

ON EVALUATIONS OF THE CUBIC CONTINUED FRACTION BY MODULAR EQUATIONS OF DEGREE 3

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ABSTRACT. We find modular equations of degree 3 to evaluate some new values of the cubic continued fraction $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{2 \cdot 4^m}{3}, \frac{1}{3 \cdot 4^m}$, and $\frac{2}{3 \cdot 4^m}$, where $m = 1, 2, 3$, or 4.

1. INTRODUCTION

Let, for $|q| < 1$,

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$$

and define

$$\chi(q) = (-q; q^2)_{\infty}.$$

The Ramanujan's cubic continued fraction $G(q)$, for $|q| < 1$, is defined by

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \dots,$$

which, in terms of $\chi(q)$, can be expressed as

$$G(q) = q^{1/3} \frac{\chi(-q)}{\chi^3(-q^3)}.$$

As mentioned in [6], in the mid 1990s some interesting numerical values of $G(q)$ were determined for $q = e^{-\pi\sqrt{n}}$ and $q = -e^{-\pi\sqrt{n}}$ with a positive rational number n . Berndt, Chan, and Zhang [3] found the values of $G(e^{-\pi\sqrt{n}})$ for $n = 2, 10, 22, 58$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5, 13, 37$ by employing Ramanujan's class invariants G_n and g_n such as

$$G_n = 2^{-1/4} q^{-1/24} \chi(q) \quad \text{and} \quad g_n = 2^{-1/4} q^{-1/24} \chi(-q),$$

Received by the editors November 18, 2017. Accepted February 23, 2018

2010 *Mathematics Subject Classification*. Primary 11F27, 33C90, Secondary 11F20, 33C05, 33C75.

Key words and phrases. continued fraction, modular equation, theta function.

*This work was supported by a grant from the Busan National University of Education (RC2016008).

where $q = e^{-\pi\sqrt{n}}$. Chan [4] applied some reciprocity theorems for $G(q)$ to compute some numerical values of $G(e^{-\pi\sqrt{n}})$ for $n = \frac{2}{9}, 1, 2, 4$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 5$. Meanwhile, in the early 2000s, Yi [7] used relations among $G(q)$, Ramanujan-Weber class invariants, and some parameters for eta function so that she systematically found the values of $G(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{1}{9}, \frac{4}{9}, 3, 6, 7, 8, 10, 12, 16, 28$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}, 2, 3, 4, 7$.

Recently, in [8], the values of $G(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3}, 1, 4, 9$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1, 4, 9$ were evaluated by applying some modular equations of degrees 3 and 9. Paek and Yi [5] evaluated $G(e^{-\pi\sqrt{n}})$ for $n = \frac{4}{3}, \frac{16}{3}, \frac{64}{3}, 36, 144, 324$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{4}{3}, \frac{16}{3}, 36$ by employing modular equations of degrees 3 and 9. Moreover, Paek and Yi [6] obtained the values of $G(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}, 1, 8, 16, 32, 64, 128, 256$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}, 8, 16, 32, 64$ by using modular equations of degree 9. The next table summarizes some known values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$.

	n
$G(e^{-\pi\sqrt{n}})$	$\frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{16}{3}, \frac{64}{3}, \frac{1}{4}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128},$ 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 16, 22, 28, 32, 36, 58, 64, 128, 144, 256, 324
$G(-e^{-\pi\sqrt{n}})$	$\frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{16}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128},$ 1, 2, 3, 4, 5, 7, 8, 9, 13, 16, 32, 36, 37, 64

In this paper, we further find some new values of $G(e^{-\pi\sqrt{n}})$ for $n = \frac{8}{3}, \frac{32}{3}, \frac{128}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192}, \frac{1}{384}$ and the values of $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{8}{3}, \frac{32}{3}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192}, \frac{1}{384}$ by using modular equations of degree 3. In addition, we show how to evaluate $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{2 \cdot 4^m}{3}, \frac{1}{3 \cdot 4^m},$ and $\frac{2}{3 \cdot 4^m}$, where m is a positive integer.

We now turn to definitions of Ramanujan's theta functions φ and ψ , for $|q| < 1$, such as

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Transcribing in terms of $(a; q)_{\infty}$ they can be written as

$$\varphi(q) = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} \quad \text{and} \quad \psi(q) = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}.$$

Let $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2) \dots (a+n-1)$ for each positive integer n .

Definition 1.1 ([1, Definition 5.1.1]). Let a , b , and c be arbitrary complex numbers except that c cannot be a non-positive integer. Then, for $|z| < 1$, the Gaussian or ordinary hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$

Definition 1.2 ([1, Definition 5.1.2]). The complete elliptic integral of the first kind is defined for $|k| < 1$ by

$$(1.1) \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The number k is called the *modulus* and the number $k' = \sqrt{1 - k^2}$ is called the *complementary modulus*.

Note that

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \varphi^2\left(e^{-\pi \frac{K'}{K}}\right),$$

where $0 < k < 1$ and $K' = K(k')$.

We now turn to the definition of a modular equation of degree n given in [5, 6, 8, 9]. Let K , K' , L , and L' denote complete elliptic integrals of the first kind associated with the moduli k , k' , l , and l' , respectively, where $0 < k < 1$ and $0 < l < 1$. Suppose that

$$(1.2) \quad \frac{L'}{L} = n \frac{K'}{K}$$

holds for some positive integer n . Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.2). Set $\alpha = k^2$ and $\beta = l^2$, then we say that β has degree n over α . In terms of complete elliptic integrals of the first kind and the terminology of hypergeometric functions, we conclude that a modular equation of degree n is an equation relating α and β that is induced by

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}.$$

Let $z_n = \varphi^2(q^n)$ and $m = \frac{z_1}{z_n}$. Then we call m the multiplier.

We end this section by recalling the definitions of the parameterizations $l_{k,n}$ and $l'_{k,n}$ for the theta function ψ from [5, 8, 9]. For any positive real numbers k and n , define $l_{k,n}$ and $l'_{k,n}$ by, for $q = e^{-\pi\sqrt{n/k}}$,

$$(1.3) \quad l_{k,n} = \frac{\psi(-q)}{k^{1/4} q^{(k-1)/8} \psi(-q^k)}$$

and

$$(1.4) \quad l'_{k,n} = \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}.$$

For brevity, throughout this paper we write l_n and l'_n instead of $l_{3,n}$ and $l'_{3,n}$, respectively.

2. PRELIMINARY RESULTS

In this section, we introduce basic theta function identities as in [5, 6, 8] to find modular equations of degree 3. Let k be the modulus as in (1.1). Set

$$(2.1) \quad x := k^2 = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}.$$

Then, by [1, Theorem 5.2.8], we have

$$(2.2) \quad z := \varphi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

where

$$(2.3) \quad q := \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-k^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)}\right) = \exp\left(-\pi \frac{K(k')}{K(k)}\right).$$

Theorem 2.1 ([1, Theorem 5.4.2]). *If x , q , and z are related by (2.1), (2.2), and (2.3), then*

$$(i) \quad \psi(q) = \sqrt{\frac{1}{2}}z \left(\frac{x}{q}\right)^{1/8},$$

$$(ii) \quad \psi(q^2) = \frac{1}{2}\sqrt{z} \left(\frac{x}{q}\right)^{1/4}.$$

The next result exhibits formulas for the values of $G(e^{-\pi\sqrt{n/3}})$ and $G(-e^{-\pi\sqrt{n/3}})$ in terms of l'_n and l_n , respectively.

Theorem 2.2 ([9, Theorem 6.2(v)]). *For any positive real number n , we have*

$$(i) \quad G^3(e^{-\pi\sqrt{n/3}}) = \frac{1}{3l_n^4 - 1},$$

$$(ii) \quad G^3(-e^{-\pi\sqrt{n/3}}) = \frac{-1}{3l_n^4 + 1}.$$

Theorem 2.3 ([7, Lemma 6.3.6]). *We have*

$$G(e^{-2\pi\sqrt{n}}) = -G(e^{-\pi\sqrt{n}})G(-e^{-\pi\sqrt{n}})$$

for any positive real number n .

3. MODULAR EQUATIONS OF DEGREE 3

In this section, we first derive a couple of modular equations of degree 3. We then apply them to find a relation between l'_n and l'_{4n} and another relation between l_n and l_{4n} .

Theorem 3.1. *If $P = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$ and $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$, then*

$$(3.1) \quad P^4(Q^2 - 1) = Q^2(Q^2 + 3).$$

Proof. By Theorem 2.1,

$$P = \sqrt{\frac{z_1}{z_3}} \left(\frac{\alpha}{\beta}\right)^{1/8} \quad \text{and} \quad Q = \sqrt{\frac{z_1}{z_3}} \left(\frac{\alpha}{\beta}\right)^{1/4},$$

where β has degree 3 over α . Thus we deduce that

$$\sqrt{\frac{z_1}{z_3}} = \frac{P^2}{Q} \quad \text{and} \quad \left(\frac{\alpha}{\beta}\right)^{1/8} = \frac{Q}{P}.$$

From the proof of Entry 5(iv) from [2, Chapter 19], we find that

$$\beta = \frac{(m-1)^3(m+3)}{16m} \quad \text{and} \quad \alpha = \frac{(m-1)(m+3)^3}{16m^3},$$

where $m = \frac{z_1}{z_3}$. Hence we deduce that

$$\sqrt{\frac{\beta}{\alpha}} = \frac{m(m-1)}{m+3}.$$

Transcribing the last equation in terms of P and Q , we have

$$\frac{1}{Q^2} = \frac{P^4 - Q^2}{P^4 + 3Q^2}.$$

The desired result now follows by rearranging the terms. \square

Under the same hypothesis as Theorem 3.1, we have another modular equation of the form

$$P^4 + \frac{9}{P^4} = \left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + 8.$$

See [5, Theorem 3.1] for details.

Corollary 3.2. *For every positive real number n , we have*

$$(3.2) \quad l_n'^4(\sqrt{3}l_{4n}'^2 - 1) = l_{4n}'^2(l_{4n}'^2 + \sqrt{3}).$$

Proof. Set $q = e^{-\pi\sqrt{n/3}}$ in (1.4), then $P = 3^{1/4}l'_n$, $Q = 3^{1/4}l'_{4n}$ from Theorem 3.1. Rewriting (3.1) in terms of l'_n and l'_{4n} , we complete the proof. \square

Theorem 3.3. *If $P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$, then*

$$(3.3) \quad P^4(Q^2 + 1) = Q^2(Q^2 - 3).$$

Proof. Let $R = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$. Then, by Theorem 3.1,

$$R^4(Q^2 - 1) = Q^2(Q^2 + 3).$$

Replacing q by $-q$, then R^4 , Q^2 are converted into $-P^4$, $-Q^2$, respectively. Now substituting in the last equality R^4 by $-P^4$ and Q^2 by $-Q^2$, we complete the proof. \square

Corollary 3.4. *For every positive real number n , we have*

$$(3.4) \quad l_n^4(\sqrt{3}l_{4n}^2 + 1) = l_{4n}^2(l_{4n}^2 - \sqrt{3}).$$

Proof. The proof of this is similar to that of Corollary 3.2. \square

4. EVALUATIONS OF l'_n AND l_n

We now evaluate $l'_{2.4^m}$, $l'_{1/4^m}$ and $l'_{2/4^m}$ for $m = 1, 2$, and 3 . We begin with $l'_{2.4^m}$ for $m = 1, 2$, and 3 .

Theorem 4.1. *We have*

- (i) $l_8^2 = (1 + \sqrt{2})(\sqrt{2} + \sqrt{3})$,
- (ii) $l_{32}^2 = (1 + \sqrt{2})^2 \left(3\sqrt{2} + \sqrt{3} + \sqrt{6} + 2\sqrt{3 + 4\sqrt{2} + 5\sqrt{3}} \right)$,
- (iii) $l_{128}^2 = \frac{\sqrt{3}(a-1) + \sqrt{(3a-1)(a-3)}}{2}$,

where

$$a = (1 + \sqrt{2})^4 \left(3\sqrt{2} + \sqrt{3} + \sqrt{6} + 2\sqrt{3 + 4\sqrt{2} + 5\sqrt{3}} \right)^2.$$

Proof. For (i), let $n = 2$ in (3.2) and put $l'_2 = \sqrt{1 + \sqrt{2}}$ from [9, Theorem 3.4(iii)], then we deduce that

$$(3 + 2\sqrt{2})(\sqrt{3}l_8^2 - 1) = l_8^2(l_8^2 + \sqrt{3}).$$

Since $l_8^2 > 1$, by solving last equation for l_8^2 , we complete the proof.

For (ii), by letting $n = 8$ in (3.2) and putting the value of l_8^2 from the result of (i), we complete the proof. For (iii), repeat the same argument as in the proof of (ii). \square

Note that Corollary 3.2 and Theorem 4.1 show that $l_{2 \cdot 4^m}^2$ can be evaluated for $m = 4, 5, 6, \dots$. Next we evaluate $l_{1/4^m}^4$ for $m = 1, 2$, and 3.

Theorem 4.2. *We have*

$$(i) \quad l_{1/4}^4 = \frac{4 + 3\sqrt{2}}{1 + \sqrt{3}},$$

$$(ii) \quad l_{1/16}^4 = \frac{4 + 3\sqrt{2} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}},$$

$$(iii) \quad l_{1/64}^4 = \frac{b(b + \sqrt{3})}{\sqrt{3}b - 1},$$

where

$$b = \sqrt{\frac{4 + 3\sqrt{2} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}}.$$

Proof. For (i), letting $n = \frac{1}{4}$ in (3.2) and putting $l_1' = \sqrt{\frac{1+\sqrt{3}}{\sqrt{2}}}$ from [9, Theorem 3.3(iii)], we deduce that

$$\left(-1 + \sqrt{3(2 + \sqrt{3})}\right) l_{1/4}^4 = \sqrt{2 + \sqrt{3}} \left(\sqrt{3} + \sqrt{2 + \sqrt{3}}\right).$$

The result follows after a brief calculation.

For (ii), letting $n = \frac{1}{16}$ in (3.2) and putting the value of $l_{1/4}'$ from the result of (i), we complete the proof. For (iii), the proof of this is similar to that of (ii). \square

Note that Corollary 3.2 and Theorem 4.2 show that $l_{1/4^m}^4$ can be evaluated for $m = 4, 5, 6, \dots$. We now determine $l_{2/4^m}^4$ for $m = 1, 2, 3$, and 4.

Theorem 4.3. *We have*

$$(i) \quad l_{1/2}^4 = \sqrt{2} + \sqrt{3},$$

$$(ii) \quad l_{1/8}^4 = \frac{\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}}}{\sqrt{3} - \sqrt{\sqrt{3} - \sqrt{2}}},$$

$$(iii) \quad l_{1/32}^4 = \frac{\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}}{-\sqrt{3} + \sqrt{\sqrt{3} - \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}},$$

$$(iv) \ l_{1/128}^4 = \frac{c(c + \sqrt{3})}{\sqrt{3}c - 1},$$

where

$$c = \sqrt{\frac{\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}}{-\sqrt{3} + \sqrt{\sqrt{3} - \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}}.$$

Proof. For (i), letting $n = \frac{1}{2}$ in (3.2) and putting the value of $l'_2 = \sqrt{1 + \sqrt{2}}$ from [9, Theorem 3.4(iii)], we deduce that

$$(-1 + \sqrt{3} + \sqrt{6})l_{1/2}^4 = (1 + \sqrt{2})(1 + \sqrt{2} + \sqrt{3}).$$

The result follows after an elementary calculation.

For (ii), letting $n = \frac{1}{8}$ in (3.2) and putting the value of $l'_{1/2}$ from the result of (i), we obtain the required result. For (iii) and (iv), the proofs of these are exactly the same as that of (ii). \square

Note also that from Corollary 3.2 and Theorem 4.3, we can further determine $l_{2/4^m}^4$ for $m = 5, 6, 7, \dots$

We now determine $l_{1/4^m}^4$ for $m = 1, 2$, and 3.

Theorem 4.4. *We have*

$$(i) \ l_{1/4}^4 = \frac{(1 + \sqrt{3})(1 + \sqrt{3} - \sqrt{6})}{2 + 3\sqrt{2} + \sqrt{6}},$$

$$(ii) \ l_{1/16}^4 = \frac{4 + 3\sqrt{2} - \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{1 + \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}},$$

$$(iii) \ l_{1/64}^4 = \frac{b(b - \sqrt{3})}{\sqrt{3}b + 1},$$

where

$$b = \sqrt{\frac{4 + 3\sqrt{2} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}}.$$

Proof. For (i), let $n = \frac{1}{4}$ in (3.4) and put the value of $l'_1 = \sqrt{\frac{1 + \sqrt{3}}{\sqrt{2}}}$ from [9, Theorem 3.3(iii)], then we find that

$$\left(\frac{3 + \sqrt{3}}{\sqrt{2}} + 1\right)l_{1/4}^4 = \frac{1 + \sqrt{3}}{\sqrt{2}} \left(\frac{1 + \sqrt{3}}{\sqrt{2}} - \sqrt{3}\right).$$

The result follows after a straightforward calculation.

For (ii), letting $n = \frac{1}{16}$ in (3.4) and putting the value of $l_{1/4}^2$ from the result of Theorem 4.2(i), we complete the proof. For (iii), the proof of this is exactly the same as that of (ii). \square

Note also that we can further determine $l_{1/4^m}^4$ for $m = 4, 5, 6, \dots$ from Corollary 3.4 and Theorem 4.2.

5. EVALUATIONS OF $G(q)$

We are ready to evaluate $G^3(e^{-\pi\sqrt{n}})$ and $G^3(-e^{-\pi\sqrt{n}})$ for $n = \frac{2 \cdot 4^m}{3}, \frac{1}{3 \cdot 4^m}$, and $\frac{2}{3 \cdot 4^m}$, where $m = 1, 2, 3$, or 4. By taking cube roots of them, the corresponding values of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ can easily be obtained. We first determine $G^3(e^{-\pi\sqrt{n}})$ for $n = \frac{2 \cdot 4^m}{3}$, where $m = 1, 2$, and 3.

Theorem 5.1. *We have*

$$\begin{aligned} \text{(i)} \quad G^3(e^{-2\sqrt{2}\pi/\sqrt{3}}) &= \frac{1}{2(22 + 15\sqrt{2} + 12\sqrt{3} + 9\sqrt{6})}, \\ \text{(ii)} \quad G^3(e^{-4\sqrt{2}\pi/\sqrt{3}}) &= -\frac{1}{4} + \frac{3(1 + \sqrt{2} + \sqrt{6})}{2\left(6 + \sqrt{2} + 3\sqrt{6} + 2\sqrt{3(3 + 4\sqrt{2} + 5\sqrt{3})}\right)}, \\ \text{(iii)} \quad G^3(e^{-8\sqrt{2}\pi/\sqrt{3}}) &= \frac{4}{3\left(\sqrt{3}(a-1) + \sqrt{(3a-1)(a-3)}\right)^2 - 4}, \end{aligned}$$

where

$$a = (1 + \sqrt{2})^4 \left(3\sqrt{2} + \sqrt{3} + \sqrt{6} + 2\sqrt{3 + 4\sqrt{2} + 5\sqrt{3}}\right)^2.$$

Proof. For (i), letting $n = 8$ in Theorem 2.2(i) and putting the value of l_8^2 from Theorem 4.1(i), we obtain the result.

For (ii) and (iii), apply the same argument as in the proof of (i). \square

We next determine $G^3(-e^{-\pi\sqrt{n}})$ for $n = \frac{2 \cdot 4^m}{3}$, where $m = 1$ and 2.

Corollary 5.2. *We have*

$$\begin{aligned} \text{(i)} \quad G^3(-e^{-2\sqrt{2}\pi/\sqrt{3}}) &= \frac{(9 - 7\sqrt{2} - 9\sqrt{3} + 6\sqrt{6}) \left(5 + 3\sqrt{3} - \sqrt{6(3 + 4\sqrt{2} + 5\sqrt{3})}\right)}{(-24 + 17\sqrt{2}) \left(1 + 3\sqrt{2} + 3\sqrt{3} + \sqrt{6(3 + 4\sqrt{2} + 5\sqrt{3})}\right)}, \end{aligned}$$

$$(ii) \ G^3(-e^{-4\sqrt{2}\pi/\sqrt{3}}) \\ = \frac{4 - 3(2 + \sqrt{2})^4 \left(3\sqrt{2} + \sqrt{3} + \sqrt{6} + 2\sqrt{3 + 4\sqrt{2} + 5\sqrt{3}}\right)^2}{3 \left(\sqrt{3}(a-1) + \sqrt{(3a-1)(a-3)}\right)^2 - 4},$$

where

$$a = (1 + \sqrt{2})^4 \left(3\sqrt{2} + \sqrt{3} + \sqrt{6} + 2\sqrt{3 + 4\sqrt{2} + 5\sqrt{3}}\right)^2.$$

Proof. The results are immediate consequences of Theorem 2.3 and Theorem 5.1. \square

We now evaluate $G^3(e^{-\pi\sqrt{n}})$ and $G^3(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3 \cdot 4^m}$, where $m = 1, 2$, and 3.

Theorem 5.3. *We have*

$$(i) \ G^3(e^{-\pi/2\sqrt{3}}) = \frac{1 + \sqrt{3}}{11 + 9\sqrt{2} - \sqrt{3}}, \\ (ii) \ G^3(e^{-\pi/4\sqrt{3}}) = \frac{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{13 + 9\sqrt{2} + \sqrt{3} + 2\sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}, \\ (iii) \ G^3(e^{-\pi/8\sqrt{3}}) = \frac{\sqrt{3}b - 1}{(\sqrt{3}b + 1)^2},$$

where

$$b = \sqrt{\frac{4 + 3\sqrt{2} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}}.$$

Proof. For (i), letting $n = \frac{1}{4}$ in Theorem 2.2(i) and putting the value of $l_{1/4}^4$ from Theorem 4.2(i), we complete the proof.

For (ii) and (iii), repeat the same process as in the proof of (i). \square

Theorem 5.4. *We have*

$$(i) \ G^3(-e^{-\pi/2\sqrt{3}}) = \frac{3 + \sqrt{2} + \sqrt{3}}{6 - 7\sqrt{2} + 2\sqrt{3} - 3\sqrt{6}}, \\ (ii) \ G^3(-e^{-\pi/4\sqrt{3}}) = -\frac{1 + \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{13 + 9\sqrt{2} + \sqrt{3} - 2\sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}, \\ (iii) \ G^3(-e^{-\pi/8\sqrt{3}}) = -\frac{\sqrt{3}b + 1}{(\sqrt{3}b - 1)^2},$$

where

$$b = \sqrt{\frac{4 + 3\sqrt{2} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}{-1 - \sqrt{3} + \sqrt{3(4 + 3\sqrt{2})(1 + \sqrt{3})}}}$$

Proof. For (i), letting $n = \frac{1}{4}$ in Theorem 2.2(ii) and putting the value of $l_{1/4}^4$ from Theorem 4.4(i), we complete the proof.

For (ii) and (iii), employing Theorem 2.2(ii) and Theorem 4.4(ii) and (iii), we finish the proof. \square

Note that we easily obtain Theorem 5.4(ii) and (iii) by Theorem 2.3 and Theorem 5.3.

We now find $G^3(e^{-\pi\sqrt{n}})$ for $n = \frac{2}{3 \cdot 4^m}$, where $m = 1, 2, 3$, and 4.

Theorem 5.5. *We have*

$$(i) \quad G^3(e^{-\pi/\sqrt{6}}) = \frac{1}{-1 + 3(\sqrt{2} + \sqrt{3})},$$

$$(ii) \quad G^3(e^{-\pi/2\sqrt{6}}) = \frac{3 + \sqrt{2} - \sqrt{3}}{9 - \sqrt{2} + \sqrt{3} + 3\sqrt{6} + 6\sqrt{3}},$$

$$(iii) \quad G^3(e^{-\pi/4\sqrt{6}}) = \frac{-\sqrt{3} + \sqrt{\sqrt{3} - \sqrt{2}} + \sqrt{6 + 3\sqrt{6}\sqrt{3} - 6}}{4\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} - \sqrt{\sqrt{3} - \sqrt{2}} + 2\sqrt{6 + 3\sqrt{6}\sqrt{3} - 6}},$$

$$(iv) \quad G^3(e^{-\pi/8\sqrt{6}}) = \frac{\sqrt{3}c - 1}{(\sqrt{3}c + 1)^2},$$

where

$$c = \sqrt{\frac{\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} + \sqrt{6 + 3\sqrt{6}\sqrt{3} - 6}}{-\sqrt{3} + \sqrt{\sqrt{3} - \sqrt{2}} + \sqrt{6 + 3\sqrt{6}\sqrt{3} - 6}}}$$

Proof. For (i), letting $n = \frac{1}{2}$ in Theorem 2.2(i) and putting the value of $l_{1/2}^4$ from Theorem 4.3(i), we complete the proof.

The proofs of (ii), (iii), and (iv) are similar to that of (i). \square

We lastly use Theorem 2.3 and Theorem 5.5 to find $G^3(-e^{-\pi\sqrt{n}})$ for $n = \frac{2}{3 \cdot 4^m}$, where $m = 2, 3$, and 4.

Corollary 5.6. *We have*

$$(i) \quad G^3(-e^{-\pi/2\sqrt{6}}) = \frac{9 - \sqrt{2} + \sqrt{3} + 3\sqrt{6} + 6\sqrt{3}}{2(3 - 4\sqrt{2} - 5\sqrt{3})},$$

$$(ii) \ G^3(-e^{-\pi/4\sqrt{6}}) \\ = \frac{(3 + \sqrt{2} - \sqrt{3}) \left(4\sqrt{3} + 3\sqrt{\sqrt{3} + \sqrt{2}} - \sqrt{\sqrt{3} - \sqrt{2}} + 2\sqrt{6 + 3\sqrt{6\sqrt{3} - 6}} \right)}{(9 - \sqrt{2} + \sqrt{3} + 3\sqrt{6 + 6\sqrt{3}}) \left(\sqrt{3} - \sqrt{\sqrt{3} - \sqrt{2}} - \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}} \right)},$$

$$(iii) \ G^3(-e^{-\pi/8\sqrt{6}}) \\ = \frac{\left(\sqrt{3} - \sqrt{\sqrt{3} - \sqrt{2}} - \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}} \right) (\sqrt{3}c + 1)^2}{\left(4\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} - \sqrt{\sqrt{3} - \sqrt{2}} + 2\sqrt{6 + 3\sqrt{6\sqrt{3} - 6}} \right) (\sqrt{3}c - 1)},$$

where

$$c = \sqrt{\frac{\sqrt{3} + \sqrt{\sqrt{3} + \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}}{-\sqrt{3} + \sqrt{\sqrt{3} - \sqrt{2}} + \sqrt{6 + 3\sqrt{6\sqrt{3} - 6}}}.$$

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