$ for all $(x,\, y) \in X \times X$. Note that F is $G-$increasing with respect to $\leq$.

Example 11. Let $X = \mathbb{N}$ endowed with the partial order defined by $x,\, y \in X \times X$, $x \preceq y$ if and only if y divides x. Define the mappings $F,\, G : X \times X \to X$ by $F(x,\, y) = x^2y^2$ and $G(x,\, y) = xy$ for all $(x,\, y) \in X \times X$. Then F is $G-$increasing with respect to $\preceq$.

Definition 12. An element $(x,\, y) \in X \times X$ is called a *coupled coincidence point* of mappings $F,\, G : X \times X \to X$ if $F(x,\, y) = G(x,\, y)$ and $F(y,\, x) = G(y,\, x)$.

Example 13. Let $F,\, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x,\, y) = xy$ and $G(x,\, y) = \frac{2}{3}(x + y)$ for all $(x,\, y) \in X \times X$. Note that $(0,\, 0)$, $(1,\, 2)$ and $(2,\, 1)$ are coupled coincidence points of F and G.

Definition 14. Let $F,\, G : X \times X \to X$ be two maps. We say that the pair $\{F,\, G\}$ is *commuting* if

$$F(G(x,\ y),\ G(y,\ x)) = G(F(x,\ y),\ F(y,\ x)),\ \text{for all } x,\ y \in X.$$

Definition 15. Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that F is $g-$*increasing* with respect to $\preceq$ if for any $x, y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x, y_1) \preceq F(x, y_2).$$

Definition 16. Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$. We say that F is *increasing* with respect to $\preceq$ if for any $x, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_1) \preceq F(x, y_2).$$

Definition 17. Let $F, G : X \times X \to X$. We say that the pair $\{F, G\}$ is *generalized compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n, y_n), G(y_n, x_n)), G(F(x_n, y_n), F(y_n, x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n, x_n), G(x_n, y_n)), G(F(y_n, x_n), F(x_n, y_n))) &= 0, \end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} G(x_n, y_n) &= \lim_{n\to\infty} F(x_n, y_n) = x, \\ \lim_{n\to\infty} G(y_n, x_n) &= \lim_{n\to\infty} F(y_n, x_n) = y, \text{ for all } x, y. \end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Recently, Samet et al. [19] defined the following definition of $\alpha-$admissible mapping and proved coupled fixed point theorems in complete metric spaces.

Definition 18. Let $F : X \times X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be two mappings. Then F is said to be $(\alpha)-$*admissible* if

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1,$$

for all $x, y, u, v \in X$.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ satisfying

(*i*) $\psi^{-1}\{0\} = 0$,

(*ii*) $\psi(t) < t$, for all $t > 0$,

(ii_φ) $\lim_{r\to t+}\psi(r) < t$ for all $t > 0$.

Lemma 19 ([15]). *If $\psi : [0, +\infty) \to [0, +\infty)$ is non-decreasing and right continuous, then $\psi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Definition 20. An ordered metric space $(X, d, \preceq)$ is a metric space (X, d) provided with a partial order $\preceq$.

In [15], Mursaleen et al. established the following result:

Theorem 21. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property of X satisfying*

(i) for all $x, y, u, v \in X$, where $x \succeq u$, $y \preceq v$, there exist $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi\left(\frac{d(x, u) + d(y, v)}{2}\right),$$

(ii) F is $(\alpha)-$admissible,

(iii) there exist $x_0, y_0 \in X$ such that

$$\begin{aligned} \alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) &\geq 1, \\ \alpha((y_0, x_0), (F(y_0, x_0), F(x_0, y_0))) &\geq 1, \end{aligned}$$

(iv) suppose that either

(a) F is continuous or

(b) if (x_n) and (y_n) are sequences in X such that

$$\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } \alpha((y_n, x_n), (y_{n+1}, x_{n+1})) \geq 1,$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} x_n = x \in X \text{ and } \lim_{n\to\infty} y_n = y \in X,$$

then

$$\alpha((x_n, y_n), (x, y)) \geq 1 \text{ and } \alpha((y_n, x_n), (y, x)) \geq 1,$$

(v) there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

then F has a coupled fixed point.

Very recently Samet et al. [17] claimed that most of the coupled fixed point theorems for single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems. There exists several coupled fixed point results for single valued mappings including [2, 4, 6, 7, 8, 9, 10, 11, 12, 15, 16, 18, 20].

In this paper, we define some new type of admissible mappings and prove a coupled coincidence point results by using newly defined concepts for generalized compatible pair of mappings satisfying $\alpha - \psi$ contraction on partially ordered metric space. We also prove the uniqueness of a coupled fixed point for such mappings. Furthermore, we give an example and an application to integral equations to demonstrate the applicability of the obtained results. We generalize the results of Bhaskar and Lakshmikantham [3], Lakshmikantham and Ciric [13], Mursaleen et al. [15] and many others.

2. Main Results

First, we introduce the following:

Definition 22. Let F, $G : X \times X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be mappings. Then F is said to be $(\alpha_G)-$*admissible* if

$$\alpha \begin{pmatrix} (G(x,\ y),\ G(y,\ x)), \\ (G(u,\ v),\ G(v,\ u)) \end{pmatrix} \geq 1 \Rightarrow \alpha \begin{pmatrix} (F(x,\ y),\ F(y,\ x)), \\ (F(u,\ v),\ F(v,\ u)) \end{pmatrix} \geq 1,$$

for all x, y, u, $v \in X$.

Definition 23. Let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ be mappings. Then F is said to be $(\alpha_g)-$*admissible* if

$$\alpha\left((gx,\ gy),\ (gu,\ gv)\right) \geq 1 \Rightarrow \alpha \begin{pmatrix} (F(x,\ y),\ F(y,\ x)), \\ (F(u,\ v),\ F(v,\ u)) \end{pmatrix} \geq 1,$$

for all x, y, u, $v \in X$.

Theorem 24. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Assume F, $G : X \times X \to X$ be two generalized compatible mappings such that F is $G-$increasing with respect to $\preceq$, G is continuous and has the mixed monotone property satisfying*
(1) *there exist two elements x_0, $y_0 \in X$ such that*

$$\begin{aligned} \alpha((G(x_0,\ y_0),\ G(y_0,\ x_0)),\ (F(x_0,\ y_0),\ F(y_0,\ x_0))) &\geq 1, \\ \alpha((G(y_0,\ x_0),\ G(x_0,\ y_0)),\ (F(y_0,\ x_0),\ F(x_0,\ y_0))) &\geq 1, \end{aligned}$$

(2) *for all x, y, u, $v \in X$, where $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, there*

exist $\psi \in \Psi$ *and* $\alpha : X^2 \times X^2 \to (0, +\infty)$ *such that*

$$\begin{aligned}\alpha\begin{pmatrix} (G(x,\ y),\ G(y,\ x)), \\ (G(u,\ v),\ G(v,\ u)) \end{pmatrix} d(F(x,\ y),\ F(u,\ v)) \\ \leq \ \psi\left(\frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2}\right),\end{aligned}$$

(3) *for any* $x, y \in X$, *there exist* $u, v \in X$ *such that*

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u),$$

(4) *suppose that either*

(*a*) F *is continuous or*

(*b*) *if* (x_n) *and* (y_n) *are sequences in* X *such that*

$$\begin{aligned}\alpha((G(x_n,\ y_n),\ G(y_n,\ x_n)),\ (G(x_{n+1},\ y_{n+1}),\ G(y_{n+1},\ x_{n+1}))) &\geq 1, \\ \alpha((G(y_n,\ x_n),\ G(x_n,\ y_n)),\ (G(y_{n+1},\ x_{n+1}),\ G(x_{n+1},\ y_{n+1}))) &\geq 1,\end{aligned}$$

for all $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} G(x_n, y_n) = x \in X$ *and* $\lim_{n\to\infty} G(y_n, x_n) = y \in X$, *then*

$$\begin{aligned}\alpha((G(x_n,\ y_n),\ G(y_n,\ x_n)),\ (G(x,\ y),\ G(y,\ x))) &\geq 1, \\ \alpha((G(y_n,\ x_n),\ G(x_n,\ y_n)),\ (G(y,\ x),\ G(x,\ y))) &\geq 1,\end{aligned}$$

(5) *there exist two elements* $x_0, y_0 \in X$ *such that*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Then F *and* G *have a coupled coincidence point.*

Proof. By hypothesis, there exist $x_0, y_0 \in X$ such that

$$\begin{aligned}\alpha((G(x_0,\ y_0),\ G(y_0,\ x_0)),\ (F(x_0,\ y_0),\ F(y_0,\ x_0))) &\geq 1, \\ \alpha((G(y_0,\ x_0),\ G(x_0,\ y_0)),\ (F(y_0,\ x_0),\ F(x_0,\ y_0))) &\geq 1.\end{aligned}$$

By using (3), select $x_1, y_1 \in X$ such that

$$G(x_1,\ y_1) = F(x_0,\ y_0) \text{ and } G(y_1,\ x_1) = F(y_0,\ x_0).$$

Continuing in this manner, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(6) $\quad G(x_{n+1},\ y_{n+1}) = F(x_n,\ y_n)$ and $G(y_{n+1},\ x_{n+1}) = F(y_n,\ x_n),$ for all $n \geq 0$.

We shall show that

(7) $\quad G(x_n,\ y_n) \preceq G(x_{n+1},\ y_{n+1})$ and $G(y_n,\ x_n) \succeq G(y_{n+1},\ x_{n+1}),$ for all $n \geq 0$.

We shall use the mathematical induction. Let $n = 0$, since

$$\begin{aligned} G(x_0,\ y_0) &\preceq F(x_0,\ y_0) = G(x_1,\ y_1), \\ G(y_0,\ x_0) &\succeq F(y_0,\ x_0) = G(y_1,\ x_1), \end{aligned}$$

we have

$$G(x_0,\ y_0) \preceq G(x_1,\ y_1) \text{ and } G(y_0,\ x_0) \preceq G(y_1,\ x_1).$$

Thus (7) hold for $n = 0$. Suppose now that (7) hold for some fixed $n \in \mathbb{N}$. Then

$$G(x_n,\ y_n) \preceq G(x_{n+1},\ y_{n+1}) \text{ and } G(y_n,\ x_n) \preceq G(y_{n+1},\ x_{n+1}).$$

Since F is $G-$increasing with respect to $\preceq$, by using (6), we have

$$\begin{aligned} G(x_{n+1},\ y_{n+1}) &= F(x_n,\ y_n) \preceq F(x_{n+1},\ y_{n+1}) = G(x_{n+2},\ y_{n+2}), \\ G(y_{n+1},\ x_{n+1}) &= F(y_n,\ x_n) \succeq F(y_{n+1},\ x_{n+1}) = G(y_{n+2},\ x_{n+2}). \end{aligned}$$

Thus by the mathematical induction we conclude that (7) hold for all $n \geq 0$. Therefore

$$G(x_0,\ y_0) \preceq G(x_1,\ y_1) \preceq ... \preceq G(x_n,\ y_n) \preceq G(x_{n+1},\ y_{n+1}) \preceq ...$$

and

$$G(y_0,\ x_0) \succeq G(y_1,\ x_1) \succeq ... \succeq G(y_n,\ x_n) \succeq G(y_{n+1},\ x_{n+1}) \succeq ...$$

Since F is $(\alpha_G)-$admissible, we have

$$\begin{aligned} &\alpha\left((G(x_0,\ y_0),\ G(y_0,\ x_0)),\ (G(x_1,\ y_1),\ G(y_1,\ x_1))\right) \\ = &\ \alpha\left((G(x_0,\ y_0),\ G(y_0,\ x_0)),\ (F(x_0,\ y_0),\ F(y_0,\ x_0))\right) \geq 1, \end{aligned}$$

which implies that

$$\begin{aligned} &\alpha\left((F(x_0,\ y_0),\ F(y_0,\ x_0)),\ (F(x_1,\ y_1),\ F(y_1,\ x_1))\right) \\ = &\ \alpha\left((G(x_1,\ y_1),\ G(y_1,\ x_1)),\ (G(x_2,\ y_2),\ G(y_2,\ x_2))\right) \geq 1. \end{aligned}$$

Thus, by the mathematical induction, for all $n \in \mathbb{N}$, we have

(8) $$\alpha\left((G(x_n,\ y_n),\ G(y_n,\ x_n)),\ (G(x_{n+1},\ y_{n+1}),\ G(y_{n+1},\ x_{n+1}))\right) \geq 1.$$

Similarly

(9) $$\alpha\left(G(y_n,\ x_n),\ (G(x_n,\ y_n)),\ (G(y_{n+1},\ x_{n+1}),\ G(x_{n+1},\ y_{n+1}))\right) \geq 1.$$

Now by (2) and (8), we have

$$\begin{aligned}
& d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})) \\
= \;& d(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)) \\
\leq \;& \alpha\left((G(x_{n-1},\ y_{n-1}),\ G(y_{n-1},\ x_{n-1})),\ (G(x_n,\ y_n),\ G(y_n,\ x_n))\right) \\
& \times d(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)) \\
\leq \;& \psi\left(\frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)) + d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n))}{2}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1})) \\
\leq \;& \psi\left(\frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)) + d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n))}{2}\right).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
& \frac{d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})) + d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1}))}{2} \\
\leq \;& \psi\left(\frac{d(G(x_{n-1},\ y_{n-1}),\ G(x_n,\ y_n)) + d(G(y_{n-1},\ x_{n-1}),\ G(y_n,\ x_n))}{2}\right).
\end{aligned}$$

Repeating the above process, we get

$$\begin{aligned}
& \frac{d(G(x_n,\ y_n),\ G(x_{n+1},\ y_{n+1})) + d(G(y_n,\ x_n),\ G(y_{n+1},\ x_{n+1}))}{2} \\
\leq \;& \psi^n\left(\frac{d(G(x_0,\ y_0),\ G(x_1,\ y_1)) + d(G(y_0,\ x_0),\ G(y_1,\ x_1))}{2}\right),
\end{aligned}$$

for all $n \in \mathbb{N}$. Without any loss of generality, we can assume that $\frac{1}{2}[d(G(x_0,\ y_0),\ G(x_1,\ y_1)) + d(G(y_0,\ x_0),\ G(y_1,\ x_1))] \neq 0$. In fact, if this is not true, then $G(x_0,\ y_0) = G(x_1,\ y_1) = F(x_0,\ y_0)$, $G(y_0,\ x_0) = G(y_1,\ x_1) = F(y_0,\ x_0)$, that is, $(x_0,\ y_0)$ is a coupled coincidence point of F and G. For $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} \psi^n\left(\frac{d(G(x_0,\ y_0),\ G(x_1,\ y_1)) + d(G(y_0,\ x_0),\ G(y_1,\ x_1))}{2}\right) < \frac{\varepsilon}{2}. \tag{10}$$

Let $n,\ m \in \mathbb{N}$ be such that $m > n > n(\varepsilon)$. Then, by using the triangle inequality and (10), we have

$$\frac{d(G(x_n,\ y_n),\ G(x_m,\ y_m)) + d(G(y_n,\ x_n),\ G(y_m,\ x_m))}{2}$$

$$\leq \sum_{k=n}^{m-1} \frac{d(G(x_k,\ y_k),\ G(x_{k+1},\ y_{k+1})) + d(G(y_k,\ x_k),\ G(y_{k+1},\ x_{k+1}))}{2}$$
$$\leq \sum_{k=n}^{m-1} \psi^k \left(\frac{d(G(x_0,\ y_0),\ G(x_1,\ y_1)) + d(G(y_0,\ x_0),\ G(y_1,\ x_1))}{2} \right)$$
$$\leq \sum_{n \geq n(\varepsilon)} \psi^n \left(\frac{d(G(x_0,\ y_0),\ G(x_1,\ y_1)) + d(G(y_0,\ x_0),\ G(y_1,\ x_1))}{2} \right) < \frac{\varepsilon}{2}.$$

This implies that

$$d(G(x_n,\ y_n),\ G(x_m,\ y_m)) + d(G(y_n,\ x_n),\ G(y_m,\ x_m)) < \varepsilon. \tag{11}$$

Thus, by (11), we get

$$d(G(x_n,\ y_n),\ G(x_m,\ y_m)) \leq \begin{bmatrix} d(G(x_n,\ y_n),\ G(x_m,\ y_m)) \\ +d(G(y_n,\ x_n),\ G(y_m,\ x_m)) \end{bmatrix} < \varepsilon,$$
$$d(G(y_n,\ x_n),\ G(y_m,\ x_m)) \leq \begin{bmatrix} d(G(x_n,\ y_n),\ G(x_m,\ y_m)) \\ +d(G(y_n,\ x_n),\ G(y_m,\ x_m)) \end{bmatrix} < \varepsilon,$$

it follows that $\{G(x_n, y_n)\}$ and $\{G(y_n, x_n)\}$ are Cauchy sequences in X. Since X is complete, therefore there is some $x, y \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x, \\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y, \end{aligned} \tag{12}$$

Since the pair $\{F, G\}$ satisfies the generalized compatibility, from (12), we get

$$\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) = 0, \tag{13}$$

and

$$\lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) = 0. \tag{14}$$

Now suppose that assumption (a) holds. Then

$$\begin{aligned} & d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\ \leq\ & d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) \\ & +d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(x,\ y)). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (12), (13) and the fact that F and G are continuous, we have

$$F(x,\ y) = G(x,\ y).$$

Similarly we can show that

$$F(y,\ x) = G(y,\ x).$$

Thus $(x,\ y)$ is a coupled coincidence point of F and G.

Now, suppose that (b) holds. Then, by (8) and (12), we have

$$\alpha((G(x_n,\ y_n),\ G(y_n,\ x_n)),\ (G(x,\ y),\ G(y,\ x))) \geq 1. \tag{15}$$

Similarly, we have

$$\alpha((G(y_n,\ x_n),\ G(x_n,\ y_n)),\ (G(y,\ x),\ G(x,\ y))) \geq 1. \tag{16}$$

Now, using (2), we get

$$\begin{aligned} & d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) \\ \leq\ & \alpha\left(\left(\begin{array}{c} G\left(G(x_n,\ y_n),\ G(y_n,\ x_n)\right), \\ G\left(G(y_n,\ x_n),\ G(x_n,\ y_n)\right) \end{array}\right),\ (G(x,\ y),\ G(y,\ x))\right) \\ & \times d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) \\ \leq\ & \psi\left(\frac{1}{2}\left[\begin{array}{c} d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\ +d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(y,\ x)) \end{array}\right]\right) \end{aligned}$$

which implies by the fact that $\psi(t) < t$ for all $t > 0$,

$$\begin{aligned} & d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) \\ \leq\ & \frac{1}{2}\left[\begin{array}{c} d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\ +d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(y,\ x)) \end{array}\right]. \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, by using (12), (13) and by the continuity of G, we have

$$\begin{aligned} & \lim_{n\to\infty} d\left(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(x,\ y)\right) \\ =\ & \lim_{n\to\infty} d\left(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(x,\ y)\right) \\ \leq\ & \frac{1}{2}\lim_{n\to\infty}\left[\begin{array}{c} d(G(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(x,\ y)) \\ +d(G(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(y,\ x)) \end{array}\right], \end{aligned}$$

which implies that

$$F(x,\ y) = G(x,\ y).$$

Similarly we can show that

$$F(y,\ x) = G(y,\ x).$$

Thus $(x,\ y)$ is a coupled coincidence point of F and G. □

Corollary 25. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space. Assume $F,\ G : X \times X \to X$ be two commuting mappings such that F is $G-$increasing with respect to $\preceq$, G is continuous and has the mixed monotone property satisfying (1) – (5). Then F and G have a coupled coincidence point.*

Now, we deduce following results which are analogous to Theorem 21:

Corollary 26. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space. Assume $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F is $g-$increasing with respect to $\preceq$ and $\{F,\ g\}$ is compatible. Suppose that*

(17) *there exist two elements $x_0,\ y_0 \in X$ such that*

$$\begin{aligned}\alpha((gx_0,\ gy_0),\ (F(x_0,\ y_0),\ F(y_0,\ x_0))) &\geq 1,\\ \alpha((gy_0,\ gx_0),\ (F(y_0,\ x_0),\ F(x_0,\ y_0))) &\geq 1,\end{aligned}$$

(18) *for all $x,\ y,\ u,\ v \in X$, where $gx \preceq gu$ and $gy \succeq gv$, there exist $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \to (0,\ +\infty)$ such that*

$$\alpha((gx,\ gy),\ (gu,\ gv))d(F(x,\ y),\ F(u,\ v)) \leq \psi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right),$$

(19) *$F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing with respect to $\preceq$,*

(20) *suppose that either*

(*a*) *F is continuous or*

(*b*) *if (x_n) and (y_n) are sequences in X such that*

$$\begin{aligned}\alpha((gx_n,\ gy_n),\ (gx_{n+1},\ gy_{n+1})) &\geq 1,\\ \alpha((gy_n,\ gx_n),\ (gy_{n+1},\ gx_{n+1})) &\geq 1,\end{aligned}$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} gx_n = x \in X \text{ and } \lim_{n\to\infty} gy_n = y \in X,$$

then

$$\begin{aligned}\alpha((gx_n,\ gy_n),\ (gx,\ gy)) &\geq 1,\\ \alpha((gy_n,\ gx_n),\ (gy,\ gx)) &\geq 1,\end{aligned}$$

(21) *there exist two elements $x_0,\ y_0 \in X$ such that*

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point.

Corollary 27. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space. Assume $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F is $g-$increasing with respect to $\preceq$ and $\{F,\ g\}$ is commuting satisfying (17) – (21), then F and g have a coupled coincidence point.*

Corollary 28. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space. Assume $F : X \times X \to X$ be an increasing mapping with respect to $\preceq$ satisfying*

(22) *there exist two elements* $x_0,\ y_0 \in X$ *such that*

$$\begin{aligned}\alpha((x_0,\ y_0),\ (F(x_0,\ y_0),\ F(y_0,\ x_0))) &\geq 1,\\ \alpha((y_0,\ x_0),\ (F(y_0,\ x_0),\ F(x_0,\ y_0))) &\geq 1,\end{aligned}$$

(23) *for all* $x,\ y,\ u,\ v \in X$, *where* $x \preceq u$ *and* $y \succeq v$, *there exist* $\psi \in \Psi$ *and* $\alpha : X^2 \times X^2 \to (0,\ +\infty)$ *such that*

$$\alpha((x,\ y),\ (u,\ v))d(F(x,\ y),\ F(u,\ v)) \leq \psi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right),$$

(24) *suppose that either*

(*a*) F *is continuous or*

(*b*) *if* (x_n) *and* (y_n) *are sequences in* X *such that*

$$\begin{aligned}\alpha((x_n,\ y_n),\ (x_{n+1},\ y_{n+1})) &\geq 1,\\ \alpha((y_n,\ x_n),\ (y_{n+1},\ x_{n+1})) &\geq 1,\end{aligned}$$

for all $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} x_n = x \in X \text{ and } \lim_{n\to\infty} y_n = y \in X,$$

then

$$\begin{aligned}\alpha((x_n,\ y_n),\ (x,\ y)) &\geq 1,\\ \alpha((y_n,\ x_n),\ (y,\ x)) &\geq 1,\end{aligned}$$

(25) *there exist two elements* $x_0,\ y_0 \in X$ *such that*

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F *has a coupled fixed point.*

Example 29. Suppose $X = [0, 1]$ provided with its usual order $\leq$ and the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $F, G : X \times X \to X$ be defined as

$$F(x,\ y) = \begin{cases} \frac{x^2-y^2}{4}, \text{ if } x \geq y,\\ 0, \text{ if } x < y,\end{cases}$$

and

$$G(x,\ y) = \begin{cases} x^2 - y^2, \text{ if } x \geq y,\\ 0, \text{ if } x < y.\end{cases}$$

Define $\alpha : X^2 \times X^2 \to (0, +\infty)$ as follows

$$\alpha((x,\ y),\ (u,\ v)) = \begin{cases} 1, \text{ if } x \geq y,\ u \geq v,\\ 0, \text{ otherwise,}\end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ as follows

$$\psi(t) = \frac{t}{2} \text{ for all } t > 0.$$

First, we shall show that F and G satisfy the contractive condition (2). Let $x, y \in X$ such that $G(x, y) \leq G(u, v)$ and $G(y, x) \geq G(v, u)$, we have

$$\begin{aligned}
& \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u,\ v),\ G(v,\ u))\right) d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \left| \frac{x^2 - y^2}{4} - \frac{u^2 - v^2}{4} \right| \\
\leq\ & \frac{1}{4} \left| G(x,\ y) - G(u,\ v) \right| \\
\leq\ & \frac{1}{4} d(G(x,\ y),\ G(u,\ v)) \\
\leq\ & \frac{1}{2} \left(\frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2} \right) \\
\leq\ & \psi \left(\frac{d(G(x,\ y),\ G(u,\ v)) + d(G(y,\ x),\ G(v,\ u))}{2} \right).
\end{aligned}$$

Thus the contractive condition (2) is satisfied for all $x, y, u, v \in X$. The other conditions of Theorem 24 are satisfied like in [10] and $z = (0, 0)$ is a coincidence point of F and G.

Now we prove the uniqueness of the coupled coincidence point. If $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow G(x,\ y) \preceq G(u,\ v) \text{ and } G(y,\ x) \succeq G(v,\ u),$$

for all $(x, y), (u, v) \in X \times X$, where $G : X \times X \to X$ is one-one.

Theorem 30. *In addition to the hypotheses of Theorem 24, suppose that for every (x, y), (x^*, y^*) in $X \times X$, there exists another (u, v) in $X \times X$ which is comparable to (x, y) and (x^*, y^*), then F and G have a unique coupled coincidence point.*

Proof. Theorem 24 shows that the set of coupled coincidence points of F and G is non-empty. Let $(x, y), (x^*, y^*) \in X \times X$ are two coupled coincidence points of F and G, that is,

$$\begin{aligned}
F(x,\ y) &= G(x,\ y) \text{ and } F(y,\ x) = G(y,\ x), \\
F(x^*,\ y^*) &= G(x^*,\ y^*) \text{ and } F(y^*,\ x^*) = G(y^*,\ x^*).
\end{aligned}$$

We shall prove that $G(x, y) = G(x^*, y^*)$ and $G(y, x) = G(y^*, x^*)$. By assumption, there exists $(u, v) \in X \times X$ which is comparable to (x, y) and (x^*, y^*). We define the sequences $\{G(u_n, v_n)\}$ and $\{G(v_n, u_n)\}$ as follows, with $u_0 = u$, $v_0 = v$:

$$G(u_{n+1},\ v_{n+1}) = F(u_n,\ v_n),\ G(v_{n+1},\ u_{n+1}) = F(v_n,\ u_n),\ n \geq 0.$$

Since (u, v) is comparable to (x, y), we may assume that $(x, y) \preceq (u, v) = (u_0, v_0)$, which implies that $G(x, y) \preceq G(u_0, v_0)$ and $G(y, x) \succeq G(v_0, u_0)$. Suppose $(x, y) \preceq (u_n, v_n)$ for some n. Since F is $G-$increasing, we have $G(x, y) \preceq G(u_n, v_n)$ implies $F(x, y) \preceq F(u_n, v_n)$ and $G(y, x) \succeq G(v_n, u_n)$ implies $F(y, x) \succeq F(v_n, u_n)$. We now prove that $(x, y) \preceq (u_{n+1}, v_{n+1})$. Therefore $G(x, y) = F(x, y) \preceq F(u_n, v_n) = G(u_{n+1}, v_{n+1})$ and $G(y, x) = F(y, x) \succeq F(v_n, u_n) = G(v_{n+1}, u_{n+1})$. Thus, we have $(x, y) \preceq (u_{n+1}, v_{n+1})$, for all n. Since for every (x, y), (x^*, y^*) in $X \times X$, there exists (u, v) in $X \times X$ such that

$$\begin{aligned} \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u,\ v),\ G(v,\ u))\right) &\geq 1, \\ \alpha\left((G(x^*,\ y^*),\ G(y^*,\ x^*)),\ (G(u,\ v),\ G(v,\ u))\right) &\geq 1. \end{aligned}$$

Since F is $(\alpha_G)-$admissible, we have

$$\begin{aligned} \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u,\ v),\ G(v,\ u))\right) &\geq 1 \\ \text{implies that } \alpha\left((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u))\right) &\geq 1, \end{aligned}$$

Put $u = u_0$ and $v = v_0$, we get

$$\begin{aligned} \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u,\ v),\ G(v,\ u))\right) &\geq 1 \\ \text{implies that } \alpha\left((F(x,\ y),\ F(y,\ x)),\ (F(u_0,\ v_0),\ F(v_0,\ u_0))\right) &\geq 1. \end{aligned}$$

Thus

$$\begin{aligned} \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u,\ v),\ G(v,\ u))\right) &\geq 1 \\ \text{implies that } \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u_1,\ v_1),\ G(v_1,\ u_1))\right) &\geq 1. \end{aligned}$$

Thus, by the mathematical induction, we obtain

$$\alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u_n,\ v_n),\ G(v_n,\ u_n))\right) \geq 1,\ \text{for all } n \in \mathbb{N}.$$

Similarly

$$\alpha\left((G(y,\ x),\ G(x,\ y)),\ (G(v_n,\ u_n),\ G(u_n,\ v_n))\right) \geq 1,\ \text{for all } n \in \mathbb{N}.$$

Now, by using (2), we have

$$\begin{aligned} & d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) \\ = \ & d(F(x,\ y),\ F(u_n,\ v_n)) \\ \leq \ & \alpha\left((G(x,\ y),\ G(y,\ x)),\ (G(u_n,\ v_n),\ G(v_n,\ u_n))\right) \\ & \times d(F(x,\ y),\ F(u_n,\ v_n)) \\ \leq \ & \psi\left(\frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2}\right). \end{aligned}$$

Thus

$$d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) \leq \psi\left(\frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2}\right). \tag{26}$$

Similarly, we have

$$d(G(y,\ x),\ G(v_{n+1},\ u_{n+1})) \leq \psi\left(\frac{d(G(y,\ x),\ G(v_n,\ u_n)) + d(G(x,\ y),\ G(u_n,\ v_n))}{2}\right). \tag{27}$$

Combining (26) and (27), we get

$$\begin{aligned} & \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(y,\ x),\ G(v_{n+1},\ u_{n+1}))}{2} \\ \leq \ & \psi\left(\frac{d(G(x,\ y),\ G(u_n,\ v_n)) + d(G(y,\ x),\ G(v_n,\ u_n))}{2}\right). \end{aligned}$$

Thus

$$\begin{aligned} & \frac{d(G(x,\ y),\ G(u_{n+1},\ v_{n+1})) + d(G(y,\ x),\ G(v_{n+1},\ u_{n+1}))}{2} \\ \leq \ & \psi^n\left(\frac{d(G(x,\ y),\ G(u_1,\ v_1)) + d(G(y,\ x),\ G(v_1,\ u_1))}{2}\right), \end{aligned}$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality and using Lemma 19, we get

$$G(x,\ y) = \lim_{n\to\infty} G(u_{n+1},\ v_{n+1}) \text{ and } G(y,\ x) = \lim_{n\to\infty} G(v_{n+1},\ u_{n+1}).$$

Similarly, we can show that

$$G(x^*,\ y^*) = \lim_{n\to\infty} G(u_{n+1},\ v_{n+1}) \text{ and } G(y^*,\ x^*) = \lim_{n\to\infty} G(v_{n+1},\ u_{n+1}).$$

Thus $G(x, y) = G(x^*, y^*)$ and $G(y, x) = G(y^*, x^*)$. □

3. Application to Integral Equations

Now, we study the existence of the solution to a Fredholm nonlinear integral equation as an application of the results obtained in the previous section. Consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))\,[f(q,\ x(q)) + g(q,\ x(q))]\,dq + h(p), \tag{28}$$

for all $p \in I = [a,\ b]$.

Let Θ denote the set of all functions $\theta : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq p$.

Assumption 31. We assume that the functions K_1, K_2, f, g fulfill the following conditions:

(i) $K_1(p,\ q) \geq 0$ and $K_2(p,\ q) \leq 0$ for all $p,\ q \in I$,

(ii) There exist positive numbers λ, μ and $\theta \in \Theta$ such that for all $x,\ y \in \mathbb{R}$ with $x \succeq y$, the following conditions hold:

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\theta(x - y), \tag{29}$$

and

$$-\mu\theta(x - y) \leq g(q,\ x) - g(q,\ y) \leq 0, \tag{30}$$

(iii)

$$\max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b [K_1(p,\ q) - K_2(p,\ q)]dq \leq \frac{1}{8}. \tag{31}$$

Definition 32 ([14])**.** A pair $(\widetilde{\alpha},\ \widetilde{\beta}) \in X \times X$ with $X = C(I,\ \mathbb{R})$, where $C(I,\ \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a *coupled lower-upper solution* of (28) if, for all $p \in I$,

$$\begin{aligned} \widetilde{\alpha}(p) \ \leq\ & \int_a^b K_1(p,\ q)[f(q,\ \widetilde{\alpha}(q)) + g(q,\ \widetilde{\beta}(q))]dq \\ & + \int_a^b K_2(p,\ q)[f(q,\ \widetilde{\beta}(q)) + g(q,\ \widetilde{\alpha}(q))]dq + h(p) \end{aligned}$$

and

$$\begin{aligned}\widetilde{\beta}(q) &\geq \int_a^b K_1(p,\ q)[f(q,\ \widetilde{\beta}(q)) + g(q,\ \widetilde{\alpha}(q))]dq \\ &+ \int_a^b K_2(p,\ q)[f(q,\ \widetilde{\alpha}(q)) + g(q,\ \widetilde{\beta}(q))]dq + h(p).\end{aligned}$$

Theorem 33. *Consider the integral equation* (28) *with* $K_1, K_2 \in C(I \times I, \mathbb{R})$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ *and* $h \in C(I,\ \mathbb{R})$. *Suppose there exists a coupled lower-upper solution* $(\widetilde{\alpha},\ \widetilde{\beta})$ *of* (28) *with* $\widetilde{\alpha} \leq \widetilde{\beta}$ *and that Assumption 31 is satisfied. Then the integral equation* (28) *has a solution in* $C(I,\ \mathbb{R})$.

Proof. Consider $X = C(I,\ \mathbb{R})$, the natural partial order relation, that is, for x, $y \in C(I,\ \mathbb{R})$,

$$x \preceq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I.$$

It is obvious that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} |x(p) - y(p)|.$$

Now define on $X \times X$ the following partial order: for $(x,\ y)$, $(u,\ v) \in X \times X$,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \text{for all } p \in I.$$

Obviously, for any $(x,\ y) \in X \times X$, the functions $\max\{x,\ y\}$ and $\min\{x,\ y\}$ are the upper and lower bounds of x and y respectively. Therefore for every $(x,\ y)$, $(u,\ v) \in X \times X$, there exists the element $(\max\{x,\ u\},\ \min\{y,\ v\})$ which is comparable to $(x,\ y)$ and $(u,\ v)$. Define $\alpha : X^2 \times X^2 \to (0,\ +\infty)$ as follows

$$\alpha((x,\ y),\ (u,\ v)) = \begin{cases} 1, \text{ if } x \geq y,\ u \geq v, \\ 0, \text{ otherwise,} \end{cases}$$

and $\psi : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$\psi(t) = \frac{t}{2} \text{ for all } t > 0.$$

Now define the mapping $F : X \times X \to X$ by

$$\begin{aligned}F(x,\ y)(p) &= \int_a^b K_1(p,\ q)\left[f(q,\ x(q)) + g(q,\ y(q))\right]dq \\ &+ \int_a^b K_2(p,\ q)\left[f(q,\ y(q)) + g(q,\ x(q))\right]dq + h(p),\end{aligned}$$

for all $p \in I$. It is easy to prove, like in [11], that F is increasing. Now for x, y, u, $v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned}
& F(x,\ y)(p) - F(u,\ v)(p) \\
= & \int_a^b K_1(p,\ q)\,[f(q,\ x(q)) + g(q,\ y(q))]\,dq \\
& + \int_a^b K_2(p,\ q)\,[f(q,\ y(q)) + g(q,\ x(q))]\,dq \\
& - \int_a^b K_1(p,\ q)\,[f(q,\ u(q)) + g(q,\ v(q))]\,dq \\
& - \int_a^b K_2(p,\ q)\,[f(q,\ v(q)) + g(q,\ u(q))]\,dq \\
= & \int_a^b K_1(p,\ q)\,[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))]\,dq \\
& - \int_a^b K_2(p,\ q)\,[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))]\,dq.
\end{aligned}$$

Thus, by using (29) and (30), we get

$$\begin{aligned}
& F(x,\ y)(p) - F(u,\ v)(p) \\
\leq\ & \int_a^b K_1(p,\ q)\,[\lambda\theta\,(x(q) - u(q)) + \mu\theta\,(v(q) - y(q))]\,dq \\
& - \int_a^b K_2(p,\ q)\,[\lambda\theta\,(v(q) - y(q)) + \mu\theta\,(x(q) - u(q))]\,dq.
\end{aligned} \tag{32}$$

Since θ is non-decreasing and $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned}
\theta\,(x(q) - u(q)) & \leq \theta\left(\sup_{p\in I} |x(q) - u(q)|\right) = \theta(d(x,\ u)), \\
\theta\,(v(q) - y(q)) & \leq \theta\left(\sup_{p\in I} |v(q) - y(q)|\right) = \theta(d(y,\ v)).
\end{aligned}$$

Hence by (32), in fact that $K_2(p,\, q) \leq 0$, we obtain

$$\begin{aligned}
& |F(x,\ y)(p) - F(u,\ v)(p)| \\
\leq\ & \int_a^b K_1(p,\ q)\,[\lambda\theta(d(x,\ u)) + \mu\theta(d(y,\ v))]\,dq \\
& - \int_a^b K_2(p,\ q)\,[\lambda\theta(d(y,\ v)) + \mu\theta(d(x,\ u))]\,dq,
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_a^b K_1(p,\ q)\,[\max\{\lambda,\ \mu\}\theta(d(x,\ u)) + \max\{\lambda,\ \mu\}\theta(d(y,\ v))]\,dq \\
&\quad - \int_a^b K_2(p,\ q)\,[\max\{\lambda,\ \mu\}\theta(d(y,\ v)) + \max\{\lambda,\ \mu\}\theta(d(x,\ u))]\,dq,
\end{aligned}
$$

since all the quantities on the right hand side of (32) are non-negative. Now, taking supremum with respect to t, by using (31), we get

$$
\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b (K_1(p,\ q) - K_2(p,\ q))\,dq.\,[\theta(d(x,\ u)) + \theta(d(y,\ v))] \\
\leq\ & \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{8}.
\end{aligned}
$$

Thus

$$
d(F(x,\ y),\ F(u,\ v)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{8}. \tag{33}
$$

Now, since θ is non-decreasing, we have

$$
\begin{aligned}
\theta(d(x,\ u)) &\leq \theta\left(d(x,\ u) + d(y,\ v)\right), \\
\theta(d(y,\ v)) &\leq \theta\left(d(x,\ u) + d(y,\ v)\right),
\end{aligned}
$$

which implies, by (ii_θ), that

$$
\begin{aligned}
\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} &\leq \theta\left(d(x,\ u) + d(y,\ v)\right) \\
&\leq d(x,\ u) + d(y,\ v).
\end{aligned}
$$

Hence

$$
\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{8} \leq \frac{d(x,\ u) + d(y,\ v)}{4}. \tag{34}
$$

Thus by (33) and (34), we have

$$
\begin{aligned}
& \alpha((x,\ y),\ (u,\ v))d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \frac{d(x,\ u) + d(y,\ v)}{4} \\
\leq\ & \frac{1}{2}\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right) \\
\leq\ & \psi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right),
\end{aligned}
$$

which is the contractive condition of Corollary 28. Now, let $(\widetilde{\alpha},\ \widetilde{\beta}) \in X \times X$ be a coupled upper-lower solution of (28), then we have $\widetilde{\alpha}(p) \leq F(\widetilde{\alpha},\ \widetilde{\beta})(p)$ and $\widetilde{\beta}(p) \geq$

$F(\widetilde{\beta}, \widetilde{\alpha})(p)$, for all $p \in I$, which shows that all the hypothesis of Corollary 28 are satisfied. This proves that F has a coupled fixed point $(x, y) \in X \times X$ which is the solution in $X = C(I, \mathbb{R})$ of the integral equation (28). □

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