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A REMARK ON GENERALIZED DERIVATIONS IN RINGS AND ALGEBRAS

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Abstract. In the present paper, we investigate the action of generalized derivation G associated with a derivation g in a (semi-) prime ring R satisfying $(G([x,y]) - [G(x),y])^n = 0$ for all $x, y \in I$, a nonzero ideal of R, where n is a fixed positive integer. Moreover, we also examine the above identity in Banach algebras.

1. Introduction

Let R be an associative ring with characteristic different from 2, $Z(R)$ its center, U its (right) Utumi quotient ring and C its extended centroid. The simple commutator $ab - ba$ will be denoted by $[a, b]$. Recall that a derivation $g : R \to R$ is an additive map satisfying the product rule $g(xy) = g(x)y + xg(y)$ for all $x, y \in R$.

A generalized derivation G on a ring R is an additive map satisfying $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$ and some derivation g of R. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such a generalized derivation is called inner. Since the sum of two generalized derivations is a generalized derivation, of course every map of the form $G(x) = cx + g(x)$ is a generalized derivation on R, where c is a fixed element of R and g is a derivation of R.

In [16] (Theorem 3) Lee proved that every generalized derivation G on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus any generalized derivation of R can be defined on the whole U, moreover it is of the form $G(x) = ax + g(x)$ for some $a \in U$ and g a derivation on U (G is said to be a generalized derivation associated with derivation g).
Many results in literature indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R.

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In [4] Bergen proved that if α is an automorphism of R such that $(\alpha(x)-x)^m = 0$, for all $x \in R$, where $m \geq 1$ is a fixed integer, then $\alpha = 1$. Bell and Daif [3] showed some results which have the same flavour, when the automorphism is replaced by a non-zero derivation g. In [3] it is proved that if R is a semiprime ring with a non-zero ideal I such that $g([x,y]) - [x,y] = 0$, or $g([x,y]) + [x,y] = 0$, for all $x, y \in I$, then I is central. Later Hongan [12] proved that if R is a 2-torsion free semiprime ring and I a non-zero ideal of R, then I is central if and only if $g([x,y]) - [x,y] \in Z(R)$, or $g([x,y]) + [x,y] \in Z(R)$, for all $x, y \in I$. Recently in [1] Ashraf and Ali obtained commutativity theorems for prime rings admitting left multipliers which satisfy similar conditions. More percisely, in [1] it is proved that if R is a prime ring, I a nonzero ideal of R and G is a generalized derivation associated with a nonzero deivation g such that any one of the following holds: (i) $G([x,y]) \pm [x,y] = 0$ (ii) $G(x \circ y) \pm (x \circ y) = 0$ for all $x, y \in I$, then R commutative.

In this line of investigation, our aim is to study the above identities (i) $G([x,y]) \pm [x,y] = 0$ (ii) $G(x \circ y) \pm (x \circ y) = 0$ with some power values, where G is a generalized derivation with associated derivation g and I is a nonzero ideal of R. More precisely, we prove the following theorems:

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R and n is a fixed positive integer. If R admits a generalized derivation G associated with a nonzero derivation g such that $(G([x,y]) - [G(x),y])^n = 0$ for all $x, y \in I$, then R is commutative.*

We also study the identity in semiprime rings and Banach algebras.

Theorem 1.2. *Let R is a semiprime ring, U the left Utumi quotient ring of R and m, n fixed positive integers. If R admits a generalized derivation G associated with a nonzero derivation g such that $(G([x,y]) - [G(x),y])^n = 0$ for all $x, y \in R$, then there exists a central idempotent element e of U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Now we shall study the identity on Banach algebra. Here $\mathcal{A}$ denotes a complex non- commutative Banach algebra. By Banach algebra we shall mean a complex normed algebra $\mathcal{A}$ whose underlying vector space is a Banach space. By $(\mathcal{A})$, we denote the Jacobson radical of $\mathcal{A}$ is said to be semisimple, if $rad(\mathcal{A}) = 0$.

Singer and Wermer in [24] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. Then Thomas [25] proved that the continuity assumption of Singer and Wermer's result can be removed. It is clear that the same result of Singer and Wermer does not hold in non-commutative Banach algebras (because of inner derivations). It is still an open question how to obtain the Singer and Wermer's result in non-commutative Banach algebras. There are some papers in which the Singer and Wermer's result are obtained in non-commutative Banach algebras under certain conditions. Let $\mathcal{A}$ be a non-commutative Banach algebra and D be a continuous derivation on A. In [19], Mathieu proved that if $[D(x), x]D(x) \in rad(\mathcal{A})$ for all $x \in A$, then D maps A into $rad(\mathcal{A})$. Further [18] proved the same conclusion under the condition $[D(X), x] \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. Recently in [21], Park proves that if g is a derivation of a non-commutative Banach algebra A, such that $[[g(x), x], g(x)] \in rad(\mathcal{A})$, then again g maps $\mathcal{A}$ into $rad(\mathcal{A})$.

In this line of investigation, we will prove the following theorem in non-commutative Banach algebra.

Theorem 1.3. *Let $\mathcal{A}$ be a non-commutative Banach algebra. Let $G = L_p + g$ be continuous generalized derivation of $\mathcal{A}$, where L_p denotes the left multiplication by an element $p \in \mathcal{A}$ and g is a nonzero derivation on $\mathcal{A}$. If $(G([x, y]) - [G(x), y])^n \in rad(\mathcal{A})$ for all $x, y \in \mathcal{A}$, where $n \geq 1$ is a fixed integer, then g maps $\mathcal{A}$ into $rad(\mathcal{A})$.*

2. Preliminaries

Let R be a prime ring and U be Utumi quotient ring of R. We have the following properties:

(1) $R \subseteq U$;
(2) U is prime ring with identity;
(3) The center of U is denoted by C and is called the extended centroid of R. C is field. (see [2] for more details).

Before starting with the proofs, we fix some well known facts, which is important for developing the proof of our main result. In particular, we will make frequent use of the following facts:

Fact 2.1 ([6])**.** *If I is a two-sided ideal of R, then R, I and U satisfy the same generalized polynomial identities (GPIs) with coefficients in U.*

Fact 2.2 ([2, Proposition 2.5.1])**.** *Every derivation g of R can be uniquely extended to a derivation of U.*

Fact 2.3 ([15])**.** *If I is a two-sided ideal of R, then R, I and U satisfy the same differential identities.*

Fact 2.4 ([16, Theorem 3])**.** *Every generalized derivation G on a dense right ideal of R can be uniquely extended to a generalized derivation of U. Furthermore, the extended generalized derivation G has the form $G(x) = ax + g(x)$ for all $x \in U$, where $a \in U$ and g is a derivation of U.*

We refer the reader to Chapter 7 in [2], for a complete and detailed description of the theory of generalized polynomial identities involving derivations. We will make frequent use of the following result due to Kharchenko [13] (see also [15]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1, \cdots x_n, d(x_1, \cdots x_n))$ be a differential identity in I, that is

$$f(r_1, \cdots r_n, d(r_1), \cdots, d(r_n)) = 0 \text{ for all } r_1, \cdots, r_n \in I.$$

One of the following holds:

(1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, \cdots, r_n, [q, r_1], \cdots, [q, r_n]) = 0;$$

(2) or, I satisfies the generalized polynomial identity

$$f(x_1, \cdots, x_n, y_1, \cdots, y_n) = 0.$$

3. MAIN RESULTS

We prove first theorem in prime rings.

Proof of Theorem 1.1. Let U be the Utumi ring of quotients of R and $C = Z(U)$. In [16, Theorem 3], Lee proved that every generalized derivation G on a dense right ideal of R can be uniquely extended to a generalized derivation of U and thus can be assumed to be defined on the whole U with the form $G(x) = ax + g(x)$ for some $a \in U$ and g is a derivation of U. In the light of this, we may assume that there exist $a \in U$ and derivation G of U such that $G(x) = ax + g(x)$. Since I, R, and U satisfies the same generalized polynomial identities by Fact 2.1 as well as the same differential identities by Fact 2.2, without loss of generality, we have $(G([x, y] - [G(x), y])^n = 0$

for all $x, y \in I$, that is

$$\begin{aligned} 0 &= (a[x,y] + g([x,y]) - [ax + g(x), y])^n \\ &= (a[x,y] + xg(y) - g(y)x - axy + yax)^n \text{ for all } x, y \in I. \end{aligned} \tag{3.1}$$

Now we divide the proof into two cases:

Case 1. If g is not an inner derivation of U, then by Kharchenko's theorem [13], we have from (3.1) that

$$(a[x,y] + xt - tx - axy + yax)^n = 0$$

for all $x, yt \in I$. In particular, for $y = 0$, we have $([x,t])^n = 0$ for all $x, t \in I$. This is a generalized polynomial identity (GPI) for U. Thus, Herstein [11, Theorem 2], we have $I \subseteq Z(R)$, and so R is commutativity by Mayne [20].

Case 2. Next, we assume that g is an inner derivation induced by an element $q \in Q$, that is $g(x) = [q,x]$ for all $x \in Q$. It follows that for any $x, y \in I$,

$$(a[x,y] + x[q,y] + [q,y]x - axy + yax)^n = 0.$$

By Chuang [6, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by Q, that is

$$(a[x,y] + x[q,y] + [q,y]x - axy + yax)^n = 0 \tag{3.2}$$

for all $x, y \in Q$. In case center C is infinite, then (3.2) is also satisfied by $Q \bigotimes_C \bar{C}$, where $\bar{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \bar{C}$ are prime and centrally closed [8, Theorem 2.5 and Theorem 3.5], we may replace R by Q or $Q \bigotimes_C \bar{C}$ according to C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e. $RC = R$) which is either finite or algebraically closed and hence R satisfy (3.2) for all $x, y \in R$. By Martindale's [17, Theorem 3], R is then a primitive ring having nonzero $soc(R)$ with C as the associated division ring. Hence by Jacobson's theorem [9], R is isomorphic to dense ring of linear transformations of a vector space $\mathcal{V}$ over C. If $dim_C V = k$, then the density of R on $\mathcal{V}$ implies that $R \cong M_k(C)$, where $k = dim_C \mathcal{V}$. If $dim_C \mathcal{V} = 1$, then R is commutative and we are done in this case.

Therefore, we assume $dim_C \mathcal{V} \geq 3$. First of all, we want to show that v and qv are liner C-dependent for all $v \in V$. If $qv = 0$ then $\{v, qv\}$ is C-dependent. Suppose that $qv \neq 0$. If $\{v, qv\}$ is C-independent, as $dim_C \mathcal{V} \geq 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ are also linear C-independent. By density of R, there exists

$x, y \in R$ such that

$$xv = 0, \quad xqv = w, \quad xw = v,$$
$$yv = 0, \quad yqv = w, \quad yw = v.$$

These imply that $0 = (a[x, y] + x[q, y] + [q, y]x - axy + yax)^n v = (-1)^n v \neq 0$, a contradiction. So we conclude that v and qv are linear C-dependent for all $v \in \mathcal{V}$.

We show here that there exists $\alpha \in C$ such that $qv = v\alpha$, for any $v \in \mathcal{V}$. Note that the arguments in [5], are still valid in the present situation. For the sake of completeness and clearness we prefer to present it. In fact, we choose $v, w \in \mathcal{V}$ linearly independent. Since $dim_C\mathcal{V} \geq 3$, there exists $u \in \mathcal{V}$ such that v, w, u are linearly independent. As already mention above, there exist $\alpha_v, \alpha_w, \alpha_u \in C$ such that

$$qv = v\alpha_v, qw = w\alpha_w, qu = u\alpha_u \text{ that is } q(v + w + u) = v\alpha_v + w\alpha_w + u\alpha_u.$$

Moreover $q(v + w + u) = (v + w + u)\alpha_{v+w+u}$, for a suitable $\alpha_{v+w+u} \in C$. Then

$$0 = v(\alpha_{v+w+u} - \alpha_v) + w(\alpha_{v+w+u} - \alpha_w) + u(\alpha_{v+w+u} - \alpha_u),$$

and, because v, w, u are linearly independent, $\alpha_u = \alpha_w = \alpha_v = \alpha_{v+w+u}$, that is, α does not depend on the choice of v. So there exists $\alpha \in C$ such that $qv = \alpha v$ for all $v \in \mathcal{V}$. Now for $r \in R, v \in \mathcal{V}$. Since $qv = v\alpha$ we have $[q, r]v = (qr)v - (rq)v = q(rv) - r(qv) = (rv)\alpha - r(v\alpha) = 0$, that is $[q, R]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q, R] = 0$, i.e., $q \in Z(R)$ and so $g = 0$, a contradiction.

Therefore $dim_C\mathcal{V}$ must be ≤ 2. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [14, Lemma 2], it follows that there exists a suitable filed F such that $R \subseteq \mathcal{M}_k(\mathbb{F})$, the ring of all $k \times k$ matrices over $\mathbb{F}$, and moreover $\mathcal{M}_k(\mathbb{F})$ satisfy the same GPI as R. Assume that $k \geq 3$, by the same argument as in the above, we can get a contradiction. If $k = 1$, then it is clear that R is commutative. Thus we may assume that $R \subseteq \mathcal{M}_2(\mathbb{F})$, where $\mathcal{M}_2(\mathbb{F})$ satisfies $(a[x, y] + x[q, y] + [q, y]x - axy + yax)^n = 0$. Denote e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Let $[x, y] = [e_{21}, e_{11}] = e_{21}$. In this case we have $(e_{21}qe_{11} - e_{21}q - e_{11}qe_{21} - e_{11}ae_{21})^n = 0$. Right multiplying by e_{21}, we get $(-1)^n(e_{21}q)^n e_{21} = 0$. Set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation we find that $(-1)^n \begin{pmatrix} 0 & 0 \\ q_{12}^n & 0 \end{pmatrix} = 0$, which implies that $q_{12} = 0$. Similarly, we can see that $q_{21} = 0$. Therefore q is diagonal in $M_2(\mathbb{F})$. Let $h \in Aut(\mathcal{M}_2(\mathbb{F}))$. Since $(h(a)[h(x), h(y)] + h(x)[h(q), h(y)] + [h(q), h(y)]h(x) - h(a)h(x)h(y) - h(y)h(a)h(x))^n = 0$, so $h(q)$ must be diagonal matrix in $\mathcal{M}_2(\mathbb{F})$. In particular, let $h(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$, then $h(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $\mathcal{M}_2(\mathbb{F})$, which leads to $g = 0$, a contradiction. □

Using the same techniques with necessary variations, we can prove the following:

Theorem 3.1. *Let R be a prime ring, I a nonzero ideal of R and n is a fixed positive integer. If R admits a generalized derivation G associated with a nonzero derivation g such that $(G(x \circ y) - (G(x) \circ y))^n = 0$ for all $x, y \in I$, then R is commutative.*

The following examples shows that the main results are not true in the case of arbitrary ring.

Example 3.1. Let S be any ring.

(*i*) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$. Then R is a ring under usual operations and I is a nonzero ideal of R. We define a map $G : R \to R$ by $G(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that G is a generalized derivation associated with a nonzero derivation $g(x) = e_{11}x - xe_{11}$. It is straightforward to check that for all positive integer n, G satisfies the properties, $(G([x, y]) - [G(x), y])^n = 0$ and $(G(x \circ y) - (G(x) \circ y))^n = 0$ for all $x, y \in I$, however R is not commutative.

(*ii*) Let $\mathbb{Z}$ be the ring of integers. Next, let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z} \right\}$. Clearly, R is a ring under the usual operations which is not prime. Define a map $G : R \to R$ by $G\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b + a \\ 0 & 0 \end{pmatrix}$. It is easy to see that G is a generalized derivation associated with a nonzero derivation $g(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & a - c \\ 0 & 0 \end{pmatrix}$. It is straightforward to check that I is a nonzero ideal of R and G satisfies the properties, $(G([x, y]) - [G(x), y])^n = 0$ and $(G(x \circ y) - (G(x) \circ y))^n = 0$ for all $x, y \in I$, but R is not commutative. Hence, the hypothesis of primeness is necessary.

To prove our next theorem for semiprime rings, we need the following:

Fact 3.1 ([2, Proposition 2.5.1])**.** *Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.*

Fact 3.2 ([7, p. 38]). *If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.*

We will prove the following:

Proof of Theorem 1.2. Since R is semiprime and F is a generalized derivation of R, by Lee [15, Theorem 3], $G(x) = ax + g(x)$ for some $a \in U$ and a derivation g on U. We have given that

$$0 = (a[x, y] + g([x, y]) - [ax + g(x), y])^n = (a[x, y] + xg(y) - g(y)x - axy + yax)^n,$$

for all $x, y \in I$. By Fact 3.2, $Z(U) = C$, the extended centroid of R, and, by Fact 3.1, the derivation d can be uniquely extended on U. By Lee [15], R and U satisfy the same differential identities. Then

$$(a[x, y] + xg(y) - g(y)x - axy + yax)^n = 0$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Therefore by Chuang [7, p.42], U is orthogonal complete B-algebra, and by [7, p.42], MU is a prime ideal of U, which is g-invariant. Denote $\overline{U} = U/MU$ and $\overline{g}$ the derivation induced by g on $\overline{U}$, i.e., $\overline{g}(\overline{u}) = \overline{g(u)}$ for all $u \in U$. For any $\overline{x}, \overline{y} \in \overline{U}$,

$$(\overline{a}[\overline{x}, \overline{y}] + \overline{xg}(\overline{y}) - \overline{g}(\overline{y})\overline{x} - \overline{axy} + \overline{yax})^n = 0.$$

It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 1.1, we have either $\overline{U}$ is commutative or $\overline{g} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, either $g(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case $g(U)[U, U] \subseteq MU$, for all M, where MU runs over all minimal prime ideals of U. Therefore $g(U)[U, U] \subseteq \bigcap_M MU = 0$, we obtain $g(U)[U, U] = 0$. By using the theory of orthogonal completion for semiprime rings [2, Chapter 3], it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1 - e)U$, g vanishes identically on eU and the ring $(1 - e)U$ is commutative. This completes the proof of the theorem. □

Using arguments similar to those used in the proof of the Theorem 1.2, we may conclude with the following (we omit the proof brevity). We can prove

Theorem 3.2. *Let R is a semiprime ring with characteristic different from 2, U the left Utumi quotient ring of R and n is a fixed positive integer. If R admits a* *generalized derivation G associated with a nonzero derivation g such that $(G(x \circ y) - (G(x) \circ y))^n = 0$ for all $x, y \in R$, then there exists a central idempotent element e of U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Now we prove our last theorem for Banach algebras.

Proof of Theorem 1.3. By hypothesis G is a continuous generalized derivation of $\mathcal{A}$. Since we know that left multiplication map is continuous, we get that g is continuous, Sinclair [23] proved that any continuous derivation of a Banach algebra leveas the primitive deals invariant. hence, for any primitive ideal P of $\mathcal{A}$, it is obvious that $G(P) \subseteq aP + g(P) \subseteq P$. It means that continuous generalized derivation G leaves the primitive ideals invariant,

Denote $\mathcal{A}/P = \overline{\mathcal{A}}$ for any primitive ideals P. Thus, we can define the generalized derivation

$$F_P : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$$

such that

$$F_P(\overline{r}) = F_P(r + P) \subseteq F(r) + P = ar + g(r) + P$$

for all $r \in \overline{\mathcal{A}}$, where $\mathcal{A}/P = \overline{\mathcal{A}}$ is a factor Banach algebra. Since P is primitive ideal, the factor algebra $\overline{\mathcal{A}}$ is primitive and so it is prime and semisimple. Then

$$(G([x, y]) - [G(x), y])^n \in rad(\mathcal{A})$$

yields that $(G([\overline{x}, \overline{y}]) - [G(\overline{x}), \overline{y}])^n = 0$ for all $\overline{x}, \overline{y} \in \overline{\mathcal{A}}$. By Theorem 1.1, we get either $\overline{g} =)$ or $\overline{\mathcal{A}}$ is commutative. By a result Johnson and Sinclair [10] every linear derivation on semisimple Banach algebra is continuous. Thus $\overline{g}$ is continuous in $\overline{\mathcal{A}}$. Singer and Wermer in [24], proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Hence in both the cases we get $\overline{g} = 0$, that is $g(\mathcal{A}) \subseteq P$ for any primitive ideal P of $\mathcal{A}$ and hence we get $g(\mathcal{A}) \in rad(\mathcal{A})$. This proves the theorem. □

In view of Theorem 1.3, we may prove the following corollary in the special case when $\mathcal{A}$ is a semisimple Banach algebra.

Corollary 3.1. *Let $\mathcal{A}$ be a non-commutative semisimple Banach algebra. Let $G = L_p + g$ be a continuous generalized derivation of $\mathcal{A}$, where L_p denotes the left multiplication by some $p \in \mathcal{A}$, g is a nonzero derivation on $\mathcal{A}$ If $(G([x, y]) - [G(x), y])^n \in rad(\mathcal{A})$ for all $x, y \in \mathcal{A}$, where ≥ 1 is fixed integer, then $g(\mathcal{A}) = 0$.*

Proof. We may prove the result in the same way as in the proof of Theorem 1.3, we omit the proof for brevity. Just let us remark that at the beginning of the proof one has to use the fact that the derivation g is continuous in a semisimple Banach algebra (see [23]). Moreover, any left multiplication map is continuous, also G is continuous. Finally, we use the fact that $rad(\mathcal{A}) = 0$, since $\mathcal{A}$ is semisimple. $\square$

Using arguments similar to those used in the proof of the Theorem 1.3, we may conclude with the following. We can prove

Theorem 3.3. *Let $\mathcal{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathcal{A})$ and n a fixed positive integers. Let $G(x) = ax + g(x)$ be a continuous generalized derivation of $\mathcal{A}$ for some element $a \in \mathcal{A}$ and some derivation g on $\mathcal{A}$. If $(G(x \circ y) - (G(x) \circ y))^n \in rad(\mathcal{A})$ for all $x, y \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

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