J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2018.25.1.7 ISSN(Online) 2287-6081
Volume 25, Number 1 (February 2018), Pages 7–16

A NOTE ON CERTAIN LAPLACE TRANSFORMS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION $_3F_3$

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Abstract. The main objective of this paper is to demonstrate how one can obtain very quickly so far unknown Laplace transforms of rather general cases of the generalized hypergeometric function $_3F_3$ by employing generalizations of classical summation theorems for the series $_3F_2$ available in the literature. Several new as well known results obtained earlier by Kim *et al.* follow special cases of main findings.

1. Introduction

The generalizations of almost all elementary functions, popularly known as the generalized hypergeometric function $_pF_q$ with p numeratorial and q denominatorial parameters is defined by ([1, 11, 12, 13])

$$(1.1)\quad {}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;x] = {}_pF_q\left[\begin{matrix}a_1, & \cdots, & a_p\\ b_1, & \cdots, & b_q\end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\frac{x^n}{n!}$$

where, for its convergence

(i) $p \le q$ and $|z| < \infty$ or
(ii) $p = q+1$ and $|z| < 1$ or
(iii) $p = q+1$, $|z| = 1$ and $Re(w) > 0$,

where

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

Received by the editors September 18, 2017. Accepted February 23, 2018
2010 *Mathematics Subject Classification.* Primary 33C20, Secondary 33C05.
Key words and phrases. hypergeometric functions, classical summation theorems, integral representation for $_2F_1$.
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Also, $(a)_n$ denotes the well known Pochhammer symbol (or the raised or the shifted factorial, since $(1)_n = n!$) defined in terms of the Gamma function by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0; a \in \mathbb{C}/\{0\}) \\ a(a+1)\cdots(a+n-1) & (n \in \mathbb{N}; a \in \mathbb{C}). \end{cases}$$

For $p = 2$ and $q = 1$, (1.1) reduces to the well known Gauss's hypergeometric function ${}_2F_1$.

It is interesting to mention here that whenever generalized hypergeometric function ${}_pF_q$ or hypergeometric function ${}_2F_1$ reduce to gamma functions, the results are very important from the applications point of view. Thus the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon and Whipple for the series ${}_3F_2$ and others play a key role.

Recently, good progress has been made in generalizing the above mentioned classical summation theorems. For this, during 1992-1996, in a series of three research papers, Lavoie, *et al.* ([8, 9, 10]) generalized the above mentioned classical summation theorems and obtained a large number of very interesting contiguous results. However, in our present investigations, we mention the following generalized summation theorems due to Lavoie, *et al.* ([8, 9, 10])

Generalized Watson's summation theorem [8]

$$(1.2) \qquad {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1) & 2c+j & \end{matrix} ; 1\right]$$

$$= A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{1}{2}(a+b+i+1))\Gamma(c+[\frac{j}{2}]+\frac{1}{2})\Gamma(c-\frac{1}{2}(a+b+|i+j|-j-1))}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{B_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{b}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}+[\frac{j}{2}]-\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{b}{2}+\frac{1}{2}+[\frac{j}{2}])}\right.$$

$$\left.+C_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))\Gamma(\frac{b}{2}+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+[\frac{j+1}{2}]+\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{b}{2}+[\frac{j+1}{2}])}\right\}.$$

provided $\mathrm{Re}(a+b-2c) < i+2j+1$ with $i, j = 0, \pm1, \pm2$. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are the same as given in the tables in [8].

For $i = j = 0$, we recover the following classical Watson's summation theorem [1].

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1) & 2c \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})\Gamma(c-\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{b}{2}+\frac{1}{2})} \tag{1.3}
$$

provided $\mathrm{Re}(2c-a-b) > -1$.

Generalized Dixon's summation theorem [9]

$$
\begin{aligned}
{}_3F_2&\left[\begin{matrix} a, & b, & c \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}; 1\right] \\
&= \frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma(b-\frac{1}{2}i+\frac{1}{2}|i|)\Gamma(c-\frac{1}{2}(i+j+|i+j|))}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)}. \\
&\times\left\{D_{i,j}\frac{\Gamma(\frac{1}{2}a-c+\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(\frac{1}{2}a-b-c+1+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b+1+[\frac{i}{2}])}\right. \\
&\left.+E_{i,j}\frac{\Gamma(\frac{1}{2}a-c+1+[\frac{i+j}{2}])\Gamma(\frac{1}{2}a-b-c+\frac{3}{2}+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b+\frac{1}{2}+[\frac{i+1}{2}])}\right\}.
\end{aligned} \tag{1.4}
$$

provided $\mathrm{Re}(a-2b-2c) > -2-2i-j$ with $i = -3,-2,-1,0,1,2$; $j = 0,1,2,3$. The coefficients $D_{i,j}$ and $E_{i,j}$ are the same as given in the tables in [9].

For $i = j = 0$, we get the following classical Dixon's summation theorem [1].

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(1+a-b-c)}. \tag{1.5}
$$

provided $\mathrm{Re}(a-2b-2c) > -2$.

Generalized Whipple's summation theorem [10]

$$
\begin{aligned}
{}_3F_2&\left[\begin{matrix} a, & b, & c \\ e, & f \end{matrix}; 1\right] \\
&= \frac{\Gamma(e)\Gamma(f)\Gamma(c-\frac{1}{2}(j+|j|))\Gamma(e-c-\frac{1}{2}(i+|i|))\Gamma(a-\frac{1}{2}(i+j+|i+j|))}{2^{2a-i-j}\Gamma(e-a)\Gamma(f-a)\Gamma(e-c)\Gamma(a)\Gamma(c)} \\
&\times\left\{F_{i,j}\frac{\Gamma(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{1}{2}f-\frac{1}{2}a)}{\Gamma(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}i+[\frac{-j}{2}])\Gamma(\frac{1}{2}f+\frac{1}{2}a-\frac{1}{2}i+\frac{1}{4}(-1)^j((-1)^i-1)+[\frac{-j}{2}])}\right. \\
&\left. G_{i,j}\frac{\Gamma(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))\Gamma(\frac{1}{2}f-\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}-\frac{1}{2}i+[\frac{-j+1}{2}])\Gamma(\frac{1}{2}f+\frac{1}{2}a-\frac{1}{2}i-\frac{1}{2}+\frac{1}{4}(-1)^j(1-(-1)^i)+[\frac{-j+1}{2}])}\right\}.
\end{aligned} \tag{1.6}
$$

for $i, j = 0, \pm1, \pm2, \pm3$ with $a+b = 1+i+j$ and $e+f = 1+2c+i$. The coefficients $F_{i,j}$ and $G_{i,j}$ are the same as given in the tables in [10].

For $i = j = 0$, we get the following classical Whipple's summation theorem [1].

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ e, & f & \end{matrix}; 1\right] = \frac{2^{1-2c}\Gamma(e)\Gamma(f)}{\Gamma(\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}a+\frac{1}{2}f)\Gamma(\frac{1}{2}b+\frac{1}{2}e)\Gamma(\frac{1}{2}b+\frac{1}{2}f)} \tag{1.7} $$

with $a + b = 1$ and $e + f = 2c + 1$ provided $\mathrm{Re}(c) > 0$, $\mathrm{Re}(s) > 0$, where $s = e + f - a - b - c$.

Remark 1.1. In the results (1.2), (1.4) and (1.6), $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$.

Next, we define the Laplace transform of a function $f(t)$ of real variable t denoted by $g(s)$ or $L\{f(t); s\}$ over a range of complex parameter s by the integral

$$ g(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt $$

provided the integral exists in the Lebesgue sense.

For more details about the Laplace transforms, we refer the standard texts ([3, 4, 5]). We also mention the following well known general result of the Laplace transform of generalized hypergeometric function ${}_pF_q$ recorded in [5].

$$ \begin{aligned} &\int_0^\infty e^{-st} t^{\mu-1} {}_pF_q\left[\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}; wt\right] dt \\ &= s^{-\mu}\Gamma(\mu) {}_{p+1}F_q\left[\begin{matrix} \mu, & a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q & \end{matrix}; \frac{w}{s}\right]. \end{aligned} \tag{1.8} $$

provided $p < q$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(s) > 0$ and w arbitrary or $p = q > 0$, $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(s) > \mathrm{Re}(w)$.

Here, we would like to mention the outlines of the proof of the general result (1.8). For this, expressing ${}_pF_q$ as a series, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series, evaluating the resulting integral with the help of the well known integral

$$ \int_0^\infty e^{-st} t^{\lambda-1} dt = \frac{\Gamma(\lambda)}{s^\lambda}, \tag{1.9} $$

provided $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(\lambda) > 0$ and finally summing up the series.

In our present investigation, we are interested in the following special case of (1.8):

$$ \int_0^\infty e^{-st} t^{d-1} {}_3F_3\left[\begin{matrix} a, & b, & c \\ d, & e, & f \end{matrix}; wt\right] dt = s^{-d}\Gamma(d) {}_3F_2\left[\begin{matrix} a, & b, & c \\ e, & f & \end{matrix}; \frac{w}{s}\right]. \tag{1.10} $$

Using a result obtained from (1.8) for $p = q = 1$, in 2011, Kim *et al.* [6] obtained explicit expressions of

(i)

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1 \left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) & \end{matrix} ; \frac{1}{2}ts \right] dt,$$

(ii)

$$\int_0^\infty e^{-st} t^{-a+i} {}_1F_1 \left[\begin{matrix} a \\ c \end{matrix} ; \frac{1}{2}ts \right] dt$$

and

(iii)

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1 \left[\begin{matrix} a \\ 1+a-b+i \end{matrix} ; -ts \right] dt$$

for $i = 0, \pm 1, \cdots, \pm 5$,

by employing generalized Gauss second, Bailey and Kummer summation theorems recorded in [6]. Recently, Choi and Rathie [2] obtained these results for any $i \in \mathbb{Z}$

By employing generalized Watson summation theorem (1.2), generalized Dixon summation theorem (1.4) and generalized Whipple summation theorem (1.6), very recently, Kim, *et al.* [7] have obtained three new and general Laplace transforms for the generalized hypergeometric function ${}_2F_2(x)$.

The main objective of this paper is to demonstrate how one can obtain very quickly so far unknown Laplace transforms of rather three general cases of the generalized hypergeometric functions ${}_3F_3$ by employing the generalizations of Watson, Dixon and Whipple summation theorems (1.2), (1.4) and (1.6). Several new as well as known results obtained earlier by Kim, *et al.* [7] follow special cases of our main findings. The results established in this paper are simple, interesting, easily established and may be potentially useful in theoretical physics, engineering and mathematics.

2. Three General Laplace Transforms involving ${}_3F_3[a, b, c; d, e, f; x]$

The three general Laplace transform involving generalized hypergeometric functions ${}_3F_3(x)$ to be established in this section are given in the following theorems.

Theorem 2.1. *For* $Re(d) > 0$, $Re(s) > 0$ *and* $Re(a + b - 2c) < i + 2j + 1$ *with* $i, j = 0, \pm 1, \pm 2$, *the following results hold.*

(2.1)

$$\int_0^\infty e^{-st}t^{d-1}{}_3F_3\begin{bmatrix}a, & b, & c \\ d, & \frac{1}{2}(a+b+i+1), & 2c+j\end{bmatrix}; \quad st\Bigg] dt$$

$$=\Gamma(d)s^{-d}A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{1}{2}(a+b+i+1))\Gamma(c+[\frac{j}{2}]+\frac{1}{2})\Gamma(c-\frac{1}{2}(a+b+|i+j|+j+1))}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\Bigg\{B_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}+[\frac{j}{2}]-\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{1}{2}b+\frac{1}{2}+[\frac{j}{2}])}$$

$$+C_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i))\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+[\frac{j+1}{2}]+\frac{1}{4}(-1)^j(1-(-1)^i))\Gamma(c-\frac{1}{2}b+[\frac{j+1}{2}])}\Bigg\}.$$

$$=\Gamma(d)s^{-d}\Omega_1.$$

Theorem 2.2. *For $Re(d)>0$, $Re(s)>0$ and $Re(a-2b-2c)>-2-2i-2j$ with $i=0,\pm1,\pm2$, $j=0,1,2,3$, the following results hold.*

(2.2)

$$\int_0^\infty e^{-st}t^{d-1}{}_3F_3\begin{bmatrix}a, & b, & c \\ d, & 1+a-b+i, & 1+a-c+i+j\end{bmatrix}; \quad st\Bigg] dt$$

$$=\Gamma(d)s^{-d}\frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma(b-\frac{1}{2}i+\frac{1}{2}|i|)\Gamma(c-\frac{1}{2}(i+j+|i+j|))}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)}$$

$$\times\Bigg\{D_{i,j}\frac{\Gamma(\frac{1}{2}a-c+\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(\frac{1}{2}a-b-c+1+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b+1+[\frac{i}{2}])}$$

$$+E_{i,j}\frac{\Gamma(\frac{1}{2}a-c+1+[\frac{i+j}{2}])\Gamma(\frac{1}{2}a-b-c+\frac{3}{2}+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b+\frac{1}{2}+[\frac{j+1}{2}])}\Bigg\}$$

$$=\Gamma(d)s^{-d}\Omega_2.$$

Theorem 2.3. *For $Re(s)>0$, $Re(f)>0$ and $Re(e+f-a-b-c)>0$ with $a+b=1+i$ and $e+f=2c+1+i+j$, for $i,j=0,\pm1,\pm2,\pm3$, the following results hold.*

(2.3)

$$\int_0^\infty e^{-st}t^{f-1}{}_3F_3\begin{bmatrix}a, & b, & c \\ d, & e, & f\end{bmatrix}; \quad st\Bigg] dt$$

$$=\Gamma(f)s^{-f}\frac{2^{-2a+i+j}\Gamma(e)\Gamma(d)\Gamma(c-\frac{1}{2}(j+|j|))\Gamma(d-c-\frac{1}{2}(i+|i|))\Gamma(a-\frac{1}{2}(i+j+|i+j|))}{\Gamma(d-a)\Gamma(e-a)\Gamma(d-c)\Gamma(a)\Gamma(c)}$$

$$\times\Bigg\{F_{i,j}\frac{\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{4}(1-(-1)^i))\Gamma(\frac{1}{2}c-\frac{1}{2}a)}{\Gamma(\frac{1}{2}d+\frac{1}{2}a-\frac{1}{2}i+[\frac{-j}{2}])\Gamma(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}i+\frac{1}{4}(-1)^j((-1)^i-1)+[\frac{-j}{2}])}$$

$$+G_{i,j}\frac{\Gamma(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2}(1+(-1)^i))\Gamma(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}d+\frac{1}{2}a-\frac{1}{2}-\frac{1}{2}i+[\frac{-j+1}{2}])\Gamma(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}i-\frac{1}{2}+\frac{1}{4}(-1)^j(1-(-1)^i)+[\frac{-j+1}{2}])}\Bigg\}$$
$$=\Gamma(f)s^{-f}\Omega_3.$$

Proof. The proofs of Theorem 2.1 to Theorem 2.3 are quite straight forward. For this, in order to prove (2.1) we proceed as follows. In the result (1.10), if we take $e=\frac{1}{2}(a+b+i+1)$ and $f=2c+j$, $w=s$, then $i,j=0,\pm1,\pm2$, we have

$$\int_0^\infty e^{-st}t^{d-1}{}_3F_3\left[\begin{matrix}a, & b, & c\\ d, & \frac{1}{2}(a+b+i+1), & 2c+j\end{matrix};\ 1\right]dt \tag{2.4}$$
$$=s^{-d}\Gamma(d){}_3F_2\left[\begin{matrix}a, & b, & c\\ \frac{1}{2}(a+b+i+1), & 2c+j\end{matrix};\ 1\right]dt.$$

We now observe that the ${}_3F_2$ appearing on the right-hand side of (2.4) can be evaluated with the help of the generalized Watson summation theorem (1.2), and we immediately arrive at the right-hand side of (2.1). This completes the proof of the result (2.1) given in Theorem 2.1. In exactly the same manner, the result (2.2) and (2.3) given in Theorems 2.2 and 2.3 can be proved with the help of the results (1.4) and (1.6), respectively. □

3. Special Cases

In this section, we shall mention some of the very interesting known as well as new results of our main findings which are also general in nature.

Corollary 3.1. *In (2.1), if we take $d=a$, we get the following general result.*

$$\int_0^\infty e^{-st}t^{a-1}{}_2F_2\left[\begin{matrix}b, & c\\ \frac{1}{2}(a+b+i+1), & 2c+j\end{matrix};\ st\right]dt=\Gamma(a)s^{-a}\Omega_1, \tag{3.1}$$

where Ω_1 is the same as given in (2.1).

Corollary 3.2. *In (2.1), if we take $d=c$, we get the following general result.*

$$\int_0^\infty e^{-st}t^{c-1}{}_2F_2\left[\begin{matrix}a, & b\\ \frac{1}{2}(a+b+i+1), & 2c+j\end{matrix};\ \frac{1}{2}st\right]dt=\Gamma(c)s^{-c}\Omega_1, \tag{3.2}$$

where Ω_1 is the same as given in (2.1).

Remark 3.3. The result (3.2) was obtained earlier by Kim, *et al.* ([6], Eqn. (2.8), p. 250).

Corollary 3.4. *In (2.2), if we take $d=a$, we get the following general result.*

$$(3.3)\qquad \int_0^\infty e^{-st}t^{a-1}{}_2F_2\left[\begin{matrix} b, & c \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}; \quad st\right]dt = \Gamma(a)s^{-a}\Omega_2,$$

where Ω_2 *is the same as given in (2.2).*

Corollary 3.5. *In (2.2), if we take* $d = b$*, we get the following general result.*

$$(3.4)\qquad \int_0^\infty e^{-st}t^{b-1}{}_2F_2\left[\begin{matrix} a, & c \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}; \quad st\right]dt = \Gamma(b)s^{-b}\Omega_2,$$

where Ω_2 *is the same as given in (2.2).*

Corollary 3.6. *In (2.2), if we take* $d = c$*, we get the following general result.*

$$(3.5)\qquad \int_0^\infty e^{-st}t^{c-1}{}_2F_2\left[\begin{matrix} a, & b \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}; \quad st\right]dt = \Gamma(c)s^{-c}\Omega_2,$$

where Ω_2 *is the same as given in (2.2).*

Remark 3.7. The result (3.5) was obatained by Kim, *et al.* ([6], Eqn. (2.9), p.250).

Corollary 3.8. *In (2.3), if we take* $f = a$*, we get the following general result.*

$$(3.6)\qquad \int_0^\infty e^{-st}t^{a-1}{}_2F_2\left[\begin{matrix} b, & c \\ d, & e \end{matrix}; \quad st\right]dt = \Gamma(a)s^{-a}\Omega_3,$$

where Ω_3 *is the same as given in (2.3).*

Corollary 3.9. *In (2.3), if we take* $f = b$*, we get the following general result.*

$$(3.7)\qquad \int_0^\infty e^{-st}t^{b-1}{}_2F_2\left[\begin{matrix} a, & c \\ d, & e \end{matrix}; \quad st\right]dt = \Gamma(b)s^{-b}\Omega_3,$$

where Ω_3 *is the same as given in (2.3).*

Corollary 3.10. *In (2.3), if we take* $f = c$*, we get the following general result.*

$$(3.8)\qquad \int_0^\infty e^{-st}t^{c-1}{}_2F_2\left[\begin{matrix} a, & b \\ d, & e \end{matrix}; \quad st\right]dt = \Gamma(c)s^{-c}\Omega_3,$$

where Ω_3 *is the same as given in (2.3).*

Remark 3.11. The result (3.8) was obtained earlier by Kim, *et al.* ([6], Eqn. (2.10), p. 250).

The general results (3.1), (3.3), (3.4), (3.6) and (3.7) are also believed to be new. Similarly, other results can be obtained.

4. Concluding Remark

In this paper, an attempt has been made to obtain systematically all the possible cases of the Laplace transforms for the generalized hypergeometric functions ${}_3F_3$ and in particular ${}_2F_2$ by employing generalizations of classical summation theorem such as those of Watson, Dixon and Whipple obtained earlier by Lavoie, *et al.* ([8, 9, 10]). The result given in this paper are simple, interesting, easily established and may be useful in theoretical physics, engineering and mathematics.

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