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BIPROJECTIVITY OF MATRIX BANACH ALGEBRAS WITH APPLICATION TO COMPACT GROUPS

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Abstract. In this paper, the necessary and sufficient conditions are considered for biprojectivity of Banach algebras $\mathfrak{C}_p(I)$. As an application, we investigate biprojectivity of convolution Banach algebras $A(G)$ and $L^2(G)$ on a compact group G.

1. Introduction

The Banach algebras $\mathfrak{C}_p(I)$, where $p \in [1,\infty] \cup \{0\}$, have been introduced and extensively studied in Section 28 of [4]. Recently, amenability, weak amenability and approximate weak amenability have been studied by H. Samea in [8](see also [5]). The present paper is going to investigate biprojectivity of Banach algebras $\mathfrak{C}_p(I)$, together with their applications to a number of convolution Banach algebras on compact groups.

Let H be an n-dimensional Hilbert space and suppose that $B(H)$ is the space of all linear operators on H. Clearly we can identify $B(H)$ with $\mathbb{M}_n(\mathbb{C})$ (the space of all $n \times n$-matrices on $\mathbb{C}$). For $A \in \mathbb{M}_n(\mathbb{C})$, let $A^* \in \mathbb{M}_n(\mathbb{C})$ by $(A^*)_{ij} = \overline{A_{ji}}$ $(1 \le i,j \le n)$, and let $|A|$ denote the unique positive-definite square root of AA^*. A is called *unitary*, if $A^*A = AA^* = I$, where I is the $n \times n$-identity matrix. For $E \in B(H)$, let $(\lambda_1, \dots, \lambda_n)$ be the sequence of eigenvalues of the operator $|E|$, written in any order. Define $\|E\|_{\varphi_\infty} = \max_{1 \le i \le n} |\lambda_i|$, and $\|E\|_{\varphi_p} = (\sum_{i=1}^n |\lambda_i|^p)^{\frac{1}{p}}$ $(1 \le p < \infty)$. For more details see Definition D.37 and Theorem D.40 of [4].

Let I be an arbitrary index set. For each $i \in I$, let H_i be a finite dimensional Hilbert space of dimension d_i, and let $a_i \ge 1$ be a real number. The $*$-algebra $\prod_{i \in I} B(H_i)$ will be denoted by $\mathfrak{C}(I)$; scaler multiplication, addition, multiplication,

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and the adjoint of an element are defined coordinate-wise. Let $E = (E_i)$ be an element of $\mathfrak{E}(I)$. We define $\|E\|_p := \left(\sum_{i\in I} a_i \|E_i\|_{\varphi_p}^p\right)^{\frac{1}{p}}$ $(1 \le p < \infty)$, and $\|E\|_\infty = \sup_{i\in I} \|E_i\|_{\varphi_\infty}$. For $1 \le p \le \infty$, $\mathfrak{E}_p(I)$ is defined as the set of all $E \in \mathfrak{E}(I)$ for which $\|E\|_p < \infty$.

For a locally compact group G and a function $f : G \to \mathbb{C}$, $\check{f}$ is defined by $\check{f}(x) = f(x^{-1})$ $(x \in G)$. Let $A(G)$ (or $\mathfrak{K}(G)$, defined in 35.16 of [4]) consists of all functions h in $C_0(G)$ that can be written in at least one way as $\sum_{n=1}^\infty f_n * \check{g_n}$, where $f_n, g_n \in L^2(G)$, and $\sum_{n=1}^\infty \|f_n\|_2\|g_n\|_2 < \infty$. For $h \in A(G)$, define

$$\|h\|_{A(G)} = \inf\left\{\sum_{n=1}^\infty \|f_n\|_2\|g_n\|_2 : h = \sum_{n=1}^\infty f_n * \check{g_n}\right\}.$$

With this norm $A(G)$ is a Banach space. For more details see 35.16 of [4].

As [1], let $(A, \|.\|)$ be a normed algebra, and let $I_1, ..., I_n$ be ideals in A, then $I_1...I_n$ is an ideal in A; we transfer the projective norm from $I_1 \otimes ... \otimes I_n$ into $I_1.....I_n$. So that, for $A \in I_1.....I_n$, we have

$$\|a\|_\pi = \inf\Big\{\sum_{j=1}^m \|a_{1,j}\|....\|a_{n,j}\| \ ; \ a = \sum_{j=1}^m a_{1,j}.....a_{n,j}, \ a_{i,j} \in I_i\Big\}.$$

Clearly $\|.\|_\pi$ is an algebra norm on $I_1.....I_n$ with $\|a\| \le \|a\|_\pi$ $(a \in I_1...I_n)$; the norm $\|.\|_\pi$ is again called the projective norm. In particular, we may consider $\|.\|_\pi$ on A^2. Let A be a Banach algebra. Then the continuous linear map $\pi_A : A\hat{\otimes}A \to A$ such that $\pi_A(x \otimes y) = ab$ $(a, b \in A)$ is the projective induced product map and $I_\pi = ker\pi_A$. The quotient norm on the image $\pi_A(A\hat{\otimes}A) \cong \frac{(A\hat{\otimes}A)}{\ker \pi_A}$ is denoted by $|||.|||_\pi$, so that

$$|||a|||_\pi = \inf\Big\{\sum_{j=1}^\infty \|a_j\|\|b_j\| \ ; \ a = \sum_{j=1}^\infty a_j b_j\Big\}(\ a \in \pi_A(A\hat{\otimes}A).$$

Note that by $2 \cdot 1 \cdot 15$ of [1],

$$\|a\| \le |||a|||_\pi \le \|a\|_\pi \ (a \in A^2). \tag{1.1}$$

A normed algebra A has S-property (π-property) if there is a constant $C > 0$ such that

$$\|a\|_\pi \le c\|a\| \ (|||a|||_\pi \le c\|a\|) \ (a \in A^2).$$

Clearly, If A has S-property, then A has π-property. A Banach algebra A is biprojective if $\pi_A : A\hat{\otimes}A \longrightarrow A$ has a bounded right inverse as an A-*bimodule* homomorphism. By proposition 2.8.41 of [1], if A is biprojective then $\pi_A(A\hat{\otimes}A) = A$ and A has π-property.

2. Main Results

In this section, among other results, we obtain the necessary and sufficient conditions such that $\mathfrak{E}_p(I)$ for $p \geq 1$, has π-property and as a result we apply π-property of $\mathfrak{E}_p(I)$ to find the necessary and sufficient conditions for biprojectivity of $\mathfrak{E}_p(I)$.

Theorem 2.1. *Suppose that $p \geq 1$ and $A \in \mathfrak{E}_{\frac{p}{2}}(I)$. Then there are $B, C \in \mathfrak{E}_p(I)$ such that $A = B.C$ and $\|B\|_p = \|C\|_p = \|A\|_{\frac{p}{2}}^{\frac{1}{2}}$.*

Proof. First suppose $p \neq \infty$. By Notation D.26 (i) of [4], for $i \in I, |A_i|$ can be written uniquely in the form $|A_i| = \sum_{j=1}^n b_i^j Q_i^j$, where the b_i^js are distinct positive numbers and Q_j^is are projections onto pairwise orthogonal nonzero subspaces of H_i and $|A_i|^{\frac{1}{2}} = \sum_{j=1}^n (b_i^j)^{\frac{1}{2}} Q_i^j$. Therefore, $|A_i| = |A_i|^{\frac{1}{2}}.|A_i|^{\frac{1}{2}}$. For $i \in I$, according to the polar decomposition, there is $W_i \in \mathcal{U}(H_i)$ (the set of all unitary operators on H_i) such that

$$A_i = |A_i|.W_i = |A_i|^{\frac{1}{2}}.|A_i|^{\frac{1}{2}}.W_i.$$

Let $B_i = |A_i|^{\frac{1}{2}}$ and $C_i = |A_i|^{\frac{1}{2}}.W_i$. By Lemma 1.1 of [?]

$$\|B_i\|_{\varphi_p}^p = \||A_i|^{\frac{1}{2}}\|_{\varphi_p}^p = \|A_i\|_{\varphi_{\frac{p}{2}}}^{\frac{p}{2}},$$

therefore,

$$\|B\|_p = \Big(\sum_i a_i \|B_i\|_{\varphi_p}^p\Big)^{\frac{1}{p}} = \Big(\sum_i a_i \|A_i\|_{\varphi_{\frac{p}{2}}}^{\frac{p}{2}}\Big)^{\frac{2}{p}\cdot\frac{1}{2}} = \|A\|_{\frac{p}{2}}^{\frac{1}{2}} < \infty.$$

So $B \in \mathfrak{E}_p(I)$. The rest of the proof follows easily from Theorem D. 41 of [4] and Lemma 1.1 of [5]. For $p = \infty$ the proof is similar. □

Corollary 2.2. *If $p \geq 1$, then $\mathfrak{E}_{\frac{p}{2}}(I) \subseteq \mathfrak{E}_p(I)\mathfrak{E}_p(I)$.*

Let A be a Banach algebra. We set $A^{[2]} = A.A = \{ab : a, b \in A\}$ and $A^2 = linA^{[2]} = linA.A = \big\{\sum_{i=1}^n \alpha_i a_i b_i : \alpha_1, ..., \alpha_n \in \mathbb{C}, a_1, ..., a_n, b_1, ..., b_n \in A\big\}$.

Theorem 2.3. *If $p \geq 1$, then $\mathfrak{E}_p^2(I) = \mathfrak{E}_p^{[2]}(I) = \mathfrak{E}_{\frac{p}{2}}(I)$.*

Proof. It is enough to show that if $E,F \in \mathfrak{E}_p(I)$, then $EF \in \mathfrak{E}_{\frac{p}{2}}(I)$. By using Theorem $2 \cdot 3$ of [5] for $p = q$, and applying Hölder inequality, we obtain

$$\begin{aligned}
\|EF\|_{\frac{p}{2}}^{\frac{p}{2}} &= \textstyle\sum_i a_i \|(EF)_i\|_{\varphi_{\frac{p}{2}}}^{\frac{p}{2}} \\
&\le \textstyle\sum_i a_i \|E_i\|_{\varphi_p}^{\frac{p}{2}} \|F_i\|_{\varphi_p}^{\frac{p}{2}} \\
&\le \textstyle\sum_i a_i^{\frac{1}{2}} \|E_i\|_{\varphi_p}^{\frac{p}{2}} a_i^{\frac{1}{2}} \|F_i\|_{\varphi_p}^{\frac{p}{2}} \\
&\le \Big(\textstyle\sum_i a_i \|E_i\|_{\varphi_p}^{p}\Big)^{\frac{1}{2}} \Big(\textstyle\sum_i a_i \|F_i\|^{p})_{\varphi_p}\Big)^{\frac{1}{2}} \\
&= \Big(\|E\|_p^p\Big)^{\frac{1}{2}} \Big(\|F\|_p^p\Big)^{\frac{1}{2}} < \infty.
\end{aligned}$$

□

Theorem 2.4. *If $r > p \ge 1$, then $\mathfrak{E}_p(I) \trianglelefteq \mathfrak{E}_r(I)$.*

Proof. By Theorem 28.32 of [4], $\mathfrak{E}_p(I) \subseteq \mathfrak{E}_r(I)$. Let $A \in \mathfrak{E}_r(I)$ and $B \in \mathfrak{E}_p(I)$. For each $i \in I$, we denote the sequence of eigenvalues of A_i by $s_j(A_i)$. Now, if $A_i, B_i \in B(H_i)$, then by 2.2 and 2.3 of [3],

$$s_j(A_iB_i) \le \|A_i\|_{\varphi_\infty}.s_j(B_i),$$

$$s_j(B_iA_i) \le \|A_i\|_{\varphi_\infty}.s_j(B_i).$$

Thus

$$\|A_iB_i\|_{\varphi_p} = (\sum_j s_j(A_iB_i)^p)^{\frac{1}{p}} \le (\sum_j \|A_i\|_\infty^p.s_j(B_i)^p)^{\frac{1}{p}} = \|A_i\|_{\varphi_\infty}\|B_i\|_{\varphi_p}.$$

But $A \in \mathfrak{E}_r(I)$, hence $A \in \mathfrak{E}_\infty(I)$ and

$$\|AB\|_p^p = \sum_i a_i\|A_iB_i\|_{\varphi_p}^p \le \sum_i a_i\|A_i\|_{\varphi_\infty}^p\|B_i\|_{\varphi_p}^p \le \|A\|_\infty^p\|B\|_{\varphi_p}^p < \infty.$$

Therefore, $AB \in \mathfrak{E}_p(I)$ and the proof is complete. □

Corollary 2.5. *If $p \ge 1$, then $\mathfrak{E}_{\frac{p}{2}}(I) \trianglelefteq \mathfrak{E}_p(I)$.*

Let $\|.\|_{\pi,p}$ and $|||.|||_{\pi,p}$ be the projective norms on $\mathfrak{E}_p(I)\mathfrak{E}_p(I)$ and the quotient norm from $\mathfrak{E}_p(I)\hat{\otimes}\mathfrak{E}_p(I)$, respectively. Let

$$\mathcal{U}(\mathfrak{E}(I)) = \{(E_i)_{i\in I} \in \mathfrak{E}(I) : E_i \in \mathcal{U}(H_i)\},$$

$U, V \in \mathcal{U}(\mathfrak{E}(I))$ and $E \in \mathfrak{E}_p(I)$. By Theorem D.41 of [4], we have

$$\|VEU\|_p = (\sum_i a_i\|V_iE_iU_i\|_{\varphi_p}^p)^{\frac{1}{p}} = (\sum_i a_i\|E_i\|_{\varphi_p}^p)^{\frac{1}{p}} = \|E\|_p.$$

By polar decomposition, for $i \in I$, there is a unitary operator U_i such that $U_iE_i = |E_i|$. Let $U = (U_i)_{i\in I}$ then

$$\|E\|_{\pi,p} = \|UE\|_{\pi,p} = \|(|E_i|)_{i\in I}\|_{\pi,p}. \tag{2.1}$$

Since the square root of a matrix is hermitian, then is diagonalizable, i.e. there is a unitary operator V_i such that $V_i^{-1}|E_i|V_i = T_i$, where T_i is a diagonal matrix. Let $V = (V_i)_{i\in I}$. Then

$$\|(|E_i|)_{i\in I}\|_{\pi,p} = \|V|E|\|_{\pi,p} = \|(T_i)_{i\in I}\|_{\pi,p}.$$

By (2.1), $\|E\|_{\pi,p} = \|(T_i)_{i\in I}\|_{\pi,p}$. By the similar procedure, we can prove that $|||E|||_{\pi,p} = |||(T_i)_{i\in I}|||_{\pi,p}$. Consequently, for analyzing $\|.\|_{\pi,p}$ and $|||.|||_{\pi,p}$ it is enough to focus on $E = (E_i)_{i\in I}$ of $\mathfrak{E}_p(I)$, where each E_i is a diagonal matrix with positive diagonal entries.
For the rest of the section we set $\tilde{p} = \max\{1, \frac{p}{2}\}$.

Theorem 2.6. *Let $2 \leq p < \infty$. Then for each $E \in \mathfrak{E}_p^2(I)$,*

$$\|E\|_{\pi,p} = |||E|||_{\pi,p} = \|E\|_{\tilde{p}}.$$

Proof. Suppose $2 \leq p < \infty$ and $E \in \mathfrak{E}_p^2(I)$. By Theorem 2.3, $E \in \mathfrak{E}_{\frac{p}{2}}(I)$. Using Theorem 2.1, it follows that $\|E\|_{\pi,p} \leq \|E\|_{\tilde{p}}$. Also, if $E = \sum_{j=1}^{\infty} F^{(j)}K^{(j)}$ in $\mathfrak{E}_p(I)$ with $\sum_{j=1}^{\infty} \|F^{(j)}\|_p\|K^{(j)}\|_p < \infty$, then by Theorem $28 \cdot 3$ of [4], we have

$$\|E\|_{\tilde{p}} = \|E\|_{\frac{p}{2}} \leq \sum_{j=1}^{\infty} \|F^{(j)}K^{(j)}\|_{\frac{p}{2}} \leq \sum_{j=1}^{\infty} \|F^{(j)}\|_p\|K^{(j)}\|_p < \infty,$$

which results $\|E\|_{\tilde{p}} \leq |||E|||_{\pi,p}$. Then the result follows from (1.1). $\square$

Theorem 2.7. *$\|.\|_p$ and $\|.\|_{\tilde{p}}$ are equivalent if and only if $p = 1$ or I is finite.*

Proof. The sufficient condition is evident. Let

$$K\|.\|_p \leq \|.\|_{\tilde{p}} \leq M\|.\|_p, \tag{2.2}$$

for some $K, M > 0$, and $p \neq 1$. If $1 < p < 2$, then $\tilde{p} = 1$ and by (2.2), $\|.\|_1 \leq M\|.\|_p$, that implies $\mathfrak{E}_p(I) \subseteq \mathfrak{E}_1(I)$ which is contradict with Theorem 28.32 of [4]. We can repeat the same argument for the case $p \geq 2$. $\square$

For each $i \in I$, and $1 \leq m, n \leq d_i$, let ε_{mn} be the elementary $d_i \times d_i$-matrix such that for $1 \leq k, l \leq d_i$,

$$(\varepsilon_{mn})_{kl} = \begin{cases} 1 & \text{if } k = m, l = n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.8. *Let $1 \leq p < 2$. If $M = \sup_{i\in I} a_i < \infty$, then for each $E \in \mathfrak{E}^2_p(I)$*

$$\|E\|_{\tilde{p}} = \|E\|_1 \leq |||E|||_{\pi,p} \leq \|E\|_{\pi,p} \leq M\|E\|_{\tilde{p}} = M\|E\|_1$$

Proof. Suppose that $E = \sum_{j=1}^{\infty} F^{(j)}K^{(j)}$ in $\mathfrak{E}_p(I)$, where $\sum_{j=1}^{\infty} \|F^{(j)}\|_p\|K^{(j)}\|_p < \infty$. Then Hölder inequality and Theorem $28 \cdot 3$ of [4], imply that

$$\|E\|_1 \leq \sum_{j=1}^{\infty} \|F^{(j)}K^{(j)}\|_1 \leq \sum_{j=1}^{\infty} \|F^{(j)}\|_2\|K^{(j)}\|_2 \leq \sum_{j=1}^{\infty} \|F^{(j)}\|_p\|K^{(j)}\|_p < \infty.$$

Therefore, $\|E\|_1 = \|E\|_{\tilde{p}} \leq |||E|||_{\pi,p}$. Let $\delta_i : I \to \mathbb{R}$ be defined by $\delta_i(j) = 1$ if $i = j$ and $\delta_i(j) = 0$ if $i \neq j$. Then $(E_i)_{i\in I} = \sum_{j\in I} E_j\delta_j$ and

$$\|(E_i)_{i\in I}\|_{\pi,p} \leq \|\sum_{j\in I} E_j\delta_j\|_{\pi,p} \leq \sum_j \|E_j\delta_j\|_{\pi,p} \tag{2.3}$$

where

$$E_j = \begin{bmatrix} \lambda^j_1 & 0 & \dots & 0 \\ 0 & \lambda^j_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^j_{d_j} \end{bmatrix}.$$

This gives

$$E_j = \sum_{1\leq k\leq d_j} \lambda^j_k \varepsilon^j_{kk},$$

and

$$\sum_j \|E_j\delta_j\|_{\pi,p} \leq \sum_j \sum_{1\leq k\leq d_j} \|\lambda^j_k \varepsilon^j_{kk}\|_{\pi,p}.$$

In addition,

$$\lambda^j_k \varepsilon^j_{kk} = \lambda^j_k \varepsilon^j_{kk} \cdot \lambda^j_k \varepsilon^j_{kk},$$

so

$$\|\lambda^j_k ({}^j\varepsilon^i_{kk})_i\|_{\pi,p} \leq \lambda^j_k a_j^{\frac{1}{p}} a_j^{\frac{1}{p}}.$$

Combining the above two inequalities, we have

$$\sum_j \|E_j\delta_j\|_{\pi,p} \leq \sum_j \sum_{1\leq k\leq d_j} \lambda^j_k a_j^{\frac{1}{p}} a_j^{\frac{1}{p}}.$$

By using (2.3)

$$\|(E_i)_{i\in I}\|_{\pi,p} \leq \sum_j \sum_{1\leq k\leq d_j} \lambda^j_k a_j^{\frac{1}{p}} a_j^{\frac{1}{p}},$$

and moreover

$$\|E\|_1 = \sum_j a_j \sum_{1\leq k\leq d_j} \lambda^j_k.$$

Now, since

$$0 \leq \frac{2}{p} - 1 \leq 1 \Longrightarrow a_j^{\frac{2}{p}-1} \leq a_j \leq M,$$

we have

$$\|(E_i)_{i\in I}\|_{\pi,p} \leq \sum_j \sum_{1\leq k\leq d_j} \lambda_k^j a_j^{\frac{2}{p}-1} a_j \leq M \sum_j a_j \sum_{1\leq k\leq d_j} \lambda_k^j = M\|E\|_1,$$

and hence

$$\|E\|_1 \leq |||E|||_{\pi,p} \leq \|E\|_{\pi,p} \leq M\|E\|_1.$$

□

The following two corollaries follow from Theorem 2.6, Proposition 2.7 and Theorem 2.8.

Corollary 2.9. *Let $p \geq 2$. Then $\mathfrak{E}_p(I)$ has S-property if and only if I is finite.*

Corollary 2.10. *Let $1 \leq p < 2$ and $\sup_{i\in I} a_i < \infty$. Then $\mathfrak{E}_p(I)$ has S-property if and only if $p = 1$.*

Remark 2.11. The above two corollaries can be similarly proved for the case π-property.

3. Biprojectivity of $\mathfrak{E}_p(I)$

In the following proposition which the proof is straightforward, we use $\oplus_1$ to denote the ℓ_1-direct sum of Banach spaces.

Theorem 3.1. *If E_α (for $\alpha \in A$) and F_β (for $\beta \in B$) are Banach spaces, then*

$$(\oplus_1 E_\alpha)\hat{\otimes}(\oplus_1 F_\beta) = \oplus_1(E_\alpha \hat{\otimes} F_\beta)$$

From now on, we put $a_i = d_i$ for each $i \in I$. Let M_i stands for the algebra of $d_i \times d_i$ matrices with $\|T\| = d_i\|T\|_1 = d_i(trace(T^*T)^{\frac{1}{2}})$, and M_{ij} for the algebra of $d_id_j \times d_id_j$ matrices with $\|T\| = d_id_j\|T\|_1$. It is easy to see that $\oplus_1 M_i$ and $\mathfrak{E}_1(I)$ are isometric. Similarly by Proposition 3.1, $\mathfrak{E}_1(I)\hat{\otimes}\mathfrak{E}_1(I)$ and $\mathfrak{E}_1(I\times I)$ are isomeric with $\oplus_1(M_i\hat{\otimes}M_j)$ and $\oplus_1 M_{ij}$ respectively. The norm-decreasing maps $\rho_{i,j} : M_i\hat{\otimes}M_j \to M_{ij}$ give a norm-decreasing map $\rho : \mathfrak{E}_1(I)\hat{\otimes}\mathfrak{E}_1(I) \to \mathfrak{E}_1(I \times I)$.

Theorem 3.2. *If $\sup_{i\in I} d_i < \infty$, then $\mathfrak{E}_1(I)\hat{\otimes}\mathfrak{E}_1(I) = \mathfrak{E}_1(I \times I)$.*

Proof. Injectivity of ρ follows from injectivity of the corresponding map between $\oplus_1(M_i\hat{\otimes}M_j)$ and $\oplus_1 M_{ij}$. But $M_{i,j}$ may be realized, as a linear space, as $M_i\hat{\otimes}M_j$. Because these spaces are finite dimensional, the linear isomorphism between $M_{i,j}$ and $M_i\hat{\otimes}M_j$ is bounded with both bounds dependant only on the dimensions. Hence if the dimensions are bounded, then the maps between the ℓ_1-direct sums enjoy the same property. Therefore, ρ^{-1} exists and is bounded. □

Theorem 3.3. *The following assertions are equivalent.*
(i)$\mathfrak{E}_1(I)$ *is biprojective.*
(ii)$\mathfrak{E}_1(I)$ *is weakly amenable.*
(iii) $\sup_{i\in I} d_i < \infty$.

Proof. By 5.3.13 of [7], (i) implies (ii) and if $\mathfrak{E}_1(I)$ is weakly amenable, then by [8], $\sup_{i\in I} d_i < \infty$. Let $\sup_{i\in I} d_i < \infty$, then by Proposition 3.2, $\mathfrak{E}_1(I)\hat{\otimes}\mathfrak{E}_1(I) = \mathfrak{E}_1(I \times I)$. Define $\varrho : \mathfrak{E}_1(I) \longrightarrow \mathfrak{E}_1(I \times I)$ by $\varrho((E_i)) = (E_i\delta_{(i,i)})$. It is easy to check that ϱ is a bounded $\mathfrak{E}_1(I)$-bimodule morphism which is the right inverse for $\pi : \mathfrak{E}_1(I)\hat{\otimes}\mathfrak{E}_1(I) \longrightarrow \mathfrak{E}_1(I)$ and so $\mathfrak{E}_1(I)$ is biprojective. □

Corollary 3.4. *$\mathfrak{E}_p(I)$ is biprojective if and only if $p = 1$ and $\sup_{i\in I} d_i < \infty$ or I is finite.*

Proof. The sufficient condition is evident. Let $p = 1$ and $\sup_{i\in I} d_i < \infty$, then by Proposition 3.3, $\mathfrak{E}_1(I)$ is biprojective. Also it is evident that $\mathfrak{E}_p(I)$ is biprojective if I is finite. Now let $\mathfrak{E}_p(I)$ is biprojective. Since $\mathfrak{E}_p(I)$ has π-property, the result can be deduced from Corollary 2.9 and Corollary 2.10. □

4. APPLICATIONS

Let G be a compact group with dual $\widehat{G}$ (the set of all irreducible representations of G). Let H_π be the representation space of π for each $\pi \in \widehat{G}$. The algebras $\mathfrak{E}(\widehat{G})$ and $\mathfrak{E}_p(\widehat{G})$ for $p \in [1,\infty] \cup \{0\}$, are defined as mentioned above with each a_π equals to the dimension d_π of $\pi \in \widehat{G}$ (c.f Definition 28.34 of [4]).

A unitary representation π of G is *primary* if the center $C(\pi)$, i.e., the space of interwining operators of the representations π and π, is trivial. The group G is said to be of *type I* if every primary representation of G is a direct sum of copies of some irreducible representations (for complete discussion and proof of the following two theorem, see [2]).

Theorem 4.1. *Every compact group is of type I.*

Theorem 4.2. *If either G_1 or G_2 is of type I, then there exists a bijection between $\widehat{G_1} \times \widehat{G_2}$ and $\widehat{G_1 \times G_2}$.*

The following proposition is a consequence of Proposition 3.2, Theorem 4.1 and Theorem 4.2.

Theorem 4.3. *If $\sup_{\pi \in \widehat{G}} d_\pi < \infty$, then $\mathfrak{E}_1(\widehat{G})\hat{\otimes}\mathfrak{E}_1(\widehat{G}) = \mathfrak{E}_1(\widehat{G \times G})$.*

Corollary 4.4. *If $\sup_{\pi \in \widehat{G}} d_\pi < \infty$, then $A(G)\hat{\otimes}A(G) = A(G \times G)$.*

Proof. By Theorem 34.32 of [4], the convolution Banach algebra $A(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_1(\widehat{G})$. □

Remark 4.5. By Theorem 1. of [6], there is an integer M such that $d(\pi) \leq M$ for all $\pi \in \widehat{G}$ if and only if there is an open abelian subgroup of finite index in G.

Corollary 4.6. *If G has an open abelian subgroup of finite index, then $A(G)\hat{\otimes}A(G) = A(G \times G)$.*

Theorem 4.7. *Let G be a compact group. Then,*
(i) $(A(G), *)$ *is biprojective if and only if* $\sup_{\pi \in \widehat{G}} d_\pi < \infty$ *and if and only if* $(A(G), *)$ *is weakly amenable.*
(ii) $(L^2(G), *)$ *is biprojective if and only if G is finite.*

Proof. By above, $(A(G), *)$ is isometrically algebra isometric with $\mathfrak{E}_1(\widehat{G})$, also by 28.43 of [4](Weyl-Peter Theorem) $(L^2(G), *)$ is isometrically algebra isometric with $\mathfrak{E}_2(\widehat{G})$. □

Corollary 4.8. *Let G be a compact group. Then $(A(G), *)$ is biprojective if and only if there is an open abelian subgroup of finite index in G.*

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