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SOME OPERATOR INEQUALITIES INVOLVING IMPROVED YOUNG AND HEINZ INEQUALITIES

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Abstract. In this work, by applying the binomial expansion, some refinements of the Young and Heinz inequalities are proved. As an application, a determinant inequality for positive definite matrices is obtained. Also, some operator inequalities around the Young's inequality for semidefinite invertible matrices are proved.

1. Introduction

The classical Young's inequality for non-negative real numbers says that if a and b are non-negative and $0 \leq r \leq 1$, then

$$a^r b^{1-r} \leq ra + (1-r)b$$

with equality if and only if $a = b$.
If $r = \frac{1}{2}$, we obtain the arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Here, by the following inequality a short proof for the Young's inequality is given,

$$e^{x-1} \geq x \quad x \in \mathbb{R}.$$

The proof of the Young's inequality:
Take $L = ra + (1-r)b$ then,

$$e^{\frac{a}{L}-1} \geq \frac{a}{L} \geq 0,$$

$$e^{\frac{b}{L}-1} \geq \frac{}{L} \geq 0.$$

So, we have

$$1 = e^{r(\frac{a}{L}-1)+(1-r)(\frac{b}{L}-1)} \geq (\frac{a}{L})^r \times (\frac{b}{L})^{1-r}.$$

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Hence,

$$a^r b^{1-r} \leq ra + (1-r)b.$$

Kittaneh and Manasrah obtained a refinement of Young's inequality as the following[5]:

$$a^r b^{1-r} \leq ra + (1-r)b - r_0(\sqrt{a} - \sqrt{b})^2 \tag{1.1}$$

where, $r_0 = \min\{r, 1-r\}$.
The Heinz's means are defined as

$$H_r(a,b) = \frac{a^r b^{1-r} + a^{1-r} b^r}{2}$$

where a, b and r has the same conditions of the Young's inequality.
The Heinz's inequality asserts that

$$\sqrt{ab} \leq H_r(a,b) \leq \frac{a+b}{2}.$$

Kittaneh and Manasrah also obtained a refinement of the Heinz's inequality as follows:

$$H_r(a,b) \leq \frac{a+b}{2} - r_0(\sqrt{a} - \sqrt{b})^2$$

where, $r_0 = \min\{r, 1-r\}$.

In this work, by applying the binomial expansion we prove some refinements of the Young and Heinz inequalities and as an application, a determinant inequality is proved by these inequalities. Also, some operator inequalities around the Young's inequality are obtained.

Theorem 1.1 (Newton's generalized binomial theorem)**.** *If x and y are real numbers with $|x| > |y|$ and r is any complex number, then*

$$\begin{aligned}(x+y)^r &= \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \\ &= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots .\end{aligned}$$

2. Main Results

In the following, at first a refinement of the Young's inequality is given. Then a refinement of the Heinz's inequality is proved by it.

Theorem 2.1. *Let a and b be positive numbers and $0 \leq r \leq 1$.*
(i) If $\frac{b}{2} < a < b$ then,

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{2}\frac{(b-a)^2}{a} + \frac{r(r-1)(r-2)}{6}\frac{(b-a)^3}{a^2}. \tag{2.1}$$

(ii) If $a < \frac{b}{2}$ then,

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{2}\frac{(b-a)^2}{b} - \frac{r(1-r^2)}{6}\frac{(b-a)^3}{b^2}. \tag{2.2}$$

Proof. By taking $y = \frac{b}{a}$, we have $y = 1 + x$. We consider two cases:
Case 1: $x < 1$.
By Theorem 1.1 we have

$$\begin{aligned}\left(\frac{b}{a}\right)^r &= y^r = (1+x)^r \\ &= 1 + rx + \frac{r(r-1)}{2}x^2 + \frac{r(r-1)(r-2)}{6}x^3 + \cdots .\end{aligned}$$

For $0 < r < 1$, after the second term this series is alternative. So, we have

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{2}\frac{(b-a)^2}{a} + \frac{r(r-1)(r-2)}{6}\frac{(b-a)^3}{a^2}.$$

Note that the condition $\frac{b}{2} < a < b$ ensures that this inequality is a refinement of the Young's inequality.
Case 2: $x > 1$.
In this case, by taking $y = \frac{a}{b} = 1 - t$, we have $\frac{1}{2} < t < 1$ and so

$$\begin{aligned}y^{1-r} = \left(\frac{a}{b}\right)^{1-r} &= (1-t)^{1-r} \\ &= 1 + (r-1)t + \frac{r(r-1)}{2}t^2 - \frac{r(1-r^2)}{6}t^3 + \cdots .\end{aligned}$$

Since all terms of the series after the first element are negative we have

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{2}\frac{(b-a)^2}{b} - \frac{r(1-r^2)}{6}\frac{(b-a)^3}{b^2}.$$

□

As a conclusion a refinement of the Heinz inequality is given.

Corollary 2.2. *Let a and b be positive numbers and $0 \leq r \leq 1$.*
(i) If $\frac{b}{2} < a < b$ then,

$$H_r(a,b) \leq \frac{a+b}{2} + \frac{r(r-1)}{2}\frac{(b-a)^2}{a} - \frac{r(r-1)}{4}\frac{(b-a)^3}{a^2}.$$

(ii) If $a < \frac{b}{2}$ *then,*

$$H_r(a,b) \leq \frac{a+b}{2} + \frac{r(r-1)}{2}\frac{(b-a)^2}{b} + \frac{r(r-1)}{4}\frac{(b-a)^3}{b^2}.$$

Proof. In (2.1) and (2.2) applying 1-r instead of r, two inequalities are obtained. By adding and dividing by 2, these inequalities are proved. □

In the following, another refinement of the Young's inequality is given. At first, this lemma is needed.

Lemma 2.3. *If* $r \in [0,1]$ *and* $x \in [-1,1]$ *then,*

$$(1+x)^r \leq 1 + rx + \frac{r(r-1)}{8}x^2.$$

Proof. By taking $f(x) = (1+x)^r - 1 - rx - \frac{r(r-1)}{8}x^2$ we have

$$f(0) = 0 \quad , f'(x) = r(1+x)^{r-1} - r - \frac{r(r-1)}{4}x$$

and

$$f''(x) = \frac{r(r-1)}{4}[4(1+x)^{r-2} - 1] < 0.$$

Consequently, f' decreases on the interval $(-1,1)$. Thus, $f'(x) > 0$ for $x \in (-1,0)$ and $f'(x) < 0$ for $x \in (0,1)$. It then follows that f attains its minimum at zero. Since, $f(0) = 0$, we have $f(x) \leq 0$ for $x \in [-1,1]$. □

Theorem 2.4. *Let* a *and* b *be positive numbers and* $0 \leq r \leq 1$.
(i) If $\frac{b}{2} < a < b$ *then,*

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{8}\frac{(b-a)^2}{a}. \tag{2.3}$$

(ii) If $a < \frac{b}{2}$ *then,*

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{8}\frac{(b-a)^2}{b}. \tag{2.4}$$

Proof. *(i)* If $\frac{b}{2} < a < b$ then, by taking $y = \frac{b}{a} = 1 + x$, we have $x \in (0,1)$. Hence, applying Lemma 2.3, the following inequality is obtained.

$$(\frac{b}{a})^r = (1+x)^r \leq 1 + r(\frac{b}{a} - 1) + \frac{r(r-1)}{8}(\frac{b}{a} - 1)^2.$$

So we have

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{8}\frac{(b-a)^2}{a}.$$

(ii) If $a < \frac{b}{2}$ then, by taking $y = \frac{a}{b} = 1 - t$, we have $t \in (\frac{1}{2}, 1)$. So, by Lemma 2.3

$$y^{1-r} = (\frac{a}{b})^{1-r} = (1-t)^{1-r} \leq 1 + (r-1)t + \frac{r(r-1)}{8}t^2.$$

This implies that

$$a^{1-r}b^r \leq (1-r)a + rb + \frac{r(r-1)}{8}\frac{(b-a)^2}{b}.$$

□

Corollary 2.5. *Let a and b be positive numbers and $0 \leq r \leq 1$.*
(i) If $\frac{b}{2} < a < b$ then,

$$H_r(a,b) \leq \frac{a+b}{2} + \frac{r(r-1)}{8}\frac{(b-a)^2}{a}.$$

(ii) If $a < \frac{b}{2}$ then,

$$H_r(a,b) \leq \frac{a+b}{2} + \frac{r(r-1)}{8}\frac{(b-a)^2}{b}.$$

Young's inequality in operator algebras has been considered in [2] and references therein. A determinant version of Young's inequality is also known ([4], P. 467):

$$\det(A^r B^{1-r}) \leq \det(rA + (1-r)B).$$

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. Recently, Kittaneh and Manasrah, by inequality (1.1) for $A, B \in M_n(\mathbb{C})$ which are positive definite, prove that

$$\det(A^r B^{1-r}) + r_0^n \det(A + B - 2A\sharp B) \leq \det(rA + (1-r)B)$$

where, $r_0 = \min\{r, 1-r\}$ and $A\sharp B = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}$ is the geometric mean of A and B.
In the following, as an application of Theorem 2.4, we prove the inequality for positive definite matrices.

Theorem 2.6. *Let $A, B \in M_n(\mathbb{C})$ be positive definite and $0 \leq r \leq 1$, if $\frac{1}{2}A < B < A$ then*

$$\det(A^r B^{1-r}) + \Big(\frac{r(r-1)}{8}\Big)^n \det(AB^{-1}A + B - 2A) \leq \det(rA + (1-r)B) \tag{2.5}$$

Also, if $B < \frac{A}{2}$ then

$$\det(A^r B^{1-r}) + \Big(\frac{r(r-1)}{8}\Big)^n \det(BA^{-1}B + A - 2B) \leq \det(rA + (1-r)B) \tag{2.6}$$

Proof. (*i*): If $\frac{1}{2}A < B < A$, then we have $\frac{1}{2}B^{-\frac{1}{2}}AB^{-\frac{1}{2}} < I < B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. So by inequality (2.3)

$$rs_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) + (1-r) \geq s_j^r(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) + \frac{r(1-r)}{8}(s_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) - 1)^2,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A. This implies that

$$\begin{aligned}
&\det\Big(rB^{-\frac{1}{2}}AB^{-\frac{1}{2}} + (1-r)I\Big) \\
&= \prod_{j=1}^{n}\Big(rs_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + 1 - r\Big) \\
&\geq \prod_{j=1}^{n} s_j^r(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) + \frac{r(1-r)}{8}\Big(s_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) - 1\Big)^2 \\
&\geq \prod_{j=1}^{n} s_j^r(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) + \Big(\frac{r(1-r)}{8}\Big)^n \prod_{j=1}^{n}\Big(s_j(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) - 1\Big)^2 \\
&= \det(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^r + \Big(\frac{r(1-r)}{8}\Big)^n \det(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - I)^2.
\end{aligned}$$

By multiplying $\det(B)$, the proof of inequality (2.5) is completed.
(*ii*) If $B < \frac{A}{2}$, then we have $\frac{I}{2} > A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. So by inequality (2.4) and similar to (i), the inequality (2.6) is proved. □

The Hilbert-Schmidt norm of a matrix $A = [a_{ij}] \in M_n(\mathbb{C})$ is defined by

$$\|A\|_2 = \Big(\sum_{i,j=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}.$$

This norm is unitarilly invariant, i.e.,

$$\|UAV\|_2 = \|A\|_2$$

for all unitary matrices $U, V \in M_n(\mathbb{C})$. By using the singular value decomposition of A, we have

$$\|A\|_2 = \Big(\sum_{j=1}^{n} s_j^2(A)\Big)^{\frac{1}{2}},$$

where, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A (the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$). In the following, the operator form of Theorem 2.4 will be given.

Theorem 2.7. *Suppose that $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite invertible matrices and $r \in [0, 1]$.*

(*i*) *If* $B < 2A < 2B$, *then*

$$\|A^{1-r}XB^r\|_2^2 \leq (1-r)\|AX\|_2^2+r\|XB\|_2^2+\frac{r(r-1)}{8}\left(\|A^{-1}XB^2\|_2^2+\|AX\|_2^2-2\|XB\|_2^2\right).$$

(*ii*) *If* $2A < B$, *then*

$$\|A^{1-r}XB^r\|_2^2 \leq (1-r)\|AX\|_2^2+r\|XB\|_2^2+\frac{r(r-1)}{8}\left(\|B^{-1}XA^2\|_2^2+\|XB\|_2^2-2\|AX\|_2^2\right).$$

Proof. There are unitary matrices U and V such that $A = U\Lambda U^*$ and $B = VMV^*$, so that

$$\lambda = diag(\lambda_1, \cdots, \lambda_n),$$

$$M = diag(m_1, \cdots, m_n)$$

where, $\lambda_i, m_j \geq 0$. Let $Y = U^*XV = [y_{ij}]$. Then

$$\|A^{1-r}XB^r\|_2^2 = \|U[\lambda_i{}^{1-r}y_{ij}m_j]V^*\|_2^2$$

$$= \sum_{i,j=1}^{n} (\lambda_i^2)^{1-r}(m_j^2)^r|y_{ij}|^2$$

$$\leq \sum_{i,j=1}^{n} (1-r)(\lambda_i^2)|y_{ij}|^2 + rm_j^2|y_{ij}|^2 + \frac{r(r-1)}{8}\frac{(m_j^2-\lambda_i^2)^2}{\lambda_i^2}|y_{ij}|^2.$$

On the other hand, since $AX = U\Lambda YV^*$ and $XB = UYMV^*$ and $A^{-1}XB^2 = U\Lambda^{-1}YM^2V^*$, we have

$$\|AX\|_2^2 = \|U\Lambda YV^*\|_2^2 = \|\Lambda Y\|_2^2 = \sum_{i,j=1}^{n} (\lambda_i^2)|y_{ij}|^2,$$

$$\|XB\|_2^2 = \|UYMV^*\|_2^2 = \|YM\|_2^2 = \sum_{i,j=1}^{n} m_i^2|y_{ij}|^2,$$

$$\|A^{-1}XB^2\| = \|U\Lambda^{-1}YM^2V^*\| = \sum_{i,j=1}^{n} \frac{m_j^4|y_{ij}|^2}{\lambda_i^2}.$$

By using the above equalities, the proof of the first part of the theorem is completed. The second part is proved similarly.

$\square$

Let A and B be two positive operators, $r \in [0,1]$. r-weighted arithmetic mean of A and B, denoted by $A\nabla_r B$ is defined as

$$A\nabla_r B = (1-r)A + rB.$$

For an invertible operator A, r-geometric mean of A and B, denoted by $A\sharp_r B$, is defined as

$$A\sharp_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}.$$

The power mean $A\sharp_r B$ is defined by F. Kubo and T. Ando [6]. In the special case $r = \frac{1}{2}$, the index is omitted.
There are several inequalities around $A\sharp_r B$ and $A\nabla_r B$. Recently, J. Zhao and J. Wu [7], for two positive invertible operators A and B of $B(H)$ and $r \in [0,1]$, proved that:
If $0 < r < \frac{1}{2}$ then

$$r_0(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + 2r(A\nabla B - A\sharp B) + A\sharp_r B \leq A\nabla_r B,$$

if $\frac{1}{2} < r < 1$, then

$$r_0(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + 2(1-r)(A\nabla B - A\sharp B) + A\sharp_r B \leq A\nabla_r B,$$

In the sequel, by the above refinements of the Young's inequality and the following lemma, some inequalities concerning $A\sharp_r B$ and $A\nabla_r B$ will be obtained.

Lemma 2.8 ([3]). *Let $X \in B(H)$ be self-adjoint and let f and g be continuous functions such that $f(t) \geq g(t)$ for all $t \in Sp(X)$(the spectrum of X). Then $f(X) \geq g(X)$.*

Theorem 2.9. *Let $A, B \in B(H)$ be two positive invertible operators and $r \in (0,1)$.*
(i) If $B < 2A < 2B$, then

$$\begin{aligned} A\sharp_r B \leq A\nabla_r B &+ \frac{r(r-1)(r-2)}{6}(BA^{-1})^2B + \frac{r(r-1)(3-r)}{2}BA^{-1}B \\ &+ \frac{r(r-1)(r-4)}{2}B + \frac{r(r-1)(5-r)}{6}I. \end{aligned} \tag{2.7}$$

(ii) If $2A < B$, then

$$\begin{aligned} A\sharp_r B \leq A\nabla_r B &+ \frac{r(1-r^2)}{6}(AB^{-1})^2A + \frac{r(r-1)(r+3)}{4}AB^{-1}A \\ &- \frac{r(r-1)(r+3)}{2}B + \frac{r(r-1)(5-r)}{6}I. \end{aligned} \tag{2.8}$$

Proof. (i) By Theorem 2.1 if $1 < x < 2$ then,

$$x^r \leq (1-r) + rx + \frac{r(r-1)}{2}(x-1)^2 + \frac{r(r-1)(r-2)}{6}(x-1)^3.$$

Taking $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have $I < X < 2I$ and so $Sp(X) \subseteq (0,1)$. Hence, by Lemma 2.8

$$X^r \leq (1-r) + rX + \frac{r(r-1)}{2}(X-1)^2 + \frac{r(r-1)(r-2)}{6}(X-1)^3.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, inequality (2.7) is deduced.
(ii) If $y > 2$, by Theorem 2.1

$$y^{1-r} \leq (1-r)y + r + \frac{r(r-1)}{2}(y-1)^2 + \frac{r(1-r^2)}{6}(y-1)^3.$$

Taking $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ we have $Y > 2I$ and so $Sp(Y) \subseteq (2,\infty)$. Hence, by Lemma 2.8

$$Y^{1-r} \leq (1-r)Y + r + \frac{r(r-1)}{2}(Y-1)^2 + \frac{r(1-r^2)}{6}(Y-1)^3.$$

Multiplying both sides of the above inequality by $B^{\frac{1}{2}}$ and since $A\sharp_r B = B\sharp_{1-r}A$, inequality (2.8) is deduced. $\square$

Theorem 2.10. *Let $A, B \in B(H)$ be two positive invertible operators and $r \in (0,1)$.*
(i) *If $B < 2A < 2B$, then*

$$\frac{A\sharp_r B + B\sharp_r A}{2} \leq A\nabla B + \frac{r(r-1)}{8}(BA^{-1}B - 2B + I). \tag{2.9}$$

(ii) *If $2A < B$, then*

$$\frac{A\sharp_r B + B\sharp_r A}{2} \leq A\nabla B + \frac{r(r-1)}{8}(AB^{-1}A - 2A + I). \tag{2.10}$$

Proof. (i) By Corollary 2.5 if $1 < x < 2$ then,

$$\frac{x^{1-r} + x^r}{2} \leq \frac{x+1}{2} + \frac{r(r-1)}{8}(x-1)^2.$$

If $A < B < 2A$ then, by taking $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have $I < X < 2I$ and so $Sp(X) \subseteq (0,1)$. Hence, by Lemma 2.8

$$\frac{X^{1-r} + X^r}{2} \leq \frac{X+1}{2} + \frac{r(r-1)}{8}(X-1)^2.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, inequality (2.9) is deduced.
(ii) If $y > 2$, by Corollary 2.5

$$\frac{y^r + y^{1-r}}{2} \leq \frac{y+1}{2} + \frac{r(r-1)}{8}(y-1)^2.$$

Taking $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ we have $Y > 2I$ and so $Sp(Y) \subseteq (2, \infty)$. Hence, by Lemma 2.8

$$\frac{Y^r + Y^{1-r}}{2} \leq \frac{Y+1}{2} + \frac{r(r-1)}{8}(Y-1)^2.$$

Multiplying both sides of the above inequality by $B^{\frac{1}{2}}$ and since $A\sharp_r B = B\sharp_{1-r}A$, inequality (2.10) is deduced. □

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