AN ELEMENTARY PROOF OF THE EFFECT OF 3-MOVE ON THE JONES POLYNOMIAL

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ABSTRACT. A mathematical *knot* is an embedded circle in $\mathbb{R}^3$. A fundamental problem in knot theory is classifying knots up to its numbers of crossing points. Knots are often distinguished by using a knot invariant, a quantity which is the same for equivalent knots. Knot polynomials are one of well known knot invariants. In 2006, J. Przytycki showed the effects of a $n - move$ (a local change in a knot diagram) on several knot polynomials. In this paper, the authors review about knot polynomials, especially Jones polynomial, and give an alternative proof to a part of the Przytychi's result for the case $n = 3$ on the Jones polynomial.

1. INTRODUCTION

A mathematical *knot* is a knotted loop of string, in other words, an embedded circle in $\mathbb{R}^3$. A useful way to visualize knots is projecting a knot onto 2-dimensional plane. In this projection, self-intersection is transversal double point which is called *crossing* point (see Figure 1(a)). To specify which arc passes over and under at a crossing point, one can create a break in the strand going underneath (see Figure 1(b)). The resulting diagram is called *knot diagram*. The study of mathematical knots is called *knot theory*. A fundamental problem in knot theory is classifying knots up to its number of crossing points. Although there are dozens of nine crossing knots, there are hundreds of ten crossing knots [20]. Hence it is very hard to classify all knots up to its crossing number.

The first thing that needs to be done is defining in which case two knots are the same: two knots are equivalent if one can be transformed to the other by a deformation of $\mathbb{R}^3$ upon itself without cutting the string or passing the string through [1]. In 1926, the German mathematician K. Reidemeister proved that two knot

diagrams of the same knot (up to planar isotopy) can be related by a sequence of the three local moves on the diagram, *Reidemeister moves* (see Figure 3)[19]. If there is a quantity that is the same for equivalent knots, it is natural that the quantity of a knot diagram must be unchanged under the Reidemeister moves. This quantity is called a *knot invariant*. By using knot invariants, one can distinguish one knot from another. As classical examples of knot invariants, there are tricolability, some integer-valued functions(crossing number, bridge number, etc.), knot polynomials(Conway, Homfly, Alexander, Jones polynomials and etc.), knot group and so on.

In this paper, in Section 2, we review some mathematical backgrounds (such as knots and links, tangles, knot polynomials), especially the Jones polynomial which is one of the most useful knot invariants. In Section 3, the effect of local moves ($n-$ *moves*, see Figure 12(a)) of a knot diagram on the Jones polynomial is introduced. Furthermore, an alternative proof of a part of the Przytychi's result for the case $n = 3$ (Theorem 3.3) is discussed.

2. Basic Definitions

Prior to introduce an elementary proof of Theorem 3.3, basic mathematical concepts are explained such as knots and links, tangles and invariant polynomials.

2.1. Knots and Links A *knot* is a knotted loop of string which is embedded in $\mathbb{R}^3$. A *link* is a finite disjoint union of knots: $L = K_1 \cup \cdots \cup K_n$. Each knots K_i is called a *component* of the link [3].

A *homotopy* of a space $X \in \mathbb{R}^3$ is a continuous map $F : X \times [0,1] \to \mathbb{R}^3$, where $F_t : X \times \{t\} \to \mathbb{R}^3$ for $t \in [0,1]$. Here, F_0 is an identity map and each F_t is continuous. In the case that F_t is one-to-one and continuous, F is called an *isotopy*. Unfortunately, two knots (or links) are isotopy doesn't mean that they are equivalent, since all knots are isotopy to each other [3]. To model the topological deformation of knots (or links), one need *ambient isotopy*: Two knots (or links) K_1, K_2 in X are *ambient isotopic* if there is an isotopy $F : X \times [0,1] \to X$ such that $F_0 = K_1$ and $F_1 = K_2$. Hence two knots (or links) are ambient isotopic applies that one can be transformed to the other by a deformation of $\mathbb{R}^3$ upon itself without cutting the string or passing the string through. That is, they are equivalent.

As a useful way to visualize knots (or links), one can project a knot (or link) onto 2-dimensional plane. A *knot projection* is a projection of a knot onto the 2-dimensional plane where under and over arcs are not specified. In this projection,

no three points correspond to one point onto the plane and arcs cross transversely. The self-intersection double points are called a *crossing* as in Figure 1(a). A *knot diagram* is a projection where at each crossing the over and under arcs are specified as in Figure 1(b).
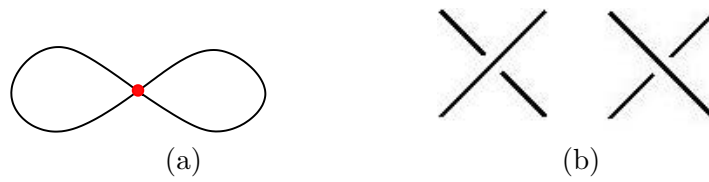

(a)   (b)

Figure 1. (a) Crossing of a knot projection (b) Possible crossings on a knot diagram

The simplest example of a knot is the *trivial knot* or *unknot* which is ambient isotopic to a zero crossing knot as in Figure 2(a). The simplest example of a non-trivial knot is the *trefoil*, see Figure 2(b). There are two types of trefoil knots, one is right handed as in Figure 2(b) and the other is left handed which is obtained by changing all under crossings to over crossings. The two trefoils are not ambient isotopic, that is, it's impossible to deform a left-handed trefoil continuously into a right-handed trefoil, or vice versa. An example of unlink with three components is shown in Figure 2(c) and nontrivial link with two components is in Figure 2(d).
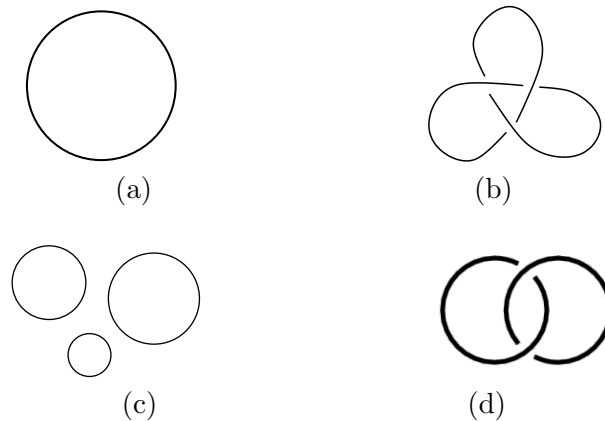

(a)   (b)

(c)   (d)

Figure 2. Examples of knots and links
(a) Unknot (b) Trefoil (c) Unlink with three components (d) Hopf link: an example of nontrivial link with two components

In 1926, the German mathematician K. Reidemeister proved that two knot diagrams of the same knot (up to planar isotopy) can be related by a sequence of the three local moves on the diagram, *R-move I, R-move II* and *R-move III* as in Figure 3. Those three moves are called *Reidemeister moves*. Hence if there is a knot (or a link) invariant, the quantity will be preserved under the Reidemeister three moves.
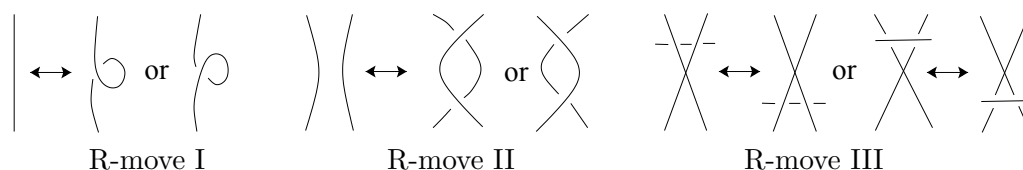

R-move I   R-move II   R-move III

Figure 3. Reidemeister moves

Now, let's consider about an orientation of a knot. An *orientation* is defined by a direction to travel along the knot diagram. The direction is our choice, so a knot has two different orientations. Then a knot with a direction is called *oriented*. One can define an *oriented link* in a similar way. By choosing a direction for each component of link diagram, one can decide the orientation of a link. For example, there are four possible orientation of the Hopf link as in Figure 4.
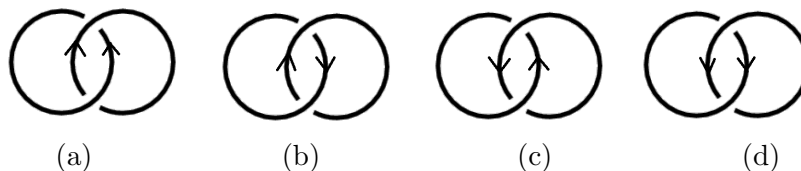

(a)   (b)   (c)   (d)

Figure 4. Four possible orientation of Hopf link

According to an observation on Figure 4, at each crossing, one of the pictures in Figure 5 holds. At a crossing, if the Figure 5(a) holds, it is called a *positive crossing* and we count *+1*. When the Figure 5(b) holds, it is called a *negative crossing* and we count *-1* at the crossing. Let L be an oriented link diagram. The sum of all those +1s and −1s of L is called *writhe* of L and denote it *w(L)*. The value *w(L)* is not a knot (or link) invariant since Reidemeister move I changes *w(L)* by +1 or -1, even though it is unchanged under Reidemeister moves II and III. The *linking number* of L, denoted by *lk(L)* is the half of the sum of the +1s and −1s at every crossing between any two components in L. A self-crossing between one component

and itself is not counted. This $lk(L)$ is a link invariant of an oriented link. The absolute value of $lk(L)$ is independent on the orientations of the link.

(a)   (b)

Figure 5. (a) Positive crossing (b) Negative crossing

2.2. Tangle In the 1960's, John Conway first introduced the concept of *tangle* while he tabulate knots and links with small crossings [2]. If we draw a 3-dimensional ball around some parts of a knot (or a link), there will be n many segments of the knot in the ball. An *n-string tangle* is the ball with n strings which are properly embedded in it. In a knot (or a link) diagram, an n-string tangle can be expressed by a circle with n segments so that each segment crosses the circle exactly $2n$ times, see Figure 6. Starting with J. Conway, tangles have been studied by many mathematicians and utilized to prove lots of results. Recently, they are applied to the topology of DNA [4] -[11], [13]-[17]. In this paper, we only use 2-string tangle, so call it just 'tangle' instead of '2-string tangle' from now on. On the tangle diagram, there are four boundary points which are called four compass directions, NW, NE, SW, and SE, respectively as in Figure 6. Two tangles are *equivalent* if one can be deformed to the other with a finite sequence of Reidemeister moves while the four end points are fixed.
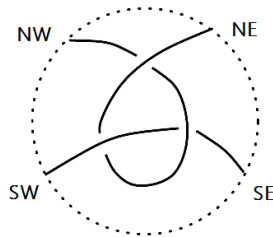

Figure 6. An example of a 2-string tangle

The tangle consisted of two vertical strings is called the ∞-*tangle* (see Figure 7(a)), and the tangle with two horizontal strings is called the 0-*tangle* (see Figure

7(b)). The *n-tangle* is the tangle which has n many horizontal right handed half twists, $n \in \mathbb{Z}$. An example of 3-tangle is shown in Figure 7(c).
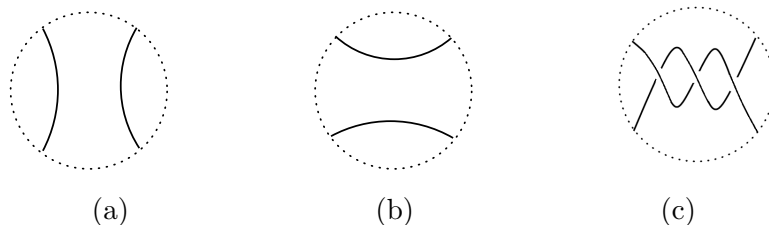

(a)   (b)   (c)

Figure 7. (a) ∞-tangle (b) 0-tangle (c) 3-tangle

Two tangles can be added or multiplied as in the manner of Figure 8. A tangle obtained by multiplication of n-tangles is called a *rational tangle*, see an example in Figure 9(a). And any tangle obtained by a sequence of multiplication and addition of rational tangle is called *algebraic tangle* such as a tangle in Figure 9(b).
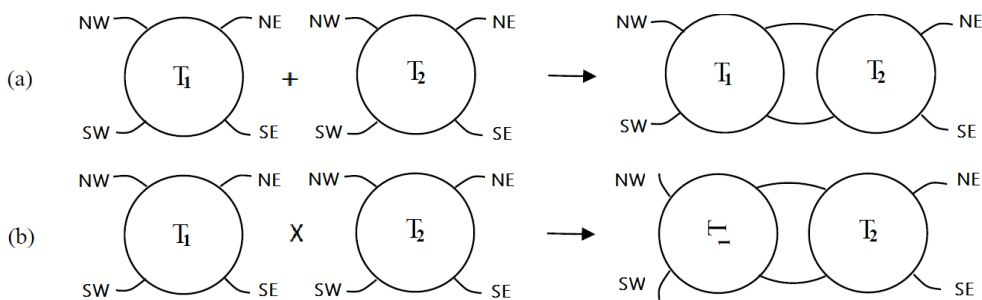

Figure 8. (a) Addition of two tangles T_1 and T_2, $T_1 + T_2$. (b) Multiplication of two tangles T_1 and T_2, $T_1 \times T_2$.
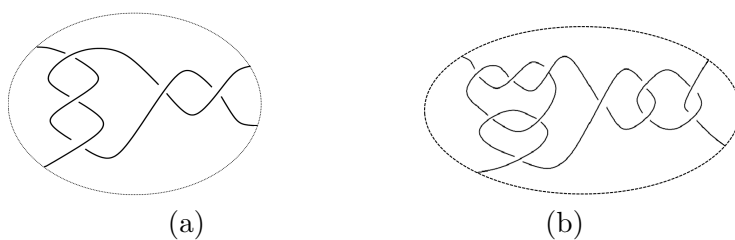

(a)   (b)

Figure 9. (a) Example of rational tangle (b) Example of algebraic tangle

Since a tangle is a part of a link diagram, a link can be obtained from tangles: The *numerator closure* of a tangle is the link obtained by connecting NW to NE,

and SW to SE with arcs as in Figure 10(a). The *denominator closure* of a tangle is the link obtained by connecting SE to NE, and SW to NW with arcs as in Figure 10(b).

(a)   (b)

Figure 10. (a) Numerator closure (b) Denominator closure

More detailed information about tangle, especially rational tangle, appears in greater detail in the books [1], [2], [3] and tangles with more than three strings are well reviewed in the articles [6], [13], [14], [15], [16].

2.3. Invariant Polynomials In this section, note that L represents an oriented link diagram, and D is the unoriented diagram obtained from L by ignoring the orientation. It can be written by $|L| = D$.

By using knot (or link) invariants, one can distinguish one knot (or link) from another. It is natural that a knot (or link) invariant value must be unchanged under the all three Reidemeister moves. In section 2.1, we mentioned that $w(L)$ is not a link invariant but $lk(L)$ is a link invariant for an oriented link diagram L.

An *invariant polynomial* is another classical link invariant. This concept was first introduced by Lou Kauffman, and he developed $X(L)$, the *Kauffman polynomial* or *X-polynomial*, for an oriented link L. Before define $X(L)$, we start with defining *bracket polynomial* in Definition 2.1.

Definition 2.1. Let D be a unoriented link diagram and $\bigcirc$ be a diagram of the unknot. $< D >$, bracket polynomial of D, is the polynomial with the variable A and satisfies the following three rules:

Rule 1: $< \bigcirc >= 1$

Rule 2: $< D \cup \bigcirc >= (-A^2 - A^{-2}) < D >$

Rule 3: (i) $< \times >= A <)(> +A^{-1} < \asymp >$

      (ii) $< \times >= A < \ominus > +A^{-1} <)(>$

One can easily check that the bracket polynomial is invariant under the Reidemeister moves II and III. However, applying R-move I changes the polynomial as in

the equation 2.1:

$$< \text{⬡} > = A < \text{⬡} > + A^{-1} < \text{⬡} >$$

(2.1)
$$= A < D > + A^{-1}(-A^2 - A^{-2}) < D >$$
$$= A < D > + (-A - A^{-3}) < D >$$
$$= -A^{-3} < D >$$

Similarly, $< \text{⬡} > = -A^3 < D >$. Remind that $w(L)$, writhe (section 2.1), has similar property with the bracket polynomial, where L is an oriented link diagram. That is, $w(L)$ is an invariant under only R-moves II and III, but not under R-move I. Motivated by this fact, one can define a link invariant under all Reidemeister three moves: the *X-polynomial* as in Definition 2.2.

Definition 2.2. Let L be an oriented link diagram, and D is the unoriented diagram obtained from L by ignoring the orientation. $< D >$ is the bracket polynomial with a variable A. Then the *X-polynomial* of L is defined by $X(L) = (-A^3)^{-w(L)} < D >$.

Since $< D >$ and $w(L)$ are both unaffected by R-move II and III, X-polynomial is not changed by the two moves. Note that R-move I changes the value of $w(L)$ by $+1$ or -1 such as in Equation 2.2:

(2.2)
$$w(\text{⬡}) = w(\text{⬡}) - 1$$

Thus the following relation (Equation 2.3) can be gotten:

$$X(\text{⬡}) = (-A^3)^{-w(\text{⬡})} < \text{⬡} > \qquad \text{- - - - -by Definition 2.2}$$

(2.3)
$$= (-A^3)^{-(w(\text{⬡})-1)} < \text{⬡} > \qquad \text{- - - - -by Equation 2.2}$$
$$= (-A^3)^{-(w(\text{⬡})-1)}(-A^{-3}) < D > \qquad \text{- - - - -by Equation 2.1}$$
$$= (-A^3)^{-w(\text{⬡})} < D >$$
$$= X(L)$$

Similarly, $X(\text{⬡}) = X(L)$. This means that $X(L)$ is also unaffected by R-move I, and thus one can conclude $X(L)$ is an invariant for any oriented link L. In 1985, Vaughan Jones defined a new polynomial, *Jones polynomial*, by replacing A by $t^{-\frac{1}{4}}$ in [12]. It is natural that the Jones polynomial is also an invariant for oriented links. Let L^+, L^0 and L^- be the three oriented link diagrams which are equivalent except where they appear as in Figure 11, then the equation in Lemma 2.3 is always true.
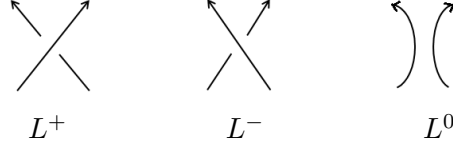
$$L^+ \qquad\qquad L^- \qquad\qquad L^0$$

Figure 11. Three link diagram which are almost identical.

Lemma 2.3. *For every oriented link L, the following equation holds:*

$$V_{L^+}(t) = t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L^0}(t) + t^2 V_{L^-}(t).$$

Proof. From the two equations of Rule 3 in Definition 2.1, $A \times$ (i) $- A^{-1} \times$ (ii), we can get the Equation 2.4:

(2.4) $$A < \times > -A^{-1} < \times > = (A^2 - A^{-2}) <)(>$$

Let D be the unoriented diagram of L. By using $w < \times > = 1$, $w < \times > = -1$, $w <)(> = 0$ and $X(L) = (-A^3)^{-w(L)} < D >$, we have the following relations:

(2.5) $$< \times > = X(L_+)(-A^3), \quad < \times > = X(L_-)(-A^3)^{-1}, \quad <)(> = X(L_0)$$

Hence the Equation 2.4 is changed to the Equation 2.6 as below:

(2.6) $$-A^4 \times X(L^+) + A^{-4} \times X(L^-) = (A^2 - A^{-2}) \times X(L^0)$$

Then, if we replace each A by $t^{-\frac{1}{4}}$, we can get the relation of Jones polynomial,

$$-t^{-1}V_{L^+}(t) + tV_{L^-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L^0}(t)$$

Therefore, $V_{L^+}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})tV_{L^0}(t) + t^2 V_{L^-}(t)$. $\qquad\qquad\square$

3. ELEMENTARY PROOF OF THE EFFECT OF $3 - move$ ON THE JONES POLYNOMIAL

If there is some local change on a knot (or a link), then the knot (or link) type can be changed. Hence it is a natural question how some local change of a knot (or link) affects its invariant values [18]. As we discussed in the previous section (Section 2.3), the Jones polynomial is a useful tool for distinguishing knots and links. In this section, we especially focus on the property of the Jones polynomial (Theorem 3.3) with related to a *3-move* (Definition 3.2).

Definition 3.1. The local change in a link diagram which replaces two parallel lines by n positive half-twists is called an *n-move*. See Figure 12(a).

Definition 3.2. Let D be a link diagram, and the diagram obtained from D by *3-move* is denoted by D_{+++}, see Figure 12(b). Especially, 3-move on an oriented link diagram L is denoted by L_{+++} as in Figure 12(c), (d).
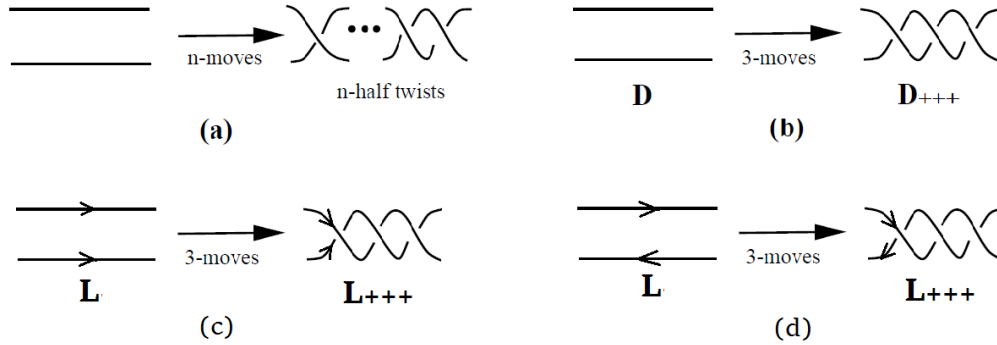

Figure 12. (a) n-move (b) 3-move (c),(d) 3-moves on an oriented link diagram.

When a link diagram is oriented, the two parallel lines in Figure 12 can have two different directions: *parallel orientation* and *anti-parallel orientation* as in Figure 13.

Figure 13. (a) parallel orientation (b) anti-parallel orientation

In [18], J. Przytychi analyzed the influence of $n - moves$ on the several knot polynomials: Homfly polynomial, Alexander polynomial, and Jones polynomial. In this paper, we prove a part of the Przytychi's result for the case $n = 3$ and the Jones polynomial by using tangle argument as in Theorem 3.3.

Theorem 3.3. *Let L be an oriented link diagram, and L_{+++} be the diagram obtained from L by 3-move. Then*

$$(3.1) \qquad V_{L_{+++}}(e^{\frac{\pi i}{3}}) = \pm i^{(com(L_{+++}) - com(L))} V_L(e^{\frac{\pi i}{3}})$$

holds, where V is the Jones polynomial, $com(L)$ is the number of components of L.

Proof. Let L be a link diagram and choose two parallel line segments for 3-move. Pull the one line segment up and pull down the other line segment as in Figure 14. Then L can be expressed by a numerator closure of a 2-string tangle T.
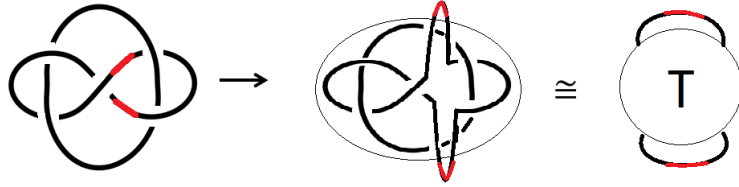
Figure 14. An example of a link diagram L such that $L = N(T)$.

Note that T can include several (knotted) circles such as in Figure 16(d). Fur-thermore, L can be considered as the numerator closure of addition of the 2-string zero tangle and T, i.e. $L = N(O + T)$ as in Figure 15. In this Figure 15, the two red segments represents two parallel lines for 3-move.
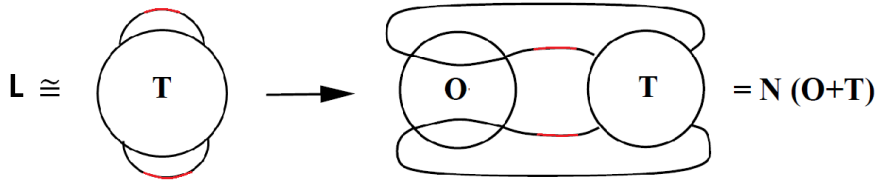

Figure 15. Any link diagram L can be expressed by $N(O+T)$, where $O =$ the 2-string zero tangle, $T =$ a 2-string tangle.

In the equation $L = N(O + T)$, a 2-string tangle T can be classified into 3 types (*Type-1, Type-2* and *Type-3*) up to the position of four boundary points of T, NW, NE, SW and SE, as in Figure 16 (a)$\sim$ (c). And examples of three types are given in Figure 16 (d)$\sim$(f).

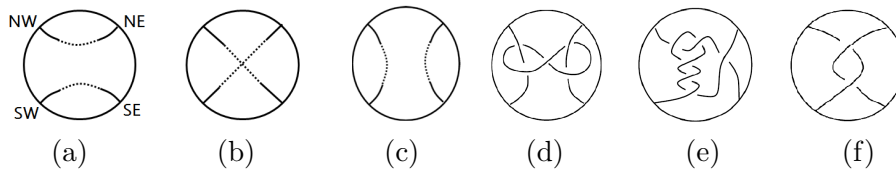(a)   (b)   (c)   (d)   (e)   (f)

Figure 16. Three types of a 2-string tangle T
(a) *Type-1*: A string connects NW and NE, and another string connects SW and SE. (b) *Type-2*: A string connects NW and SE, and another string connects NE and SW. (c) *Type-3*: A string connects NW and SW, and another string connects NE and SE. (d) An example of *Type-1* tangle (e) An example of *Type-2* tangle (f) An example of *Type-3* tangle.

It is sufficient to show that Equation 3.1 holds for all three types of T:

- **[Type-1]**

  Let $com(L)$ be the number of components of a link L. If T is a *Type-1* tangle, as one can see from Figure 17, the $com(L_{+++}) - com(L) = -1$. The tangle T may include some (knotted) circles which are not drawn in the Figure 17, but the number of circle components are not changed after 3-move.
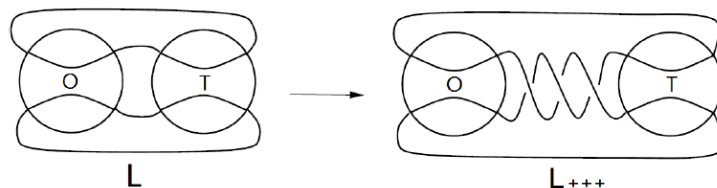

Figure 17. If T is a *Type-1* tangle, then $com(L_{+++}) - com(L) = -1$.

If the orientation of the parallel two lines between O and T is anti-parallel, we can easily figure out that L_{+++} is an unoriented link. Therefore the orientation of the parallel two lines must be parallel. Hence we can consider the following 2 cases:

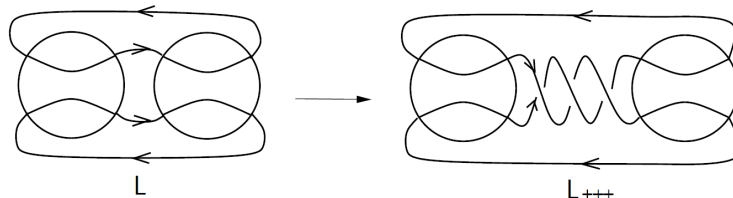- **Case 1** Two parallel lines for 3-move are as in Figure 18.

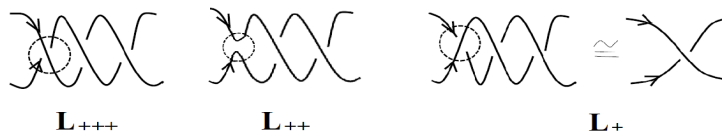*Figure* 18. A diagram of L when T is a tangle in *Case 1* of *Type-1*.

Figure 19. Three oriented diagrams, L_{+++}, L_{++} and L_{+} for *Case 1* of *Type-1* .

Now, let L_{+++}, L_{++} and L_{+} be three oriented link diagrams which are equivalent to L_{+++} except where they appear inside the dotted circle

in Figure 19. Hence L_{+++}, L_{++} and L_+ corresponds to L^+, L^0 and L^- in Figure 11, respectively. By using Lemma 2.3, one can notice the following two equations hold:

(3.2)
$$V_{L_{+++}}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})tV_{L_{++}}(t) + t^2 V_{L_+}(t)$$

and

(3.3)
$$V_{L_{++}}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})tV_{L_+}(t) + t^2 V_L(t).$$

By eliminating $V_{L_{++}}(t)$ from Equation 3.2 and Equation 3.3, the following equation can be obtained:

(3.4)
$$\begin{aligned} V_{L_{+++}}(t) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})tV_{L_{++}}(t) + t^2 V_{L_+}(t) \\ &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t((t^{\frac{1}{2}} - t^{-\frac{1}{2}})tV_{L_+}(t) + t^2 V_L(t)) + t^2 V_{L_+}(t) \\ &= (1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)t^2 V_{L_+}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^3 V_L(t) \end{aligned}$$

If $t = e^{\frac{\pi i}{3}}$, then $t^3 = -1$ and $t^{\frac{1}{2}} - t^{-\frac{1}{2}} = i$.

Therefore, $V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}}) = i^{com(L_{+++})-com(L)}V_L(e^{\frac{\pi i}{3}})$.

- **Case 2** Two parallel lines for 3-move are as in Figure 20.
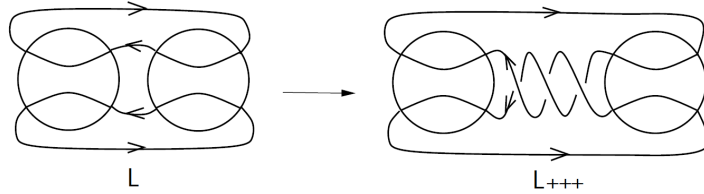

Figure 20. A diagram of L when T is *Case 2* of *Type-1*

And L_{+++}, L_{++} and L_+ of this case are defined as in Figure 21.

Figure 21. Three oriented diagrams, L_{+++}, L_{++} and L_+ for Case 2 of *Type-1*

Since the Equation 3.2 and 3.3 are also true for this case, by similar calculation of *Case 1*, Equation 3.5 holds:

(3.5)
$$V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}}) = i^{com(L_{+++})-com(L)}V_L(e^{\frac{\pi i}{3}})$$

- **[Type-2]**

  If T is a *Type-2* tangle, then $com(L_{+++}) - com(L) = 1$ as in Figure 22. As in the *Type-1*, T may include some (knotted) circles which are not drawn in the Figure 22, but the number of circle components are not changed after 3-move.
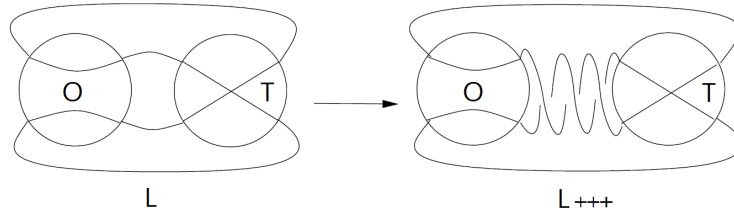

Figure 22.   If T is a *Type-2* tangle, then $com(L_{+++}) - com(L) = 1$.

Note that L_{+++} is an unoriented link if the orientation of two parallel lines between O and T is anti-parallel. Therefore the orientation of the two parallel lines must have parallel orientation. Hence we only have to consider the following 2 cases:

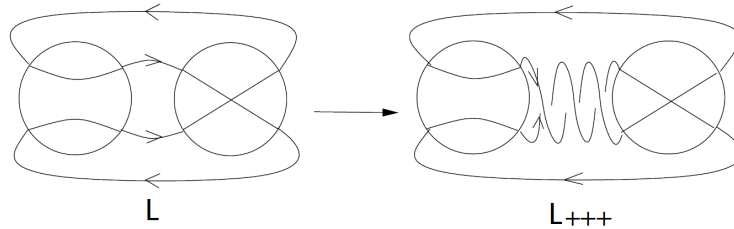- **Case 1** Two parallel lines for 3-move are as in Figure 23.

Figure 23.   A diagram of L when T is the tangle in *Case 1* of *Type-2*

  In this case, L_{+++}, L_{++} and L_{+} are defined as in Figure 19. And the remaining steps are exactly same as *Case 1* of *Type-1*, see Equation 3.4. Therefore, the relation between $V_{L_{+++}}$ and V_L is $V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}})$. Since $com(L_{+++}) - com(L) = 1$, we can conclude the following Equation 3.6:

$$(3.6) \qquad V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}}) = -i^{com(L_{+++})-com(L)}V_L(e^{\frac{\pi i}{3}})$$

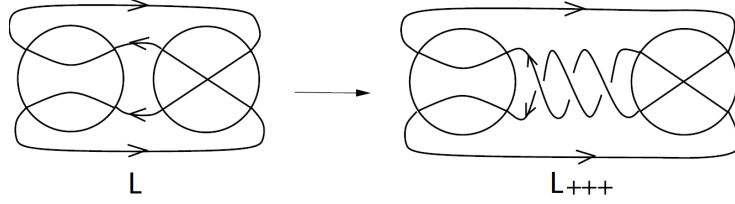- **Case 2** Two parallel lines for 3-move are as in Figure 24.

Figure 24. A diagram of L when T is the tangle in *Case 2* of *Type-2*

In this case, L_{+++}, L_{++} and L_+ are defined as in Figure 21. Similar to *Case 2* of *Type-1*, see Lemma 2.3 and Equation 3.2 $\sim$ 3.4, $V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}})$. Since $com(L_{+++}) - com(L) = 1$, we can get the following Equation 3.7 holds:

$$(3.7) \qquad V_{L_{+++}}(e^{\frac{\pi i}{3}}) = -iV_L(e^{\frac{\pi i}{3}}) = -i^{com(L_{+++})-com(L)}V_L(e^{\frac{\pi i}{3}}).$$

- **[Type-3]**

  If T is a *Type-3* tangle, $com(L_{+++}) - com(L) = 0$ as in Figure 25. T may include some (knotted) circles which are not drawn in the Figure 25, but the number of circle components are not changed after 3-move.

  Note that L_{+++} is not always isotopic to L in Figure 26, even though they looks like isotopic. This is also due to the possibility of some (knotted) circle components in T.
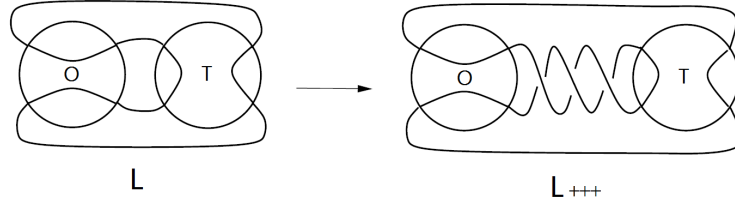

Figure 25. If T is a *Type-3* tangle, then $com(L_{+++}) - com(L) = 0$.

  If the orientation of two parallel lines between 0 and T is parallel, then L and L_{+++} both are unoriented links. Therefore the orientation of the two parallel lines must be anti-parallel. Hence we can only consider the following 2 cases:

  - **Case 1** Two parallel lines for 3-move are as in Figure 26.
    In this case, one can consider three oriented link diagrams, L_{+++}, L_{++} and L_+, as in Figure 27. And L_{+++}, L_{++} and L_+ corresponds to L^-, L^0 and L^+ in Figure 11, respectively.
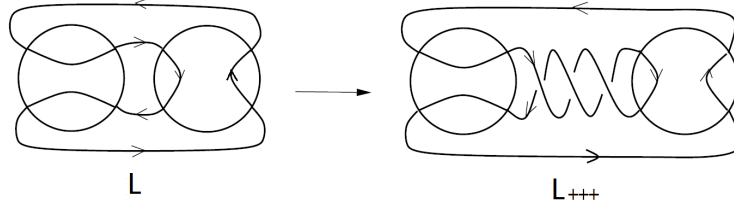
Figure 26. A diagram of L when T is the tangle in *Case 1* of *Type-3*

Figure 27. Three oriented diagrams, L_{+++}, L_{++} and L_+ for Case 1 of *Type-3*

Then the following Equation 3.8 holds by Lemma 2.3:

(3.8)
$$V_{L_+}(t) = t(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_{++}}(t) + t^2 V_{L_{+++}}(t)$$

To use a similar argument with *Type-1* and *Type-2*, we need the relation among $V_{L_{++}}$, V_{L_+} and V_L. Unfortunately, that is impossible at this time.

Now, let's consider the relation between $< D_{+++} >$ and $< D >$, where D is the link diagram after ignoring the orientation of L. From the two equations of the *Rule* 3 of Definition 2.1, a calculation $A^{-1} \times (i) - A \times (ii)$ gives the Equation 3.9:

(3.9)
$$A^{-1} < \times > - A^{-1} < \times >= (A^{-2} - A^2) < \ominus >$$

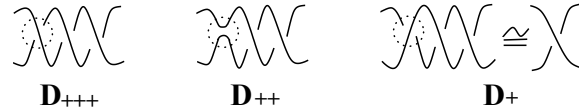Now, consider unoriented diagrams D_{+++}, D_{++} and D_+ as in Figure 28.

Figure 28.

By applying Equation 3.9 to Figure 28, we can get the Equation 3.10 and 3.11:

(3.10)
$$A^{-1} < D_+ > - A < D_{+++} >= (A^{-2} - A^2) < D_{++} >$$

and so

(3.11) $\qquad < D_{+++} >= A^{-2} < D_+ > -A^{-1}(A^{-2} - A^2) < D_{++} > .$

Furthermore, we can get the following Equation 3.12 with D_{++}, D_+ and D:

(3.12) $\qquad < D_{++} >= A^{-2} < D > -A^{-1}(A^{-2} - A^2) < D_+ >$

By plugging Equation 3.12 to Equation 3.11, Equation 3.13 can be obtained:

(3.13)
$$< D_{+++} >=A^{-2}(1 + (A^{-2} - A^2)^2) < D_+ >$$
$$- A^{-3}(A^{-2} - A^2) < D >$$

Then one can consider the writhe the oriented links: $w(L_{+++}) = w(L) - 3, w(L_+) = w(L) - 1$. Since $X(L) = (-A^3)^{-w(L)} < D >$, Equation 3.13 turns into Equation 3.14:

(3.14)
$$X(L_{+++}) \times (-A^3)^{-3} =A^{-2}(1 + (A^{-2} - A^2)^2)X(L_+)$$
$$\times (-A^3)^{-1} - A^{-3}(A^{-2} - A^2)X(L)$$

After replacing each A by $t^{-\frac{1}{4}}$, one can get the relation of Jones polynomial:

(3.15) $\qquad V_{L_{+++}}(t) \times -t^{\frac{9}{4}} = -t^{\frac{5}{4}}(1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)V_{L_+}(t) - t^{\frac{3}{4}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_L(t).$

Therefore,

(3.16) $\qquad V_{L_{+++}}(t) = t^{-1}(1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2)V_{L_+}(t) + t^{-\frac{3}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_L(t).$

Plugging $t = e^{\frac{\pi i}{3}}$ gives that $t^{\frac{1}{2}} - t^{-\frac{1}{2}} = i$ and $t^{-\frac{3}{2}} = e^{-\frac{\pi i}{2}} = -i$.

Therefore, $V_{D_{+++}}(e^{\frac{\pi i}{3}}) = e^{-\frac{\pi i}{3}}(1 + i^2)V_{D_+}(e^{\frac{\pi i}{3}}) + (-i) \times iV_D(e^{\frac{\pi i}{3}}).$

Thus we can get the desired equation:

(3.17) $\qquad V_{D_{+++}}(e^{\frac{\pi i}{3}}) = V_D(e^{\frac{\pi i}{3}}) = i^{com(D_{+++})-com(D)}V_D(e^{\frac{\pi i}{3}}).$

– **<u>Case 2</u>** Two parallel lines for 3-move are as in Figure 29.

As we mentioned in *Case 1* of *Type-3*, L_{+++} is not always isotopic to L in Figure 26, even though they looks like isotopic.

The only difference with *Case1* of *Type-3* is the orientation which affects the value of the writhe, w. However, because $w(\diagup\!\!\!\!\diagdown) = w(\diagup\!\!\!\!\diagdown)$, we can
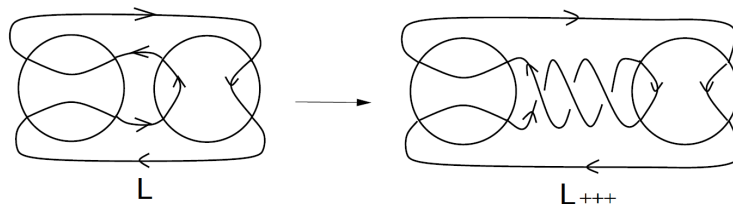
Figure 29.   A diagram of L when T is the tangle in *Case 2* of *Type-3*

notice that Equation 3.17 still holds. I.e.,

$$(3.18) \qquad V_{D_{+++}}(e^{\frac{\pi i}{3}}) = V_D(e^{\frac{\pi i}{3}}) = i^{com(D_{+++})-com(D)}V_D(e^{\frac{\pi i}{3}}).$$

So far, we showed that Equation 3.1 holds for all three types of the tangle T. Hence we conclude that Equation 3.1 holds for any oriented link diagram L. $\square$

ACKNOWLEDGMENT

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