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CENTRAL INDEX BASED SOME COMPARATIVE GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS FROM THE VIEW POINT OF L^*-ORDER

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Abstract. In this paper, we discuss central index oriented and slowly changing function based some growth properties of composite entire functions.

1. Introduction, Definitions and Notations

Let f be an *entire function* defined in the *open complex plane* $\mathbb{C}$. For entire $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the *maximum modulus* symbolized as $M_f(r)$, the *maximum term* denoted as $\mu_f(r)$ and the *central index* indicated as $\nu_f(r)$ are respectively defined as $\max_{|z|=r} |f(z)|$, $\max_{n\geq 0}(|a_n| r^n)$ and $\max\{m, \mu_f(r) = |a_m| r^m\}$. Therefore, *central index* $\nu_f(r)$ of an *entire function* f is the greatest exponent m such that $|a_m| r^m = \mu_f(r)$. Obviously $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real and increasing function of r. For another entire function g, $M_g(r)$ and $\mu_g(r)$ are also defined and the ratios $\frac{M_f(r)}{M_g(r)}$ when $r \to \infty$ as well as $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ are called the *comparative growth* of f with respect to g in terms of their *maximum moduli* and *the maximum term* respectively. The prime object of the study of the *growth* investigation of *entire functions* has usually been done through their *maximum moduli* and *maximum term*. Though $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_g(r)$ in some sense, from another angle of view $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to \infty$ is also called the *growth* of f with respect to g where $\nu_g(r)$ denotes the *central index* of entire g. Considering this, here we compare the *central index* of *composition* of two *entire functions* with their corresponding left and right factors under the treatment of the theories of *slowly changing functions* which in fact means

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that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a i.e., $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$. Actually in this paper we attempt to prove some results related to the growth rates of composite *entire functions* on the basis of *central index* using the idea of L^**-order* (respectively, L^**-lower order)* of an *entire function* where L^* is nothing but a weaker assumption of L.Our notations are standard within the theory of *Nevanlinna's value distribution* of entire functions and therefore we do not explain those in detail as those are available in [8]. To start our paper we just recall the following definitions which will be needed in the sequel:

Definition 1. The *order* ρ_f and *lower order* λ_f of an entire function f are define as

$$\begin{matrix} \rho_f \\ \lambda_f \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log r} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Therefore it seems reasonable to state suitably an alternative definition of *order* and *lower order* of entire function in terms of its central index. *He and Xiao* [3] introduced such a definition in the following way:

$$\begin{matrix} \rho_f \\ \lambda_f \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \nu_f(r)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Considering $L(r) = \log r$ and $a = 10^{20}$, one can easily show that $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$. *Somasundaram and Thamizharasi* [6] introduced the notions of L*-order* (respectively L*-lower order*) of *entire functions.* The more generalized concept for L*-order* and L*-lower order* for entire functions is L^**-order* and L^**-lower order* whose definition are as follows:

Definition 2 ([6])**.** The L^*-order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\begin{matrix} \rho_f^{L^*} \\ \lambda_f^{L^*} \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log\left[re^{L(r)}\right]} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Taking $f(z) = \exp z$ and $L(r) = \log r$, one can easily verify that $\rho_f = \lambda_f = 1$ and $\rho_f^{L^*} = \lambda_f^{L^*} = \frac{1}{2}$.

In terms of *central index* of *entire functions*, Definition 2 can be reformulated as:

Definition 3. The growth indicators $\rho_f^{L^*}$ and $\lambda_f^{L^*}$ of an entire function f are defined as:

$$\begin{matrix} \rho_f^{L^*} \\ \lambda_f^{L^*} \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \nu_f(r)}{\log\left[re^{L(r)}\right]}.$$

The concept of (p,q)-φ order of entire function was introduced by Shen et al. [5] where $p \geq q \geq 1$ and $\varphi : [0,+\infty) \rightarrow (0,+\infty)$ be a non-decreasing unbounded function. Shen et al. [5] also established the equivalence of the definition of (p,q)-φ order of entire function in terms of *maximum modulus* and *central index* under some certain condition. For details about it, one may see [5]. For particular if we consider $p = 1$, $q = 1$ and $\varphi(r) = re^{L(r)}$, then in view of Proposition 1.2 of [5], we can write that

$$\begin{matrix} \rho_f^{L^*} \\ \lambda_f^{L^*} \end{matrix} = \lim_{r\rightarrow+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M_f(r)}{\log\left[re^{L(r)}\right]} = \lim_{r\rightarrow+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \nu_f(r)}{\log\left[re^{L(r)}\right]} .$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([3, Theorems 1.9 and 1.10] or [4, Satz 4.3 and 4.4]). *Let f be any entire function, then*

$$\log \mu_f(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \text{ where } a_0 \neq 0,$$

and for $r < R$,

$$M_f(r) < \mu_f(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\} .$$

Lemma 2 ([1]). *Let f and g are any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)) .$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right)\right) .$$

Lemma 3. *Let f be an entire function with $0 < \lambda_f \leq \rho_f < \infty$. Also let g be an entire function with non zero finite lower order. If $0 < \alpha < \lambda_g$, then for all sufficiently large values of r,*

$$\nu_{f\circ g}(r) > \nu_f(\exp(r^\alpha)) .$$

Proof. For any constant E, we get from the second part of Lemma 1, that

$$\log M_f(r) < \nu_f(r) \log r + \log \nu_f(2r) + E \ \{cf. [2]\} .$$

Therefore from above we obtain that

$$\begin{aligned}
\log M_f(r) &< \nu_f(2r)\log r + \nu_f(2r) + E \\
i.e.,\ \log M_f(r) &< \nu_f(2r)(1+\log r) + E \\
i.e.,\ \log M_f(r) &< \nu_f(2r)\log(e\cdot r) + E \\
i.e.,\ \log M_f\left(\frac{r}{2}\right) &< \nu_f(r)\log\left(e\cdot\frac{r}{2}\right) + E
\end{aligned}$$

$$i.e.,\ \log M_f\left(\frac{r}{2}\right) < \nu_f(r)\log\left(e\cdot\frac{r}{2}\right)\left(1+\frac{E}{\nu_f(r)\log\left(e\cdot\frac{r}{2}\right)}\right). \tag{1}$$

In view of (1) we get for all sufficiently large values of r that

$$\log\nu_{f\circ g}(r) > \log^{[2]} M_{f\circ g}(\frac{r}{2}) - \log^{[2]}\left(e.\frac{r}{2}\right) - \log\left(1+\frac{E}{\nu_{f\circ g}(r)\log\left(e.\frac{r}{2}\right)}\right)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log\nu_{f\circ g}(r)}{\log\nu_f(\exp(r^\alpha))} \geq \liminf_{r\to\infty}\frac{\log^{[2]} M_{f\circ g}(\frac{r}{2})}{\log\nu_f(\exp(r^\alpha))} -$$

$$\limsup_{r\to\infty}\frac{\log^{[2]}\left(e.\frac{r}{2}\right)}{\log\nu_f(\exp(r^\alpha))} - \limsup_{r\to\infty}\frac{\log\left(1+\frac{E}{\nu_{f\circ g}(r)\log\left(e.\frac{r}{2}\right)}\right)}{\log\nu_f(\exp(r^\alpha))}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log\nu_{f\circ g}(r)}{\log\nu_f(\exp(r^\alpha))} \geq \liminf_{r\to\infty}\frac{\log^{[2]} M_{f\circ g}(\frac{r}{2})}{\log\nu_f(\exp(r^\alpha))}\ . \tag{2}$$

Further in view of Lemma 2, we obtain for all sufficiently large values of r that

$$\log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right) \geq \log^{[2]} M_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)$$

$$i.e.,\ \log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right) \geq (\lambda_f-\varepsilon)\frac{1}{8} + (\lambda_f-\varepsilon)\left(\frac{r}{4}\right)^{\lambda_g-\varepsilon}\ . \tag{3}$$

where we choose ε in such a way that $0<\varepsilon<\min(\lambda_f,\lambda_g)$.

Again from the definition of λ_f of entire function in terms of central index, we obtain for all sufficiently large values of r that

$$\begin{aligned}
\log\nu_f(\exp(r^\alpha)) &\leq (\rho_f+\varepsilon)\log\exp(r^\alpha) \\
i.e.,\ \log\nu_f(\exp(r^\alpha)) &\leq (\rho_f+\varepsilon)\, r^\alpha\ .
\end{aligned} \tag{4}$$

Now from (3) and (4) it follows for all sufficiently large values of r that

$$\frac{\log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right)}{\log\nu_f(\exp(r^\alpha))} \geq \frac{(\lambda_f-\varepsilon)\frac{1}{8} + (\lambda_f-\varepsilon)\left(\frac{r}{4}\right)^{\lambda_g-\varepsilon}}{(\rho_f+\varepsilon)\, r^\alpha}$$

(5) $$\liminf_{r\to\infty} \frac{\log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right)}{\log \nu_f(\exp(r^\alpha))} \geq \liminf_{r\to\infty} \frac{(\lambda_f - \varepsilon)\frac{1}{8} + (\lambda_f - \varepsilon)\left(\frac{r}{4}\right)^{\lambda_g - \varepsilon}}{(\rho_f + \varepsilon)\, r^\alpha}$$

As $\alpha < \lambda_g$ we can choose $\varepsilon\,(> 0)$ in such a way that

(6) $$\alpha < \lambda_g - \varepsilon \ .$$

Thus from (5) and (6) we get that

(7) $$\lim_{r\to\infty} \frac{\log^{[2]} M_{f\circ g}\left(\frac{r}{2}\right)}{\log \nu_f(\exp(r^\alpha))} = \infty \ .$$

Therefore from (2) and (7) we obtain that

$$i.e.,\ \lim_{r\to\infty} \frac{\log \nu_{f\circ g}(r)}{\log \nu_f(\exp(r^\alpha))} = \infty \ .$$

So from above we obtain for all sufficiently large values of r and $K > 1$ that

$$\begin{aligned} \log \nu_{f\circ g}(r) &> K \log \nu_f(\exp(r^\alpha)) \\ i.e.,\ \log \nu_{f\circ g}(r) &> \log \{\nu_f(\exp(r^\alpha))\}^K \\ i.e.,\ \nu_{f\circ g}(r) &\geq \nu_f(\exp(r^\alpha)) \ . \end{aligned}$$

This proves the theorem. □

In the line of Lemma 3, one can easily verify the following corollary and therefore its proof is omitted.

Corollary 1. *Let f be an entire function with non zero lower order. Also let g be an entire function with $0 < \lambda_g \leq \rho_g < \infty$. If $0 < \alpha < \lambda_g$, then for all sufficiently large values of r,*

$$\nu_{f\circ g}(r) > \nu_g(\exp(r^\alpha)) \ .$$

3. Results

In this section we present the main results of the paper.

Theorem 4. *Let f be an entire function with non zero finite order and lower order and g be an entire function with non zero finite lower order. If $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$, then for any $A > 0$*

$$\lim_{r\to\infty} \frac{\log^{[2]} \nu_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log \nu_f\left(\exp\left(r^\alpha\right)\right) + K\left(r, A; L\right)} = \infty \ ,$$

where $0<\alpha<\lambda_g$ *and* $K(r,A;L)=\begin{cases} 0 \text{ if } r^{\alpha}=o\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right) \text{ otherwise }. \end{cases}$

Proof. Let $0<\alpha<\alpha'<\lambda_g$. Now from the definition of L^*-lower order we obtain in view of Lemma 3, for for all sufficiently large values of r that

$$\log \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \geq \log \nu_f\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)$$

$$\begin{aligned} & i.e.,\ \log \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \\ \geq\ & \left(\lambda_f^{L^*}-\varepsilon\right).\log\left\{\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}.\exp L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)\right\} \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \\ \geq\ & \left(\lambda_f^{L^*}-\varepsilon\right).\left\{\left(\exp\left(r^{A}\right)\right)^{\alpha'}+L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)\right\} \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \\ \geq\ & \left(\lambda_f^{L^*}-\varepsilon\right).\left\{\left(\exp\left(r^{A}\right)\right)^{\alpha'}\left(1+\frac{L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)}{\left(\exp\left(r^{A}\right)\right)^{\alpha'}}\right)\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[2]} \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \geq\ & O(1)+\alpha'\log\exp\left(r^{A}\right) \\ & +\log\left\{1+\frac{L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)}{\left(\exp\left(r^{A}\right)\right)^{\alpha'}}\right\} \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log^{[2]} \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \geq O(1)+\alpha' r^{A} \\ & \qquad\qquad +\log\left\{1+\frac{L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\alpha'}\right)}{\left(\exp\left(r^{A}\right)\right)^{\alpha'}}\right\} \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log^{[2]} \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \geq O(1)+\alpha' r^{A} \\ & \qquad\qquad +\log\left[1+\frac{L\left(\exp\left(\exp\left(\alpha' r^{A}\right)\right)\right)}{\exp\left(\alpha' r^{A}\right)}\right] \end{aligned}$$

$$\begin{aligned} & i.e.,\ \log^{[2]} \nu_{f\circ g}\left(\exp\left(r^{A}\right)\right) \geq O(1)+\alpha' r^{A}+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right) \\ & \qquad -\log\left[\exp\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\}\right] \\ & \qquad\qquad +\log\left[1+\frac{L\left(\exp\left(\exp\left(\alpha' r^{A}\right)\right)\right)}{\exp\left(\alpha' r^{A}\right)}\right] \end{aligned}$$

$$i.e.,\ \log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)\geq O\left(1\right)+\alpha' r^{A}+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)$$
$$+\log\left[\frac{\exp\left(\alpha' r^{A}\right)+L\left(\exp\left(\exp\left(\alpha' r^{A}\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\}\cdot\exp\left(\alpha' r^{A}\right)}\right]$$

$$(8)\qquad i.e.,\ \log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)\geq O\left(1\right)+\alpha' r^{(A-\alpha)}.r^{\alpha}$$
$$+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\ .$$

Again we have for all sufficiently large values of r that

$$\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)\leq\left(\rho_{f}^{L^{*}}+\varepsilon\right)\log\left\{\exp\left(r^{\alpha}\right)e^{L\left(\exp\left(r^{\alpha}\right)\right)}\right\}$$
$$i.e.,\ \log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)\leq\left(\rho_{f}^{L^{*}}+\varepsilon\right)\left\{\log\exp\left(r^{\alpha}\right)+L\left(\exp\left(r^{\alpha}\right)\right)\right\}$$
$$i.e.,\ \log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)\leq\left(\rho_{f}^{L^{*}}+\varepsilon\right)\left\{r^{\alpha}+L\left(\exp\left(r^{\alpha}\right)\right)\right\}$$

$$(9)\qquad i.e.,\ \frac{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)-\left(\rho_{f}^{L^{*}}+\varepsilon\right)L\left(\exp\left(r^{\alpha}\right)\right)}{\left(\rho_{f}^{L^{*}}+\varepsilon\right)}\leq r^{\alpha}\ .$$

Now from (8) and (9) it follows for all sufficiently large values of r that

$$(10)\qquad \log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)$$
$$\geq O\left(1\right)+\left(\frac{\alpha' r^{(A-\alpha)}}{\rho_{f}^{L^{*}}+\varepsilon}\right)\left[\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)-\left(\rho_{f}^{L^{*}}+\varepsilon\right)L\left(\exp\left(r^{\alpha}\right)\right)\right]$$
$$+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)$$

$$(11)\qquad i.e.,\ \frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}\geq\frac{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)+O\left(1\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}$$
$$+\frac{\mu' r^{(A-\alpha)}}{\rho_{f}^{L^{*}}+\varepsilon}\left\{1-\frac{\left(\rho_{f}^{L^{*}}+\varepsilon\right)L\left(\exp\left(r^{\alpha}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}\right\}\ .$$

Again from (10) we get for all sufficiently large values of r that

$$\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}\geq\frac{O\left(1\right)-\alpha' r^{(A-\mu)}L\left(\exp\left(r^{\alpha}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}$$
$$+\frac{\left(\frac{\mu' r^{(A-\alpha)}}{\rho_{f}^{L^{*}}+\varepsilon}\right)\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}+\frac{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}$$

$$(12)\qquad i.e., \frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}\geq\frac{\frac{O(1)-\alpha' r^{(A-\mu)}L(\exp(r^{\alpha}))}{L(\exp(\exp(\alpha r^{A})))}}{\frac{\log\nu_{f}(\exp(r^{\alpha}))}{L(\exp(\exp(\alpha r^{A})))}+1}$$

$$+\frac{\left(\frac{\mu' r^{(A-\alpha)}}{\rho_{f}^{L^{*}}+\varepsilon}\right)\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}{1+\frac{L(\exp(\exp(\alpha r^{A})))}{\log\nu_{f}(\exp(r^{\alpha}))}}+\frac{1}{1+\frac{\log\nu_{f}(\exp(r^{\alpha}))}{L(\exp(\exp(\alpha r^{A})))}}\ .$$

Case I. If $r^{\alpha}=o\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\}$ then it follows from (11) that

$$\lim_{r\to\infty}\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}=\infty\ .$$

Case II. $r^{\alpha}\neq o\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\}$ then two sub cases may arise.

Sub case (a). If $L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)=o\left\{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)\right\}$, then we get from (12) that

$$\limsup_{r\to\infty}\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}=\infty\ .$$

Sub case (b). If $L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\sim\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)$ then

$$\lim_{r\to\infty}\frac{L\left\{\exp\left(\exp\left(\alpha r^{A}\right)\right)\right\}}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)}=1$$

and we obtain from (12) that

$$\lim_{r\to\infty}\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)}=\infty\ .$$

Combining Case I and Case II we may obtain that

$$\lim_{r\to\infty}\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{f}\left(\exp\left(r^{\alpha}\right)\right)+K\left(r,A;L\right)}=\infty\ ,$$

where $K\left(r,A;L\right)=\begin{cases}0 \text{ if } r^{\mu}=o\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\} \\ \qquad\qquad\text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right) \text{ otherwise }.\end{cases}$

This proves the theorem. □

Theorem 5. *Let f be an entire function with non zero finite order and lower order and g be an entire function with non zero finite lower order. If $\lambda_{f}^{L^{*}}>0$ and $\rho_{g}^{L^{*}}<\infty$ then for any $A>0$*

$$\lim_{r\to\infty}\frac{\log^{[2]}\nu_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log\nu_{g}\left(\exp\left(r^{\alpha}\right)\right)+K\left(r,A;L\right)}=\infty\ ,$$

$$\textit{where } 0 < \alpha < \lambda_g \textit{ and } K(r, A; L) = \begin{cases} 0 \textit{ if } r^{\alpha} = o\left\{L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{as } r \to \infty \\ L\left(\exp\left(\exp\left(\alpha r^{A}\right)\right)\right) \textit{ otherwise .} \end{cases}$$

The proof is omitted because it can be carried out in the line of Theorem 4.

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