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# THE PRIOR SET TAKING A MAXIMAL SCENARIO IN THE REPRESENTATION OF COHERENT RISK MEASURE

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Abstract. It is proved that 'maximum' is actually attained in the following risk measure representation

$$\rho_m(X) = \max_{Q\in\mathcal{Q}_m} E_Q[-X].$$

## 1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space. Kim [5] showed that the set of priors in the representation of Choquet expectation [2] is the one of equivalent martingale measures under some conditions, when the distortion is submodular. That is, if a capacity $c$ is submodular, then the coherent risk measure is represented as

$$\rho(X) := \int X\,dc = \max_{Q\in\mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T), \tag{1.1}$$

where $\mathcal{Q}_c$ is defined as

$$\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} : Q[A] \le c(A) \quad \forall A \in \mathcal{F}_T\} \tag{1.2}$$

that is equal to the maximal set $\mathcal{Q}_{max}$ representing $\rho$. The set $\mathcal{Q}_{max}$ is defined in (1.7). Here $\mathcal{M}_1 := \mathcal{M}_1(\Omega, \mathcal{F})$ is the set of all probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{M}_{1,f} := \mathcal{M}_{1,f}(\Omega, \mathcal{F})$ is the set of all finitely additive normalized set functions $Q : \mathcal{F} \to [0, 1]$. $E_Q[X]$ is denoted by the integral of $X$ with respect to $Q \in \mathcal{M}_{1,f}$.

By using $g$-expectation [7] and related topics [1, 9], Kim [5] showed that $Q_c$ equals to $\mathcal{Q}^\theta$ where $\mathcal{Q}^\theta$ and $\Theta^g$ are respectively defined as

$$\mathcal{Q}^\theta := \left\{Q^\theta : \theta \in \Theta^g,\ \frac{dQ^\theta}{dP}\Big|_{\mathcal{F}_t} = exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right)\right\} \tag{1.3}$$

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and

(1.4) $\Theta^g = \{(\theta_t)_{t\in[0,T]} :\quad \theta \text{ is } \mathbb{R} - \text{valued, progressively measurable } \& \; |\theta_t| \le \nu_t\},$

for a continuous function $\nu_t$ for $t \in [0,T]$.

We consider the Banach spaces $L^p(\Omega, \mathcal{F}, P)$ for $1 \le p < \infty$. Let $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and define

$$\mathcal{M}_1^q(P) := \left\{ Q \in \mathcal{M}_{1,f}(P) \,\middle|\, \frac{dQ}{dP} \in L^q \right\}.$$

It is well-known in the literature [4, 6, 8] that the coherent risk measures $\rho_m$ defined as

$$\rho_m(X) := \int_{(0,1]} AV@R_\lambda(X)\, m(d\lambda)$$

for $m$ which is a probability measure defined on $(0,1]$, can be expressed as Choquet expectation and consequently

(1.5) $$\rho_m(X) = \sup_{Q\in\mathcal{Q}_m} E_Q[-X],$$

where the set $\mathcal{Q}_m$ is defined as

(1.6)
$$\mathcal{Q}_m := \left\{ Q \in \mathcal{M}_1^2(P) \,\middle|\, \varphi := \frac{dQ}{dP} \text{ satisfies } \int_t^1 q_\varphi(s)ds \le \psi(1-t) \text{ for } t \in (0,1) \right\},$$

and $q_\varphi$ is a quantile function, $\psi$ is a increasing concave functions $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$.

The Average Value at Risk at level $\lambda \in (0,1]$ of a position $X$ is defined as

$$AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\lambda(X)d\gamma.$$

In those papers, it is shown that $\mathcal{Q}_{c_\psi} = \mathcal{Q}_{max} = \mathcal{Q}_m$ as maximal representing set,

(1.7) $$\mathcal{Q}_{max} = \{Q \in \mathcal{M}_1^2(P) \,|\, \alpha_{min}(Q) = 0\}.$$

The minimal penalty function $\alpha_{min}$ is defined as

$$\alpha_{min}(Q) := \sup_{X\in\mathcal{A}_\rho} E_Q[-X] \quad \text{ for } Q \in \mathcal{M}_1^2(P),$$

where $\mathcal{A}_\rho$ is the acceptance set of $\rho$ on a measurable set $\mathcal{X}$ defined as

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \,|\, \rho(X) \le 0\}.$$

It was not clearly shown that '*Maximum*' is attained in (1.5). We will show in this paper that '*Maximum*' is actually taken in (1.5).

## 2. Weakly compact set

We will show that the set of densities

$$\mathcal{D} := \left\{ \frac{dQ}{dP} \,\Big|\, Q \in \mathcal{Q}_m \right\} \tag{2.1}$$

is weakly compact subset in $L^2(\Omega, \mathcal{F}, P)$.

The following theorem and proposition are well-known( See [3]).

**Theorem 2.1.** *Let $C$ be a nonempty convex set in a Banach space. Then $C$ is strongly closed if and only if it is weakly closed.*

**Proposition 2.2.** *Let $Y$ be a normed space. A sequence $(Z_n)_{n\in\mathbb{N}}$ converges weakly to $Z$ in $Y$ if and only if*

$$\forall f \in Y^* \quad f(Z_n) \to f(Z) \text{ as } n \to \infty,$$

*where $Y^*$ is the set of all continuous linear functionals defined on $Y$.*

**Proposition 2.3.** *The subset $\mathcal{D}$ is a convex set and is strongly closed. So it is weakly closed.*

*Proof.* If $Q \in \mathcal{D}$, then $Q \in \mathcal{M}_1^2(P)$ by the definition of $\mathcal{Q}_m$, and so $\frac{dQ}{dP} \in L^2$. Thus we have $\mathcal{D} \subset L^2(\Omega, \mathcal{F}, P)$.

Let $Q_1, Q_2 \in \mathcal{D}$. Let $\lambda \in [0, 1]$. Then it holds that $d(\lambda Q_1 + (1-\lambda)Q_2)/dP \in L^2$ by the Minkowski inequality. We also have

$$\begin{aligned} \alpha_{min}(\lambda Q_1 + (1-\lambda)Q_2) &:= \sup_{X\in\mathcal{A}_\rho} E_{\lambda Q_1+(1-\lambda)Q_2}[-X] \\ &= \lambda \sup_{X\in\mathcal{A}_\rho} E_{Q_1}[-X] + (1-\lambda) \sup_{X\in\mathcal{A}_\rho} E_{Q_1}[-X] = 0. \end{aligned}$$

Hence $\lambda Q_1 + (1-\lambda)Q_2 \in \mathcal{D}$ and so $\mathcal{D}$ is convex set.

Let $Z_n = dQ_n/dP \in \mathcal{D}$ be a sequences converging to $Z = dQ/dP$ in $L^2$. Since $L^2$ is a Banach space, $Z \in L^2$. By the Hölder's inequality, we have

$$0 \le |E[XZ_n] - E[XZ]| = |E[X(Z_n - Z)]| \le \|X\|_{L^2}\|Z_n - Z\|_{L^2} \to 0 \text{ as } n \to \infty.$$

Thus it follows that

$$0 = \alpha_{min}(Q_n) := \sup_{X\in\mathcal{A}_\rho} E_{Q_n}[-X] \to \sup_{X\in\mathcal{A}_\rho} E_Q[-X] =: \alpha_{min}(Q) \text{ as } n \to \infty.$$

Therefore, $\mathcal{D}$ is strongly closed and so weakly closed by Theorem 2.1. Thus $\mathcal{D}$ of $L^2$ is a weakly compact set. □

## 3. The Main Results

In this section, the main theorem is given and proved.

**Theorem 3.1** (James' Theorem)**.** *A weakly closed subset $C$ of a Banach space $B$ is weakly compact if and only if each continuous linear functional on $B$ attains a maximum on $C$.*

*Proof.* See [3]. □

**Theorem 3.2.** *'Maximum' is actually taken in the following equation*

$$\rho_m(X) = \max_{Q \in \mathcal{Q}_m} E_Q[-X].$$

*Proof.* For $X \in L^2(\Omega, \mathcal{F}, P)$, define the linear functional $J_X$ as

$$J_X(Z) := E[-XZ] \quad \forall Z \in \mathcal{D}.$$

By the Hölder's inequality, we have

$$|J_X(Z)| \leq E[|XZ|] \leq \left(\int |X|^2 dP\right)^{1/2} \cdot \left(\int |Z|^2 dP\right)^{1/2} < +\infty.$$

Thus the set $J_X$ is bounded and so continuous on $L^2(\Omega, \mathcal{F}, P)$.

By Theorem 3.1, the linear functional $J_X$ attains a maximum on $\mathcal{D}$. □

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