J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
https://doi.org/10.7468/jksmeb.2018.25.4.297
Volume 25, Number 4 (November 2018), Pages 297–336

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SOME RESULTS ON (p,q)-TH RELATIVE RITT ORDER AND (p,q)-TH RELATIVE RITT TYPE OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES

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Abstract. In this paper we wish to establish some basic properties of entire functions represented by a vector valued Dirichlet series on the basis of (p,q)-th relative Ritt order, (p,q)-th relative Ritt type and (p,q)-th relative Ritt weak type where p and q are integers such that $p \geq 0$ and $q \geq 0$.

1. Introduction and Definitions

Suppose $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent *vector valued Dirichlet series* briefly known as *VVDS*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)$, $\lambda_n \to +\infty$ as $n \to +\infty$ and satisfy the conditions $\overline{\lim\limits_{n\to+\infty}} \frac{\log n}{\lambda_n} = D < +\infty$ and $\overline{\lim\limits_{n\to+\infty}} \frac{\log\|a_n\|}{\lambda_n} = -\infty$. If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = +\infty$. The function $M_f(\sigma)$ known as *maximum modulus* function corresponding to an entire function $f(s)$ defined by (1), is written as follows

$$M_f(\sigma) = \underset{-\infty<t<+\infty}{l.u.b.} \|f(\sigma + it)\| \ .$$

In this connection the following definition is well known:

Received by the editors March 27, 2018. Accepted November 24, 2018.
2010 *Mathematics Subject Classification.* 30B50, 30D15, 30D99.
Key words and phrases. vector valued Dirichlet series (VVDS), (p,q)-th relative Ritt order, (p,q)-th relative Ritt lower order, (p,q)-th relative Ritt type, (p,q)-th relative Ritt weak type, growth.

Definition 1. A non-constant entire function $f(s)$ defined by *VVDS* is said to have Property (A), if for any $\delta > 1$ and $\sigma > \sigma_0(\delta)$

$$[M_f(\sigma)]^2 \leq M_f(\sigma\delta) .$$

Property (A) has been closely studied by Bernal [1], [2].

Now we state the following two notations which are frequently used in our subsequent study:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \log^{[0]} x &= x, \log^{[-1]} x = \exp x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \exp^{[0]} x &= x, \exp^{[-1]} x = \log x . \end{aligned}$$

Further we assume that throughout the present paper p, q, m and l always denote integers. However, Juneja, Nandan and Kapoor [8] first introduced the concept of (p,q)*-th order* and (p,q)*-th lower order* of an *entire Dirichlet series* where $p \geq q + 1 \geq 1$. In the line of Juneja et al. [8], one can define the (p,q)*-th Ritt order* (respectively (p,q)*-th Ritt lower order*) of an entire function f represented by *VVDS* in the following way:

$$\rho^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]} \sigma}, \text{ respectively } \lambda^{(p,q)}(f) = \underline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]} \sigma},$$

where $p \geq q + 1 \geq 1$.

In this connection let us recall that if $0 < \rho^{(p,q)}(f) < \infty$, then the following properties hold

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } \quad n = 1, 2, \cdots \end{cases} .$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } \quad n = 1, 2, \cdots \end{cases} .$$

An entire function f (represented by *VVDS)* of *index-pair* (p,q) is said to be of *regular* (p,q) *Ritt growth* if its (p,q)*-th Ritt order* coincides with its (p,q)*-th Ritt lower order*, otherwise f is said to be of *irregular* (p,q) *Ritt growth.*

Now to compare the relative growth of two entire functions represented by *VVDS* having same non zero finite (p,q)-*th Ritt order*, one may introduce the definition of (p,q)-*th Ritt type* (respectively (p,q)-th *Ritt lower type*) in the following manner:

Definition 2. The (p,q)-*th Ritt type* (respectively (p,q)-th *Ritt lower type*) respectively denoted by $\Delta_f(p,q)$ (respectively $\overline{\Delta}_f(p,q)$) of an entire function f represented by *VVDS* when $0<\rho_f(p,q)<+\infty$ is defined as follows:

$$\Delta^{(p,q)}(f)=\overline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\rho_f(p,q)}}$$

$$\left(\text{respectively } \overline{\Delta}^{(p,q)}(f)=\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\rho_f(p,q)}}\right),$$

where $p\geq q+1\geq 1$.

Analogously to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite (p,q)-th *Ritt lower order,* one may introduce the definition of (p,q)-th *Ritt weak type* in the following way:

Definition 3. The (p,q)-th *Ritt weak type* denoted by $\tau_f(p,q)$ of an entire function f represented by *VVDS* is defined as follows:

$$\tau^{(p,q)}(f)=\underline{\lim}_{\sigma\to+\infty}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\lambda_f(p,q)}}\ ,\ 0<\ \lambda_f(p,q)\ <+\infty\ .$$

Also one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function f represented by *VVDS* in the following manner :

$$\overline{\tau}^{(p,q)}(f)=\overline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\lambda_f(p,q)}}\ ,\ 0<\ \lambda_f(p,q)\ <+\infty,$$

where $p\geq q+1\geq 1$.

The above definitions extend the *generalized Ritt growth indicators* of an entire function f represented by *VVDS* for each integer $p\geq 2$ and $q=0$. Also for $p=2$ and $q=0$, the above definitions reduce to the classical definitions of an entire function f represented by *VVDS.*

G. S. Srivastava [12] introduced the *relative Ritt order* between two entire functions represented by *VVDS* to avoid comparing growth just with $\exp\exp z$ In the case of *relative Ritt order,* it therefore seems reasonable to define suitably the (p, q)-*th relative Ritt order* of two entire functions represented by *VVDS*. Recently, Datta and Biswas [7] introduced the concept of (p, q)-*th relative Ritt order* $\rho_g^{(p,q)}(f)$ of an entire function f represented by *VVDS* with respect to another entire function g which is also represented by *VVDS*, in the light of index-pair which is as follows:

Definition 4 ([7])**.** Let f and g be any two entire functions represented by *VVDS* with index-pair (m, q) and (m, p), respectively, where p, q, m are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then the (p, q)-*th relative Ritt order* (respectively (p, q)-*th relative Ritt lower order)* of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[q]}\sigma}$$

$$\left(\text{respectively } \lambda_g^{(p,q)}(f) = \underline{\lim}_{\sigma\to+\infty} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[q]}\sigma}\right).$$

In this connection, we intend to give a definition of relative index-pair of an entire function with respect to another entire function (both of which are represented by *VVDS)* which is relevant in the sequel :

Definition 5. Let f and g be any two entire functions both represented by *VVDS* with index-pairs (m, q) and (m, p) respectively where $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then the entire function f is said to have relative index-pair (p, q) with respect to another entire function g, if $b < \rho_g^{(p,q)}(f) < \infty$ and $\rho_g^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q = m$ and $b = 0$ otherwise. Moreover if $0 < \rho_g^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda_g^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Further an entire function f (represented by *VVDS)* for which (p,q)*-th relative Ritt order* and (p,q)*-th relative Ritt lower order* with respect to another entire function g (represented by *VVDS)* are the same is called a function of *regular relative* (p,q) *Ritt growth* with respect to g. Otherwise, f is said to be of *irregular relative* (p,q) *Ritt growth.*with respect to g.

Now in order to compare the relative growth of two entire functions represented by *VVDS* having same non zero finite (p,q)*-th relative Ritt order* with respect to another entire function represented by *VVDS,* one may introduce the concepts of (p,q)*-th relative Ritt-type* (respectively (p,q)-th *relative Ritt lower type*) which are as follows:

Definition 6 ([3])**.** Let f and g be any two entire functions represented by *VVDS* with index-pair (m,q) and (m,p), respectively, where p,q,m are positive integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$ and $0 < \rho_g^{(p,q)}(f) < +\infty$. Then the (p,q)*-th relative Ritt type* (respectively (p,q)*-th relative Ritt lower type)* of f with respect to g are defined as

$$\Delta_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}}$$

$$\left(\text{respectively } \overline{\Delta}_g^{(p,q)}(f) = \underline{\lim}_{\sigma\to+\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}}\right).$$

Analogously to determine the relative growth of two entire functions represented by *VVDS* having same non zero finite (p,q)*-th relative Ritt lower order* with respect to another entire function represented by *VVDS*, one may introduce the definition of (p,q)*-th relative Ritt weak type* in the following way:

Definition 7 ([3])**.** Let f and g be any two entire functions represented by *VVDS* with index-pair (m,q) and (m,p), respectively, where p,q,m are positive integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$. Then (p,q)*-th relative Ritt weak type* denoted by $\tau_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g is defined as follows:

$$\tau_g^{(p,q)}(f) = \underline{\lim}_{\sigma\to+\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\lambda_g^{(p,q)}(f)}}\ ,\ 0 < \lambda_g^{(p,q)}(f) < +\infty\ .$$

Similarly the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g both represented by *VVDS* is defined in the following manner :

$$\overline{\tau}_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\lambda_g^{(p,q)}(f)}} \text{ , } 0 < \lambda_g^{(p,q)}(f) < +\infty \text{ .}$$

If f and g have got index-pair $(m, 0)$ and (m, l), respectively, then Definition 4, Definition 6 and Definition 7 reduces to the definition of *generalized relative Ritt growth indicators* such as *generalized relative Ritt order* $\rho_g^{[l]}(f)$, *generalized relative Ritt type* $\Delta_g^{[l]}(f)$ etc. If the entire functions f and g have the same index-pair $(p, 0)$ where p is any positive integer, we get the definitions of *relative Ritt growth indicators* such as *relative Ritt order* $\rho_g(f)$, *relative Ritt type* $\Delta_g(f)$ etc introduced by Srivastava [12] and Datta et al. [5]. Further if $g = \exp^{[m]} z$, then Definition 4, Definition 6 and Definition 7 reduces to the (m, q)*-th Ritt growth indicators* of an entire function f represented by *VVDS.* Also for $g = \exp^{[m]} z$, *relative Ritt growth indicators* reduces to the definition of *generalized Ritt growth indicators.*such as *generalized Ritt order* $\rho_g^{[m]}(f)$, *generalized Ritt type* $\Delta_g^{[m]}(f)$ etc. Moreover, if f is an entire function with index-pair $(2, 0)$ and $g = \exp^{[2]} z$, then Definition 4, Definition 6 and Definition 7 becomes the definitions of Ritt order, Ritt type, Ritt weak type etc. f represented by *VVDS.* For details about Ritt type, Ritt weak type etc., one may see [6].

In this connection we state the following definition which will be needed in the sequel:

Definition 8. A pair of entire functions f and g represented by *VVDS* are mutually said to have Property (X) if for all sufficiently large values of σ, both

$$M_{f\cdot g}(\sigma) > M_f(\sigma)$$

and

$$M_{f\cdot g}(\sigma) > M_g(\sigma)$$

hold simultaneously.

However, during the past decades, several authors (cf. [5, 6, 9, 10, 11, 13, 14, 15]) made closed investigations on the properties of entire Dirichlet series in different directions using the *growth indicator* such as *Ritt order*. In the present paper we wish to establish some basic properties of entire functions represented by a *VVDS* on the basis of (p,q)-th relative Ritt order, (p,q)-th relative Ritt type and (p,q)-th relative Ritt weak type where $p \geq 0$ and $q \geq 0$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([12]). *Suppose that f be an entire function represented by* VVDS *given in* (1)*, $\alpha > 1, 0 < \beta < \alpha$, $b > 1$ and $0 < \mu < \lambda$. Then*
(a) $M_f(\alpha\sigma) > e^{\beta\sigma} M_f(\sigma)$ *for all large σ and*
(b) $\lim\limits_{\sigma\to\infty} \frac{M_f(b\sigma)}{M_f(\sigma)} = \infty = \lim\limits_{\sigma\to\infty} \frac{M_f(\lambda\sigma)}{M_f(\mu\sigma)}$.

Lemma 2 ([12]). *Let f be an entire function represented by* VVDS *given in* (1) *satisfy the Property (A), then for any positive integer n and for all sufficiently large σ,*

$$(M_f(\sigma))^n \leq M_f(\delta\sigma)$$

holds where $\delta > 1$.

3. Main Results

In this section we present our main results.

Theorem 9. *Let us consider f_1, f_2 and g_1 be any three entire functions VVDS defined by* (1)*. Also let at least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 where $p \geq 0$ and $q \geq 0$. Then*

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}.$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_1 where $i,j = 1,2$ and $i \neq j$.

Proof. If $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = 0$, then the result is obvious. So we suppose that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) > 0.$$

We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k = 1,2$. Further let

$$\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} = \Delta$$

and f_2 is of regular relative (p,q) Ritt growth with respect to g_1. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_1}^{(p,q)}(f_1)$, we have for a sequence values of σ tending to infinity that

$$M_{f_1}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\log^{[q]}\sigma\right)\right)$$

$$i.e.,\ M_{f_1}(\sigma) \leq M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\sigma\right]\right] \ . \tag{2}$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_{g_1}^{(p,q)}(f_2)\left(=\lambda_{g_1}^{(p,q)}(f_2)\right)$, we obtain for all sufficiently large values of σ that

$$M_{f_2}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\log^{[q]}\sigma\right)\right) \tag{3}$$

$$i.e.,\ M_{f_2}(\sigma) \leq M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\sigma\right]\right] \ . \tag{4}$$

Since for all large σ, $M_{f_1 \pm f_2}(\sigma) \leq M_{f_1}(\sigma) + M_{f_2}(\sigma)$, we obtain from (2) and (4) for a sequence values of σ tending to infinity that

$$M_{f_1 \pm f_2}(\sigma) < 2M_{g_1}\left(\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\sigma\right)\right) \ . \tag{5}$$

Therefore in view of Lemma 1 (a), and for any $\beta > 2$, we obtain from (5) for a sequence values of σ tending to infinity that

$$M_{f_1 \pm f_2}\left(\frac{\sigma}{\beta}\right) < M_{g_1}\left(\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\sigma\right)\right)$$

$$i.e.,\ \frac{\log^{[p]} M_{g_1}^{-1}\left(M_{f_1 \pm f_2}\left(\frac{\sigma}{\beta}\right)\right)}{\log^{[q]}\sigma} < (\Delta+\varepsilon) \ .$$

Since $\beta > 2$ and $\varepsilon > 0$ are arbitrary, we get from above that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}.$$

Similarly, if we consider that f_1 is of regular relative (p,q) Ritt growth with respect to g_1 or both f_1 and f_2 are of regular relative (p,q) Ritt growth with respect to g_1, then one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}. \tag{6}$$

Now let $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p,q) Ritt growth with respect to g_1. As $\varepsilon\,(>0)$ is arbitrary, from the definition of $\lambda_{g_1}^{(p,q)}(f_1)$, it follows that for all sufficiently large values of σ

$$M_{f_1}(\sigma) \geq M_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q)}(f_1)-\varepsilon\right)\log^{[q]}\sigma\right]\right]. \tag{7}$$

Therefore in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we obtain for all sufficiently large values of σ that

$$M_{f_1}(\sigma) \geq M_{g_1}\left[\exp^{[p]}\left[(\Delta - \varepsilon)\log^{[q]}\sigma\right]\right]. \tag{8}$$

Now we consider the expression

$$\frac{M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_1) - \varepsilon\right)\log^{[q]}\sigma\right)\right)}{M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon\right)\log^{[q]}\sigma\right)\right)}. \tag{9}$$

Therefore in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and Lemma 1 (b), we obtain from (9) that

$$\lim_{\sigma\to\infty}\frac{M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_1) - \varepsilon\right)\log^{[q]}\sigma\right)\right)}{M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon\right)\log^{[q]}\sigma\right)\right)} = \infty\ . \tag{10}$$

Now (10) can also be written as

$$\lim_{\sigma\to\infty}\frac{M_{g_1}\left(\exp^{[p]}\left((\Delta - \varepsilon)\log^{[q]}\sigma\right)\right)}{M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon\right)\log^{[q]}\sigma\right)\right)} = \infty, \tag{11}$$

So from (11), we obtain for all sufficiently large values of σ that

$$\begin{aligned} M_{g_1}&\left(\exp^{[p]}\left((\Delta - \varepsilon)\log^{[q]}\sigma\right)\right) \\ &> 2M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon\right)\log^{[q]}\sigma\right)\right)\ . \end{aligned} \tag{12}$$

Thus from (3), (8) and (12) we get for all sufficiently large values of σ that

$$\begin{aligned} &M_{f_1}(\sigma) > 2M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon\right)\log^{[q]}\sigma\right)\right) \\ &i.e.,\ M_{f_1}(\sigma) > 2M_{f_2}(\sigma)\ . \end{aligned} \tag{13}$$

Since for all large σ, $M_{f_1\pm f_2}(\sigma) \geq M_{f_1}(\sigma) - M_{f_2}(\sigma)$, we obtain from (8), (13) and in view of Lemma 1 (a) for all sufficiently large values of σ and $\beta > 2$ that

$$M_{f_1\pm f_2}(\sigma) \geq \frac{1}{2}M_{f_1}(\sigma)$$

$$i.e.,\ M_{f_1\pm f_2}(\sigma) \geq \frac{1}{2}M_{g_1}\left(\exp^{[p]}\left((\Delta - \varepsilon)\log^{[q]}\sigma\right)\right)$$

$$i.e.,\ M_{f_1\pm f_2}(\beta\sigma) \geq M_{g_1}\left(\exp^{[p]}\left((\Delta - \varepsilon)\log^{[q]}\sigma\right)\right)$$

$$i.e.,\ \frac{\log^{[p]} M_{g_1}^{-1}\left(M_{f_1\pm f_2}(\beta\sigma)\right)}{\log^{[q]}\sigma} \geq (\Delta + \varepsilon)\ .$$

As $\beta > 2$ and $\varepsilon > 0$ are arbitrary, we get from above that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \geq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}.$$

If we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1, then one can also verify that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \geq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}. \tag{14}$$

So the conclusion of the second part of the theorem follows from (6) and (14). □

Now we state the following theorem without its proof as it can easily be carried out by a similar method in Theorem 9.

Theorem 10. *Let us consider f_1, f_2 and g_1 be any three entire functions VVDS defined by* (1). *Also let f_1 and f_2 be entire functions with relative index-pair (p,q) with respect to entire g_1 where $p \geq 0$ and $q \geq 0$. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\right\}.$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1) \neq, \rho_{g_1}^{(p,q)}(f_2)$.

Theorem 11. *Let us consider f_1, g_1 and g_2 be any three entire functions VVDS defined by* (1). *Also let $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where $p \geq 0$ and $q \geq 0$. Then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\}.$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$.

Proof. If $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \infty$, then the result is obvious. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) < \infty$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$. Further let $\Psi = \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_k}^{(p,q)}(f_1)$, we have for all sufficiently large values of σ that

$$M_{g_k}\left(\exp^{[p]}\left(\left(\lambda_{g_k}^{(p,q)}(f_1) - \varepsilon\right)\log^{[q]}\sigma\right)\right) \leq M_{f_1}(\sigma) \quad \text{where } k = 1, 2 \tag{15}$$

$$i.e., \ M_{g_k}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\sigma\right)\right) \leq M_{f_1}(\sigma) \quad \text{where } k = 1, 2$$

Since for all large σ, $M_{g_1 \pm g_2}(\sigma) \leq M_{g_1}(\sigma) + M_{g_2}(\sigma)$, we obtain from above and first part of Lemma 1(a) for all sufficiently large values of σ and $\beta > 2$ that

$$\begin{aligned}
&M_{g_1 \pm g_2}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\sigma\right)\right) \\
&< M_{g_1}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\sigma\right)\right) + M_{g_2}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\sigma\right)\right)
\end{aligned}$$

$$i.e.,\ M_{g_1 \pm g_2}\left(\exp^{[p]}\left((\Psi-\varepsilon)\log^{[q]}\sigma\right)\right) < 2M_{f_1}(\sigma)$$

$$i.e.,\ M_{g_1 \pm g_2}\left(\exp^{[p]}\left((\Psi-\varepsilon)\log^{[q]}\sigma\right)\right) < M_{f_1}(\beta\sigma)$$

$$i.e.,\ \frac{\log^{[p]} M^{-1}_{g_1 \pm g_2}\left(M_{f_1}(\beta\sigma)\right)}{\log^{[q]}\sigma} > \Psi - \varepsilon\ .$$

Since $\beta > 2$ and $\varepsilon > 0$ are arbitrary, we get from above that

$$\lambda^{(p,q)}_{g_1 \pm g_2}(f_1) \geq \Psi = \min\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\right\}\ . \tag{16}$$

Now let $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$ holds. As $\varepsilon\,(>0)$ is arbitrary, from the definition of $\lambda^{(p,q)}_{g_k}(f_1)$, it follows that for a sequence of values of σ tending to infinity

$$M_{f_1}(\sigma) \leq M_{g_k}\left(\exp^{[p]}\left(\left(\lambda^{(p,q)}_{g_k}(f_1)+\varepsilon\right)\log^{[q]}\sigma\right)\right) \text{ for } k = 1, 2$$

$$i.e.,\ M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda^{(p,q)}_{g_k}(f_1)+\varepsilon\right)}\right)\right) \leq M_{g_k}(\sigma) \text{ for } k = 1, 2\ . \tag{17}$$

Therefore in view of $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$, we obtain from above for a sequence of values of σ tending to infinity that

$$M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi+\varepsilon)}\right)\right) \leq M_{g_1}(\sigma)\ . \tag{18}$$

Now we consider the expression

$$\frac{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda^{(p,q)}_{g_1}(f_1)+\varepsilon\right)}\right)\right)}{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda^{(p,q)}_{g_2}(f_1)-\varepsilon\right)}\right)\right)}\ . \tag{19}$$

Therefore in view of $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$ and Lemma 1 (b), we obtain from (19) that

$$\lim_{\sigma\to\infty}\frac{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda^{(p,q)}_{g_1}(f_1)+\varepsilon\right)}\right)\right)}{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda^{(p,q)}_{g_2}(f_1)-\varepsilon\right)}\right)\right)} = \infty\ . \tag{20}$$

Now (20) can also be written as

$$\lim_{\sigma\to\infty}\frac{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi+\varepsilon)}\right)\right)}{M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)\right)}=\infty\ . \tag{21}$$

So from (21), we obtain for all sufficiently large values of σ that

$$M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi+\varepsilon)}\right)\right)>2M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)\right)\ . \tag{22}$$

Now from (15), it follows for all sufficiently large values of σ that

$$M_{g_2}(\sigma)\le M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)\right)\ . \tag{23}$$

Thus from (18), (22) and (23) we get for a sequence of values of σ tending to infinity that

$$M_{g_1}(\sigma)>2M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{\left(\lambda_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)\right)$$

$$i.e.,\ M_{g_1}(\sigma)>2M_{g_2}(\sigma)\ . \tag{24}$$

Since for all large σ, $M_{g_1\pm g_2}(\sigma)\ge M_{g_1}(\sigma)-M_{g_2}(\sigma)$, we obtain from (18), (24) and in view of Lemma 1 (a) for a sequence of values of σ tending to infinity and $\beta>2$ that

$$M_{g_1\pm g_2}(\sigma)\ge\frac{1}{2}M_{g_1}(\sigma)$$

$$i.e.,\ M_{g_1\pm g_2}(\sigma)\ge\frac{1}{2}M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi+\varepsilon)}\right)\right)$$

$$i.e.,\ M_{g_1\pm g_2}(\sigma)\ge M_{f_1}\left(\frac{\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi+\varepsilon)}\right)}{\beta}\right)$$

$$i.e.,\ \exp^{[p]}\left((\Psi+\varepsilon)\left(\log^{[q]}(\beta\sigma)\right)\right)\ge M_{g_1\pm g_2}^{-1}M_{f_1}(\sigma)$$

$$\frac{\log^{[p]}M_{g_1\pm g_2}^{-1}(M_{f_1}(\sigma))}{\log^{[q]}(\beta\sigma)}\le(\Psi+\varepsilon)\ .$$

Since $\beta>2$ and $\varepsilon>0$ are arbitrary, we get from above that

$$\lambda_{g_1\pm g_2}^{(p,q)}(f_1)\le\Psi=\min\left\{\lambda_{g_1}^{(p,q)}(f_1),\lambda_{g_2}^{(p,q)}(f_1)\right\}\ .$$

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can also derive that

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \leq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} . \tag{25}$$

So the conclusion of the second part of the theorem follows from (16) and (25). □

Theorem 12. *Let f_1, g_1 and g_2 be any three entire functions VVDS defined by* (1). *Also let the relative index-pair of f_1 with respect to g_1 and g_2 is (p,q) where $p \geq 0$ and $q \geq 0$. Also let f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2. Then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}.$$

The equality holds when $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) growth with respect to g_j where $i,j = 1,2$ and $i \neq j$.

We omit the proof of Theorem 12 as it can easily be carried out in the line of Theorem 11.

Theorem 13. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1). *Then for $p \geq 0$ and $q \geq 0$*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) \leq \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) Ritt growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i) < \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) < \rho_{g_2}^{(p,q)}(f_j)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 14. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1). *Then for $p \geq 0$ and $q \geq 0$,*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) \geq \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_1 for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) $\lambda_{g_2}^{(p,q)}(f_i) > \lambda_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_2 for $i = 1, 2$, $j = 1,2$ and $i \neq j$.

The equality holds when $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ *and* $\lambda_{g_i}^{(p,q)}(f_2) < \lambda_{g_j}^{(p,q)}(f_2)$ *hold simultaneously for* $i = 1, 2$; $j = 1, 2$ *and* $i \neq j$.

Theorem 13 and Theorem 14 can be prove using the similar arguments adopted in the proofs of Theorem 5 and Theorem 6 of [4] respectively. We omit the details.

Theorem 15. *Let* f_1, f_2 *and* g_1 *be any three entire functions VVDS defined by* (1). *Also let at least* f_1 *or* f_2 *is of regular relative* (p, q) *Ritt growth with respect to* g_1 *where* $p \geq 0$ *and* $q \geq 0$. *Then*

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}.$$

provided g_1 *satisfy the Property (A). The equality holds when* f_1 *and* f_2 *satisfy Property (X).*

Proof. Suppose that $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) > 0$. Otherwise if $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) = 0$ then the result is obvious. Let us consider that f_2 is of regular relative (p, q) Ritt growth with respect to g_1.Also suppose that $\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\} = \Delta$. We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k = 1, 2$. Now for any arbitrary $\frac{\varepsilon}{2} > 0$, it follows from the definition of $\rho_{g_1}^{(p,q)}(f_1)$, for a sequence values of σ tending to infinity that

$$M_{f_1}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right)$$

$$\text{i.e., } M_{f_1}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\Delta + \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right). \tag{26}$$

Also for any arbitrary $\frac{\varepsilon}{2} > 0$, we obtain from the definition of $\rho_{g_1}^{(p,q)}(f_2)\left(= \lambda_{g_1}^{(p,q)}(f_2)\right)$, for all sufficiently large values of σ that

$$M_{f_2}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\lambda_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right)$$

$$\text{i.e., } M_{f_2}(\sigma) \leq M_{g_1}\left(\exp^{[p]}\left(\left(\Delta + \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right). \tag{27}$$

Observe that

$$\frac{\Delta + \varepsilon}{\Delta + \frac{\varepsilon}{2}} > 1 .$$

Therefore we consider the expression $\frac{\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\sigma\right)}{\exp^{[p]}\left(\left(\Delta+\frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)}$ for all sufficiently large values of σ. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of σ, say $\sigma \geq \sigma_1 \geq \sigma_0$ that

$$\frac{\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\sigma_0\right)}{\exp^{[p]}\left((\Delta+\frac{\varepsilon}{2})\log^{[q]}\sigma_0\right)} = \delta\ . \tag{28}$$

Since for all large σ, $M_{f_1 \cdot f_2}(\sigma) < M_{f_1}(\sigma) \cdot M_{f_2}(\sigma)$, we have from (26), (27) for a sequence values of σ tending to infinity that

$$M_{f_1 \cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p]}\left(\left(\Delta+\frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right)\right)^2\ .$$

Also in view of Lemma 2, we obtain from above for a sequence values of σ tending to infinity that

$$M_{f_1 \cdot f_2}(\sigma) < M_{g_1}\left(\delta\left(\exp^{[p]}\left(\left(\Delta+\frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right)\right),$$

since g_1 has the Property (A) and $\delta > 1$. Therefore in view of (28), it follows from above for a sequence values of σ tending to infinity that

$$M_{f_1 \cdot f_2}(\sigma) < M_{g_1}\left[\left(\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\sigma\right)\right)\right]\ . \tag{29}$$

So from above we get for a sequence values of σ tending to infinity that

$$\frac{\log^{[p]} M_{g_1}^{-1}\left(M_{f_1 \cdot f_2}(\sigma)\right)}{\log^{[q]}\sigma} \leq (\Delta+\varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary, we obtain from above that

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}\ .$$

Similarly, if we consider that f_1 is of regular relative (p,q) Ritt growth with respect to g_1 or both f_1 and f_2 are of regular relative (p,q) Ritt growth with respect to g_1, then also one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}. \tag{30}$$

Now let f_1 and f_2 are satisfy Property (X), then of course we have $M_{f_1 \cdot f_2}(\sigma) > M_{f_1}(\sigma)$ and $M_{f_1 \cdot f_2}(\sigma) > M_{f_2}(\sigma)$ for all sufficiently large values of σ. Therefore for all sufficiently large values of σ we get that

$$\frac{\log^{[p]} M_{g_1}^{-1}\left(M_{f_1 \cdot f_2}(\sigma)\right)}{\log^{[q]}\sigma} \geq \frac{\log^{[p]} M_{g_1}^{-1}\left(M_{f_1}(\sigma)\right)}{\log^{[q]}\sigma}.$$

So $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_1)$ and similarly, $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_2)$.

Hence the theorem follows. □

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 15.

Theorem 16. *Let f_1, f_2 and g_1 be any three entire functions VVDS defined by* (1)*. Also let f_1 and f_2 have relative index-pair (p,q) with respect to g_1 where $p \geq 0$ and $q \geq 0$. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\right\},$$

provided g_1 satisfy the Property (A). The equality holds when f_1 and f_2 satisfy Property (X).

Theorem 17. *Let f_1, g_1 and g_2 be any three entire functions VVDS defined by* (1)*. Also let $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where $p \geq 0$ and $q \geq 0$. Then*

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\},$$

provided f_1 satisfy the Property (A). The equality holds when g_1 and g_2 satisfy Property (X).

Proof. Suppose that $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) < \infty$. Otherwise if $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = \infty$ then the result is obvious. Also suppose that $\min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} = \Psi$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$. Now for any arbitrary $\varepsilon > 0$, with $\varepsilon < \Psi$, we obtain for all sufficiently large values of σ that

$$M_{g_k}\left(\exp^{[p]}\left(\left(\lambda_{g_k}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right) \leq M_{f_1}(\sigma) \quad \text{where } k = 1, 2$$

$$i.e.,\ M_{g_k}\left(\exp^{[p]}\left(\left(\Psi - \frac{\varepsilon}{2}\right)\log^{[q]}\sigma\right)\right) \leq M_{f_1}(\sigma) \quad \text{where } k = 1, 2$$

$$(31) \qquad i.e.,\ M_{g_k}(\sigma) \leq M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi - \frac{\varepsilon}{2})}\right)\right) \quad \text{where } k = 1, 2\ .$$

Observe that

$$\frac{\Psi - \frac{\varepsilon}{2}}{\Psi - \varepsilon} > 1\ .$$

Now we consider the expression $\dfrac{\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi-\varepsilon)}\right)}{\exp^{[q]}\left(\frac{\log^{[p]}\sigma}{(\Psi-\frac{\varepsilon}{2})}\right)}$ for all sufficiently large values of σ. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of σ, say $\sigma \geq \sigma_1 \geq \sigma_0$ that

$$(32) \qquad \frac{\exp^{[q]}\left(\frac{\log^{[p]}\sigma_0}{(\Psi-\varepsilon)}\right)}{\exp^{[q]}\left(\frac{\log^{[p]}\sigma_0}{(\Psi-\frac{\varepsilon}{2})}\right)} = \delta\ .$$

Since for all large σ, $M_{g_1 \cdot g_2}(\sigma) < M_{g_1}(\sigma) \cdot M_{g_2}(\sigma)$, we get from (31) for all sufficiently large values of σ that

$$M_{g_1 \cdot g_2}(\sigma) < \left(M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]} \sigma}{(\Psi - \frac{\varepsilon}{2})} \right) \right) \right)^2 . \tag{33}$$

Also in view of Lemma 2, we obtain from above for all sufficiently large values of σ that

$$M_{g_1 \cdot g_2}(\sigma) < M_{f_1}\left(\delta \left(\exp^{[q]}\left(\frac{\log^{[p]} \sigma}{(\Psi - \frac{\varepsilon}{2})} \right) \right) \right),$$

since f_1 has the Property (A) and $\delta > 1$. Therefore in view of (32), it follows from above for all sufficiently large values of σ that

$$M_{g_1 \cdot g_2}(\sigma) < M_{f_1}\left(\exp^{[q]}\left(\frac{\log^{[p]} \sigma}{(\Psi - \varepsilon)} \right) \right)$$

$$i.e, \ \frac{\log^{[p]} M_{g_1 \cdot g_2}^{-1}(M_{f_1}(\sigma))}{\log^{[q]} \sigma} > (\Psi - \varepsilon) \ .$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \Psi = \min\left\{ \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1) \right\} . \tag{34}$$

Now let g_1 and g_2 are satisfy Property (X), then of course we have $M_{g_1 \cdot g_2}(\sigma) > M_{g_1}(\sigma)$ and $M_{g_1 \cdot g_2}(\sigma) > M_{g_2}(\sigma)$ for all sufficiently large values of σ. Therefore for all sufficiently large values of σ, we obtain that $M_{g_1 \cdot g_2}^{-1}(\sigma) \leq M_{g_1}^{-1}(\sigma)$ and $M_{g_1 \cdot g_2}^{-1}(\sigma) \leq M_{g_2}^{-1}(\sigma)$. Hence it follows that for all sufficiently large values of σ

$$\frac{\log^{[p]} M_{g_1 \cdot g_2}^{-1}(M_{f_1}(\sigma))}{\log^{[q]} \sigma} \leq \frac{\log^{[p]} M_{g_1}^{-1}(M_{f_1}(\sigma))}{\log^{[q]} \sigma}.$$

So $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \leq \lambda_{g_1}^{(p,q)}(f_1)$ and similarly, $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \leq \lambda_{g_2}^{(p,q)}(f_1)$.

Thus the theorem follows. □

Theorem 18. *Let f_1, g_1 and g_2 be any three entire functions VVDS defined by* (1). *Also let the relative index-pair of f_1 with respect to g_1 and g_2 is (p,q) where $p \geq 0$ and $q \geq 0$. Then*

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\left\{ \rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1) \right\},$$

provided f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2 and f_1 satisfy the Property (A). The equality holds when g_1 and g_2 satisfy Property (X).

We omit the proof of Theorem 18 as it can easily be carried out in the line of Theorem 17.

Now we state the following two theorems without their proofs as those can easily be carried out with the help of Theorem 15, Theorem 16, Theorem 17 and Theorem 18 and in the line of Theorem 13 and Theorem 14 respectively.

Theorem 19. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*. Also let $g_1 \cdot g_2$, f_1 and f_2 be satisfy the Property (A). Then for $p \geq 0$ and $q \geq 0$,*

$$\begin{aligned}&\rho^{(p,q)}_{g_1 \cdot g_2}(f_1 \cdot f_2)\\&= \max\left[\min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\}, \min\left\{\rho^{(p,q)}_{g_1}(f_2), \rho^{(p,q)}_{g_2}(f_2)\right\}\right],\end{aligned}$$

when the following two conditions holds:
(i) f_1 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(ii) f_2 is of regular relative (p,q) growth with respect to at least any one of g_1 or g_2;
(iii) f_1 and f_2 satisfy Property (X); and
(iv) g_1 and g_2 satisfy Property (X).

Theorem 20. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*. Also let $g_1 \cdot g_2$, f_1 and f_2 be satisfy the Property (A). Then for $p \geq 0$ and $q \geq 0$,*

$$\begin{aligned}&\lambda^{(p,q)}_{g_1 \cdot g_2}(f_1 \cdot f_2)\\&= \min\left[\max\left\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_1}(f_2)\right\}, \max\left\{\lambda^{(p,q)}_{g_2}(f_1), \lambda^{(p,q)}_{g_2}(f_2)\right\}\right]\end{aligned}$$

when the following two conditions holds:
(i) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_1;
(ii) At least f_1 or f_2 is of regular relative (p,q) growth with respect to g_2;
(iii) f_1 and f_2 satisfy Property (X); and
(iv) g_1 and g_2 satisfy Property (X).

Theorem 21. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*. Also let $\rho^{(p,q)}_{g_1}(f_1)$, $\rho^{(p,q)}_{g_1}(f_2)$, $\rho^{(p,q)}_{g_2}(f_1)$ and $\rho^{(p,q)}_{g_2}(f_2)$ are all non zero and finite where $p \geq 0$ and $q \geq 0$.*
(A) *If $\rho^{(p,q)}_{g_1}(f_i) > \rho^{(p,q)}_{g_1}(f_j)$ for i, $j = 1, 2$ and $i \neq j$, then*

$$\Delta^{(p,q)}_{g_1}(f_1 \pm f_2) = \Delta^{(p,q)}_{g_1}(f_i) \text{ and } \overline{\Delta}^{(p,q)}_{g_1}(f_1 \pm f_2) = \overline{\Delta}^{(p,q)}_{g_1}(f_i) \mid i = 1, 2\ .$$

(B) *If $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_j for i, $j = 1, 2$ and $i \neq j$, then*

$$\Delta^{(p,q)}_{g_1 \pm g_2}(f_1) = \Delta^{(p,q)}_{g_i}(f_1) \text{ and } \overline{\Delta}^{(p,q)}_{g_1 \pm g_2}(f_1) = \overline{\Delta}^{(p,q)}_{g_i}(f_1) \mid i = 1, 2\ .$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ *with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;*
(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ *with at least f_2 is of regular relative (p,q) Ritt growth with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;*
(iii) $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ *and* $\rho_{g_2}^{(p,q)}(f_i) > \rho_{g_2}^{(p,q)}(f_j)$ *holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$;*
(iv) $\rho_{g_m}^{(p,q)}(f_l) = \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l = m = 1, 2$*;*
then we have

$$\Delta_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2$$

and

$$\overline{\Delta}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2\ .$$

Proof. From the definition of relative (p,q)-th Ritt type and relative (p,q)-th lower Ritt type of entire function VVDS defined by (1), we have for all sufficiently large values of σ that

$$M_{f_k}(\sigma) \leq M_{g_l}\left(\exp^{[p-1]}\left(\left(\Delta_{g_l}^{(p,q)}(f_k) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right)\right), \tag{35}$$

$$M_{f_k}(\sigma) \geq M_{g_l}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_l}^{(p,q)}(f_k) - \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right)\right) \tag{36}$$

$$i.e.,\ M_{g_l}(\sigma) \leq M_{f_k}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\Delta}_{g_l}^{(p,q)}(f_k) - \varepsilon\right)}\right)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\right)\right), \tag{37}$$

and for a sequence of values of σ tending to infinity, we obtain that

$$M_{f_k}(\sigma) \geq M_{g_l}\left(\exp^{[p-1]}\left(\left(\Delta_{g_l}^{(p,q)}(f_k) - \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right)\right) \tag{38}$$

$$i.e.,\ M_{g_l}(\sigma) \leq M_{f_k}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_l}^{(p,q)}(f_k) - \varepsilon\right)}\right)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\right)\right), \tag{39}$$

and

$$M_{f_k}(\sigma) \leq M_{g_l}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_l}^{(p,q)}(f_k) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right)\right), \tag{40}$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1$, 2 and $l = 1, 2$.

CASE I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Also let $\varepsilon (> 0)$ be arbitrary. Since for all large σ, $M_{f_1 \pm f_2}(\sigma) \leq M_{f_1}(\sigma) + M_{f_2}(\sigma)$, we get in view of (35) and for all sufficiently large values of σ that

$$M_{f_1 \pm f_2}(\sigma) \leq \tag{41}$$
$$M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)(1 + A)$$

where $A = \frac{M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right)\right)}{M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1)$ $> \rho_{g_1}^{(p,q)}(f_2)$, and for all sufficiently large values of σ, we can make the term A sufficiently small. Hence for any $\alpha > 1 + \varepsilon_1$ where $\varepsilon_1 = A$, it follows from Lemma 1 (a) and (41) for all sufficiently large values of σ that

$$M_{f_1 \pm f_2}(\sigma) \leq M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)(1 + \varepsilon_1)$$

$$i.e.,\ M_{f_1 \pm f_2}(\sigma) \leq M_{g_1}\left(\exp^{[p-1]}\left(\alpha\left(\Delta_{g_1}^{(p,q)}(f_1) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right).$$

Therefore in view of Theorem 10 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we get from above for all sufficiently large values of σ that

$$\frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1 \pm f_2}(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}} \leq \alpha\left(\Delta_{g_1}^{(p,q)}(f_1) + \varepsilon\right).$$

Hence making $\alpha \to 1+$, we obtain from above for all sufficiently large values of σ that

$$\overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1 \pm f_2}(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}} \leq \Delta_{g_1}^{(p,q)}(f_1)$$

$$i.e.,\ \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta_{g_1}^{(p,q)}(f_1). \tag{42}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Then $\Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta_{g_1}^{(p,q)}(f_1)$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 10 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we obtain that $\rho_{g_1}^{(p,q)}(f) > \rho_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of (42) $\Delta_{g_1}^{(p,q)}(f_1) \leq \Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1) \Rightarrow \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\Delta_{g_1}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_1}^{(p,q)}(f_2)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Also let $\varepsilon\,(>0)$ are arbitrary. Since for all large σ, $M_{f_1 \pm f_2}(\sigma) \leq M_{f_1}(\sigma) + M_{f_2}(\sigma)$, we get from (35) and (40) for a sequence of values of σ tending to infinity that

$$(43)\qquad M_{f_1 \pm f_2}(\sigma_n) \leq$$

$$M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)(1+B)$$

where $B = \frac{M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right)\right)}{M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we can make the term B sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (43) that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ when $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_2)$.

Thus combining Case I and Case II, we obtain the first part of the theorem

Case III. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_2. As for all large σ, $M_{g_1 \pm g_2}(\sigma) \leq M_{g_1}(\sigma) + M_{g_2}(\sigma)$, we get from (37) and (39) for a sequence of values of σ tending to infinity that

$$(44)\qquad M_{g_1 \pm g_2}(\sigma_n) \leq$$

$$M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)(1+C),$$

where $C = \frac{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\overline{\Delta}_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right)}{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)}$. Since $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term C sufficiently small by taking n sufficiently large. Hence in view of Lemma 1 (a) and Theorem 12, we get from (44) for a sequence of values of

σ tending to infinity that

$$M_{g_1 \pm g_2}(\sigma_n) < M_{f_1}\left(\exp^{[q-1]}\left(\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)(1+\varepsilon_1)$$

$$M_{g_1 \pm g_2}\left(\frac{\sigma_n}{\alpha}\right) < M_{f_1}\left(\exp^{[q-1]}\left(\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1 \pm g_2}^{(p,q)}(f_1)}}\right)\right)\right),$$

where $\varepsilon_1 = C$ and $\alpha > (1+\varepsilon_1)$. Hence, making $\alpha \to 1+$, we obtain from above for a sequence of values of σ tending to infinity that

$$\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\rho_{g_1 \pm g_2}^{(p,q)}(f_1)} < \log^{[p-1]} M_{g_1 \pm g_2}^{-1}\left(M_{f_1}(\sigma_n)\right) .$$

Since $\varepsilon > 0$ is arbitrary, we find that

$$\Delta_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \Delta_{g_1}^{(p,q)}(f_1) . \tag{45}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) Ritt growth with respect to g_2. Then $\Delta_{g}^{(p,q)}(f_1) = \Delta_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \Delta_{g_1}^{(p,q)}(f_1)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 12 and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we obtain that $\rho_{g}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ as at least f_1 is of regular relative (p,q) Ritt growth with respect to g_2. Hence in view of (45), $\Delta_{g_1}^{(p,q)}(f_1) \geq \Delta_{g}^{(p,q)}(f_1) = \Delta_{g_1 \pm g_2}^{(p,q)}(f_1)$. Therefore $\Delta_{g}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1) \Rightarrow \Delta_{g_1 \pm g_2}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1)$.

Similarly if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1, then $\Delta_{g_1 \pm g_2}^{(p,q)}(f_1) = \Delta_{g_2}^{(p,q)}(f_1)$.

CASE IV. In this case suppose that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_2. Therefore from (37), we get for all sufficiently large values of σ that

$$M_{g_1 \pm g_2}(\sigma) \leq \tag{46}$$

$$M_{f_1}\left(\exp^{[q-1]}\left(\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)(1+D) ,$$

where $D = \dfrac{M_{f_1}\left(\exp^{[q-1]}\left(\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\Delta}_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right)\right)}{M_{f_1}\left(\exp^{[q-1]}\left(\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term D sufficiently small by taking σ sufficiently large and therefore using the similar technique for as executed in the proof of Case III we get from (46) that $\overline{\Delta}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ where $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least f_1 is of regular relative (p,q) Ritt growth with respect to g_2. Likewise if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1, then $\overline{\Delta}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_2}^{(p,q)}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 13 and the first part and second part of the theorem. Hence its proof is omitted. $\square$

Theorem 22. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by (1). Also let $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ are all non zero and finite where $p \geq 0$ and $q \geq 0$.*

(A) *If $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_1 for i, $j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_i) \ \text{ and } \ \overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \mid i = 1,2 \ .$$

(B) *If $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ for i, $j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1\pm g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \ \text{ and } \ \overline{\tau}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1) \mid i = 1,2 \ .$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_1 for i, $j = 1,2$ and $i \neq j$;
(ii) $\rho_{g_2}^{(p,q)}(f_i) > \rho_{g_2}^{(p,q)}(f_j)$ with at least f_j is of regular relative (p,q) Ritt growth with respect to g_2 for i, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ and $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ holds simultaneously for i, $j = 1,2$ and $i \neq j$;
(iv) $\lambda_{g_m}^{(p,q)}(f_l) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l = m = 1,2$;
then we have

$$\tau_{g_1\pm g_2}^{(p,q)}(f_1 \pm f_2) = \tau_{g_m}^{(p,q)}(f_l) \mid l = m = 1,2$$

and

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q)}\left(f_1 \pm f_2\right) = \overline{\tau}_{g_m}^{(p,q)}\left(f_l\right) \mid l = m = 1, 2 \ .$$

Proof. For any arbitrary positive number $\varepsilon(>0)$, we have for all sufficiently large values of σ that

$$M_{f_k}\left(\sigma\right) \leq M_{g_l}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_l}^{(p,q)}\left(f_k\right) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right)\right), \tag{47}$$

$$M_{f_k}\left(\sigma\right) \geq M_{g_l}\left(\exp^{[p-1]}\left(\left(\tau_{g_l}^{(p,q)}\left(f_k\right) - \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right)\right) \tag{48}$$

$$i.e.,\ M_{g_l}\left(\sigma\right) \leq M_{f_k}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_l}^{(p,q)}\left(f_k\right) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\right)\right), \tag{49}$$

and for a sequence of values of σ tending to infinity we obtain that

$$M_{f_k}\left(\sigma\right) \geq M_{g_l}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_l}^{(p,q)}\left(f_k\right) - \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right)\right) \tag{50}$$

$$i.e.,\ M_{g_l}\left(\sigma\right) \leq M_{f_k}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\tau}_{g_l}^{(p,q)}\left(f_k\right) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\right)\right), \tag{51}$$

and

$$M_{f_k}\left(\sigma\right) \leq M_{g_l}\left(\exp^{[p-1]}\left(\left(\tau_{g_l}^{(p,q)}\left(f_k\right) + \varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_l}^{(p,q)}(f_k)}\right)\right), \tag{52}$$

where $k = 1, 2$ and $l = 1, 2$.

CASE I. Let $\lambda_{g_1}^{(p,q)}\left(f_1\right) > \lambda_{g_1}^{(p,q)}\left(f_2\right)$ with at least f_2 is of regular relative (p, q) Ritt growth with respect to g_1. Also let $\varepsilon\left(>0\right)$ be arbitrary.

Since for all large σ, $M_{f_1 \pm f_2}\left(\sigma\right) \leq M_{f_1}\left(\sigma\right) + M_{f_2}\left(\sigma\right)$, we get in view of (47) and (52) for a sequence $\{r_n\}$ of values of σ tending to infinity that

$$M_{f_1 \pm f_2}\left(\sigma_n\right) \leq M_{g_1}\left(\exp^{[p-1]}\left(\left(\tau_{g_1}^{(p,q)}\left(f_1\right) + \varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)\left(1 + E\right), \tag{53}$$

where $E = \frac{M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\lambda_{g_1}^{(p,q)}(f_2)}\right)\right)}{M_{g_1}\left(\exp^{[p-1]}\left(\left(\tau_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma_n\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we can make the term E sufficiently small by taking n sufficiently large. Therefore with the help of Lemma 1 (a), Theorem 9 and using the similar technique of Case I of Theorem 21, we get from (53) that

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q)}(f_1)\ . \tag{54}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p,q) Ritt growth with respect to g_1. Then $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q)}(f_1)$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 9, $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p,q) Ritt growth with respect to g_1, we obtain that $\lambda_{g_1}^{(p,q)}(f) > \lambda_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of (54), $\tau_{g_1}^{(p,q)}(f_1) \leq \tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1 then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_2)$.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_2 is of regular relative (p,q) Ritt growth with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. Since for all large σ, $M_{f_1 \pm f_2}(\sigma) \leq M_{f_1}(\sigma) + M_{f_2}(\sigma)$, we get in view of (47) for all sufficiently large values of σ that

$$M_{f_1 \pm f_2}(\sigma) \leq \tag{55}$$

$$M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)(1+F),$$

where $F = \frac{M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_i}^{(p,q)}(f_2)}\right)\right)}{M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, one can make the term F sufficiently small by taking σ sufficiently large and therefore for similar reasoning of Case I we get from (55) that $\overline{\tau}_{g_1}^{(p_i,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least f_2 is of regular relative (p,q) Ritt growth with respect to g_1.

Likewise, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1 then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_2)$

Thus combining Case I and Case II, we obtain the first part of the theorem.

CASE III. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. As for all large σ, $M_{g_1 \pm g_2}(\sigma) \leq M_{g_1}(\sigma) + M_{g_2}(\sigma)$, we get from (49) for all sufficiently large values of σ that

$$(56) \qquad M_{g_1 \pm g_2}(\sigma) \leq$$

$$M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_1}^{(p,q)}(f_1) - \varepsilon\right)}\right)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\right)\right)(1+G) ,$$

where $G = \dfrac{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right)}{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\right)\right)}$, and as $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we can make the term G sufficiently small by taking σ sufficiently large. Now with the help of Lemma 1 (a) and Theorem 11 and using the similar technique of Case III of Theorem 21, we get from (56) that

$$(57) \qquad \tau_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1) \ .$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, so $\tau_g^{(p,q)}(f_1) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1)$. Also let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 11 and $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ we obtain that $\lambda_g^{(p.q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ holds. Hence in view of (57) $\tau_{g_1}^{(p,q)}(f_1) \geq \tau_g^{(p,q)}(f_1) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1)$. Therefore $\tau_g^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1)$.

Likewise, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$.

CASE IV. In this case further we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$.

As for all large σ, $M_{g_1 \pm g_2}(\sigma) \leq M_{g_1}(\sigma) + M_{g_2}(\sigma)$, we obtain from (49) and (51) for a sequence $\{r_n\}$ of values of r tending to infinity that

$$(58) \qquad M_{g_1 \pm g_2}(\sigma_n) \leq$$

$$M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right)(1+H)\ ,$$

where $H = \dfrac{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}\right)\right)}{M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right)}$, and in view of $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we can make the term H sufficiently small by taking n sufficiently large and therefore using the similar technique as executed in the proof of Case IV of Theorem 21, we get from (58) that $\overline{\tau}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$.

Similarly, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_2}^{(p,q)}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 14 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 9 to Theorem 12 under somewhat different conditions.

Theorem 23. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*. Also let $p \geq 0$ and $q \geq 0$.*

(A) *The following condition is assumed to be satisfied:*

(*i*) *Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_1}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1}^{(p,q)}(f_2)$ holds, then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)\ .$$

(B) *The following conditions are assumed to be satisfied:*

(*i*) *Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_1)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1)$ holds;*

(*ii*) *f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2, then*

$$\rho_{g_1\pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)\ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by (1) satisfy the conditions of the theorem.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2) < \infty)$. Now in view of Theorem 10 it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \rho_{g_1}^{(p,q)}(f_1) =$

$\rho_{g_1}^{(p,q)}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) < \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) \ . \tag{59}$$

Let $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_1}^{(p,q)}(f_2)$. Then in view of the first part of Theorem 21 and (59) we obtain that $\Delta_{g_1}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \Delta_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2)$. Similarly with the help of the first part of Theorem 21, one can obtain the same conclusion under the hypothesis $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

CASE II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ $(0 < \rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1) < \infty)$ and f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$. Therefore in view of Theorem 12, it follows that $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1) \ . \tag{60}$$

Let us consider that $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_1)$. Then. in view of the proof of the second part of Theorem 21 and (60) we obtain that $\Delta_{g_1}^{(p,q)}(f_1) = \Delta_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \Delta_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1)$. Also in view of the proof of second part of Theorem 21 one can derive the same conclusion for the condition $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 24. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*. Also let $p \geq 0$ and $q \geq 0$.*

(A) *The following conditions are assumed to be satisfied:*

(i) $(f_1 \pm f_2)$ is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2;

(ii) Either $\Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \Delta_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1 \pm f_2)$;

(iii) Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_1}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1}^{(p,q)}(f_2)$;

(iv) Either $\Delta_{g_2}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_2}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*

(i) f_1 and f_2 are of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2;

(ii) Either $\Delta_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \Delta_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1 \pm g_2}^{(p,q)}(f_2)$;

(*iii*) *Either* $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_1)$ *or* $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1)$*;*
(*iv*) *Either* $\Delta_{g_1}^{(p,q)}(f_2) \neq \Delta_{g_2}^{(p,q)}(f_2)$ *or* $\overline{\Delta}_{g_1}^{(p,q)}(f_2) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_2)$*; then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2) \ .$$

We omit the proof of Theorem 24 as it is a natural consequence of Theorem 23.

Theorem 25. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1)*.*
(A) *The following conditions are assumed to be satisfied:*
(*i*) *At least any one of f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 where $p \geq 0$ and $q \geq 0$;*
(*ii*) *Either* $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$ *holds, then*

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(*i*) *f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists where $p \geq 0$ and $q \geq 0$;*
(*ii*) *Either* $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ *holds, then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1) \ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions functions VVDS defined by (1) satisfy the conditions of the theorem.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2) < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular relative (p,q) Ritt growth with respect to g_1. Now, in view of Theorem 9, it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) \ . \tag{61}$$

Let $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$. Then in view of the proof of the first part of Theorem 22 and (61) we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2) = \tau_{g_1}^{(p,q)}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2)$. Similarly in view of the proof of the first part of Theorem 22 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1) < \infty$. Therefore in view of Theorem 11, it follows that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \lambda_{g_1}^{(p,q)}(f_1) =$

$\lambda_{g_2}^{(p,q)}(f_1)$ and if possible let

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)\ . \tag{62}$$

Suppose $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$. Then in view of the second part of Theorem 22 and (62), we obtain that $\tau_{g_1}^{(p,q)}(f_1) = \tau_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_1)$. Analogously with the help of the second part of Theorem 22, the same conclusion can also be derived under the condition $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 26. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1).

(A) *The following conditions are assumed to be satisfied:*

(*i*) *At least any one of f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 and g_2 where $p \geq 0$ and $q \geq 0$*

(*iii*) *Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_1}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2)$;*

(*iv*) *Either $\tau_{g_2}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$; then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2)\ .$$

(B) *The following conditions are assumed to be satisfied:*

(*i*) *At least any one of f_1 or f_2 are of regular relative (p,q) Ritt growth with respect to $g_1 \pm g_2$ where $p \geq 0$ and $q \geq 0$;*

(*ii*) *Either $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \tau_{g_1 \pm g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_2)$ holds;*

(*iii*) *Either $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ holds;*

(*iv*) *Either $\tau_{g_1}^{(p,q)}(f_2) \neq \tau_{g_2}^{(p,q)}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$ holds, then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2)\ .$$

We omit the proof of Theorem 26 as it is a natural consequence of Theorem 25.

Theorem 27. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1). *Also let $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ are all non zero where $p \geq 0$ and $q \geq 0$.*

(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*

(*i*) *g_1 satisfies the Property (A) and*

(*ii*) *f_1 and f_2 satisfy Property (X); then*

$$\Delta_{g_1}^{(p,q)}(f_1 \cdot f_2) = \Delta_{g_1}^{(p,q)}(f_i) \ \ \textit{and} \ \ \overline{\Delta}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_i)\,.$$

(B) *Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*
(i) f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2 and f_1 satisfy the Property (A) and
(ii) g_1 and g_2 satisfy Property (X); then

$$\Delta_{g_1\cdot g_2}^{(p,q)}(f_1) = \Delta_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\Delta}_{g_1\cdot g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1 \cdot g_2$, f_1 and f_2 satisfy the Property (A);
(ii) f_1 and f_2 satisfy Property (X);
(iii) g_1 and g_2 satisfy Property (X);
(iv) f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2;
(v) f_2 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2;
(vii) $\rho_{g_m}^{(p,q)}(f_l) = \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l, m = 1, 2$; then

$$\Delta_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2) = \Delta_{g_m}^{(p,q)}(f_l) \text{ and } \overline{\Delta}_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2) = \overline{\Delta}_{g_m}^{(p,q)}(f_l).$$

Proof. Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$. Also let g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we have from (35) for all sufficiently large values of r that

$$(63) \quad M_{f_1\cdot f_2}(\sigma) \le M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)$$
$$\times M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right)\right).$$

Since $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we get that

$$\lim_{r\to+\infty}\frac{\left(\Delta_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}}{\left(\Delta_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_2)}} = \infty .$$

As $M_{g_1}(\sigma)$ is an increasing function of σ, therefore we get from (63) for all sufficiently large values of σ that

$$(64) \quad M_{f_1\cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)\right)^2 .$$

Let us observe that

$$\delta_1 := \frac{\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon}{\Delta_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}} > 1$$

$$\Rightarrow \frac{\exp^{[p-2]}\left(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}}{\exp^{[p-2]}\left(\Delta_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}} = \delta\,(\text{say}) > 1\ . \tag{65}$$

Since g_1 satisfy the Property (A), so we obtain from (64) and (65) for all sufficiently large values of σ that

$$M_{f_1\cdot f_2}(\sigma) < M_{g_1}\left(\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)^{\delta}\right)$$

$$i.e.,\ M_{f_1\cdot f_2}(\sigma) < M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)\ .$$

Now in view of Theorem 16, we get from above for all sufficiently large values of σ that

$$M_{f_1\cdot f_2}(\sigma) < M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1\cdot f_2)}\right)\right)\ .$$

$$i.e.,\ \frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1\cdot f_2}(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1\cdot f_2)}} < \left(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon\right)$$

$$i.e.,\ \Delta_{g_1}^{(p,q)}(f_1\cdot f_2) \le \Delta_{g_1}^{(p,q)}(f_1)\ . \tag{66}$$

Now we establish the equality of (66). Since f_1 and f_2 satisfy Property (X), then of course we have $(M_{f_1\cdot f_2}(\sigma)) > M_{f_1}(\sigma)$ for all sufficiently large values of σ and therefore

$$\frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1\cdot f_2}(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1\cdot f_2)}} \ge \frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1}(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}}$$

as $|M_{g_1}^{-1}(\sigma)$ is an increasing function of r. So $\Delta_{g_1}^{(p,q)}(f_1\cdot f_2) \ge \Delta_{g_1}^{(p,q)}(f_1)$. Hence $\Delta_{g_1}^{(p,q)}(f_1\cdot f_2) \le \Delta_{g_1}^{(p,q)}(f_1)$. Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can verify that $\Delta_{g_1}^{(p,q)}(f_1\cdot f_2) = \Delta_{g_1}^{(p,q)}(f_2)$.

CASE II. Let $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we have from (35) and (40) for a sequence of values of σ tending to infinity that

$$M_{f_1 \cdot f_2}(\sigma) < M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right) \tag{67}$$

$$\times M_{g_1}\left(\exp^{[p-1]}\left(\left(\Delta_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_2)}\right)\right).$$

Now in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we get that

$$\lim_{r\to+\infty} \frac{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}}{\left(\Delta_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_2)}} = \infty .$$

As $M_{g_1}(\sigma)$ is an increasing function of σ, therefore it follows from (67) for a sequence of values of σ tending to infinity that

$$M_{f_1 \cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)\right)^2 .$$

Now using the similar technique for a sequence of values of σ tending to infinity as explored in the proof of Case I, one can easily verify that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ under the conditions specified in the theorem.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can verify that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_2)$.

Therefore the first part of theorem follows from Case I and Case II.

Case III. Let f_1 satisfy the Property (A) and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2. Therefore in view of (37) and (39), we obtain for a sequence of values of σ tending to infinity that

$$M_{g_1 \cdot g_2}(\sigma) \le M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right) \tag{68}$$

$$\times M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\Delta}_{g_2}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right).$$

Now in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we obtain that

$$\lim_{r\to+\infty} \frac{\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}}{\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_2}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}} = \infty\ .$$

As $M_{f_1}(\sigma)$ is an increasing function of σ, therefore it follows from (68) for a sequence of values of σ tending to infinity that

$$M_{g_1\cdot g_2}(\sigma) \le \left(M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)^2 . \tag{69}$$

Now we observe that

$$\delta_1 := \frac{\Delta_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}}{\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon} > 1$$

$$\Rightarrow \frac{\exp^{[q-2]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)}{\exp^{[q-2]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)} = \delta\,(\text{say}) > 1\ . \tag{70}$$

Since f_1 satisfy the Property (A), therefore we obtain from (69) and (70) for a sequence of values of σ tending to infinity that

$$M_{g_1\cdot g_2}(\sigma) < M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)^{\delta}$$

$$i.e.,\ M_{g_1\cdot g_2}(\sigma) < M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right) .$$

Now we get in view of Theorem 18 and from above for a sequence of values of σ tending to infinity that

$$M_{g_1\cdot g_2}(\sigma) < M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{g_1\cdot g_2}^{(p,q)}(f_1)}}\right)\right)$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\Delta_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \Delta_{g_1}^{(p,q)}(f_1) \ . \tag{71}$$

Now we establish the equality of (71). Since g_1 and g_2 satisfy Property (X), then of course we have $M_{g_1 \cdot g_2}(\sigma) > M_{g_1}(\sigma)$ for all sufficiently large values of σ and therefore $M_{g_1 \cdot g_2}^{-1}(\sigma) < M_{g_1}^{-1}(\sigma)$. Hence

$$\frac{\log^{[p-1]} M_{g_1 \cdot g_2}^{-1}(M_{f_1}(\sigma))}{\left[\log^{[q-1]} r\right]^{\rho_{g_1 \cdot g_2}^{(p,q)}(f_1)}} \leq \frac{\log^{[p-1]} M_{g_1}^{-1}(M_{f_1}(\sigma))}{\left[\log^{[q-1]} r\right]^{\rho_{g_1}^{(p,q)}(f_1)}}$$

as $M_{f_1}(\sigma)$ is an increasing function of σ. So $\Delta_{g_1 \cdot g_2}^{(p,q)}(f_1) \leq \Delta_{g_1}^{(p,q)}(f_1)$. So $\Delta_{g_1 \cdot g_2}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1)$.

Case IV. Suppose f_1 satisfy the Property (A). Also let $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with f_1 is of regular relative (p,q) Ritt growth with respect to at least any one of g_1 or g_2. Therefore in view of (37), we obtain for all sufficiently large values of σ that

$$\begin{aligned} M_{g_1 \cdot g_2}(\sigma) < M_{f_1} & \left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} \sigma}{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right) \\ & \times M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} \sigma}{\left(\overline{\Delta}_{g_2}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)\right) . \end{aligned} \tag{72}$$

Now in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we get that

$$\lim_{r \to +\infty} \frac{\left(\frac{\log^{[p-1]} \sigma}{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}}{\left(\frac{\log^{[p-1]} \sigma}{\left(\overline{\Delta}_{g_2}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}} = \infty \ .$$

As $M_{f_1}(\sigma)$ is an increasing function of σ, therefore it follows from (72) for all sufficiently large values of σ that

$$M_{g_1 \cdot g_2}(\sigma) < \left(M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]} \sigma}{\left(\overline{\Delta}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)^2 .$$

Now using the similar technique for all sufficiently large values of σ as explored in the proof of Case III, one can easily verify that $\overline{\Delta}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least f_1 is of regular relative (p,q) Ritt growth with respect to g_1, then one can verify that $\overline{\Delta}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_2}^{(p,q)}(f_1)$.

Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 19 and the above cases. □

Theorem 28. *Let f_1, f_2, g_1 and g_2 be any four entire functions VVDS defined by* (1). *Also let $\lambda_{g_1}^{(p,q)}(f_{1i})$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ are all non zero and finite where $p \geq 0$ and $q \geq 0$.*

(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*

(*i*) *At least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 and g_1 satisfy the Property (A) and*

(*ii*) *f_1 and f_2 satisfy Property (X); then*

$$\overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \text{ and } \tau_{g_1}^{(p,q)}(f_1 \cdot f_2) = \tau_{g_1}^{(p,q)}(f_i).$$

(B) *Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*

(*i*) *f_1 satisfy the Property (A) and*

(*ii*) *g_1 and g_2 satisfy Property (X); then*

$$\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1) \text{ and } \tau_{g_1 \cdot g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1).$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*

(*i*) *$g_1 \cdot g_2$, f_1 and f_2 are satisfy the Property (A);*

(*ii*) *f_1 and f_2 satisfy Property (X);*

(*iii*) *g_1 and g_2 satisfy Property (X);*

(*iv*) *At least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;*

(*v*) *At least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;*

(*vi*) $\lambda_{g_m}^{(p,q)}(f_l) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\right\}\right] \mid l, m = 1, 2$; *then*

$$\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \text{ and } \tau_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2) = \tau_{g_m}^{(p,q)}(f_l).$$

Proof. CASE I. Suppose $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we obtain from (47) and (52) for a sequence values of σ

tending to infinity that

$$M_{f_1\cdot f_2}(\sigma) \le M_{g_1}\left(\exp^{[p-1]}\left(\left(\tau_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right) \tag{73}$$
$$\times M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_2)}\right)\right) .$$

Now in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we get that

$$\lim_{r\to+\infty}\frac{\left(\tau_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_2)}} = \infty .$$

As$M_{g_1}(\sigma)$ is an increasing function of σ, therefore we get from (73) for a sequence of values of σ tending to infinity that

$$M_{f_1\cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p-1]}\left(\left(\tau_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)\right)^2 . \tag{74}$$

Now using the similar technique as explored in the proof of Case I of Theorem 27 we obtain from (74) that

$$\tau_{g_1}^{(p,q)}(f_1\cdot f_2) = \tau_{g_1}^{(p,q)}(f_1) .$$

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1, then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1\cdot f_2) = \tau_{g_1}^{(p,q)}(f_2)$.

Case II. Let $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we get from (47) for all sufficiently large values of σ that

$$M_{f_1\cdot f_2}(\sigma) \le M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right) \tag{75}$$
$$\times M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_2)}\right)\right) .$$

Now in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we get that

$$\lim_{r\to+\infty}\frac{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_2)}} = \infty .$$

As $M_{g_1}(\sigma)$ is an increasing function of σ, therefore we get from (75) for all sufficiently large values of σ that

$$M_{f_1\cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p-1]}\left(\left(\overline{\tau}_{g_1}^{(p,q)}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_{g_1}^{(p,q)}(f_1)}\right)\right)\right)^2 . \tag{76}$$

Now using the similar technique as explored in the proof of Case I of Theorem 28 we obtain from (76) that $\overline{\tau}_{g_1}^{(p,q)}(f_1\cdot f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ under the conditions specified in the theorem.

Likewise, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least f_1 or f_2 is of regular relative (p,q) Ritt growth with respect to g_1, then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1\cdot f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_2)$.

Therefore the first part of theorem follows Case I and Case II.

CASE III. Let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and f_1 satisfy the Property (A). Therefore in view of (49) we obtain for all sufficiently large values of σ that

$$\begin{aligned} M_{g_1\cdot g_2}(\sigma) &< M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right) \\ &\times M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_2}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}\right)\right) . \end{aligned} \tag{77}$$

Now in view of $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we get that

$$\lim_{r\to+\infty} \frac{\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}}{\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_2}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}} = \infty .$$

As $M_{f_1}(\sigma)$ is an increasing function of σ, therefore it follows from (77) for all sufficiently large values of σ that

$$M_{g_1\cdot g_2}(\sigma) < \left(M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_1}^{(p,q)}(f_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)^2 . \tag{78}$$

Now using the similar technique as explored in the proof of Case III of Theorem 27 we obtain from (78) that $\tau_{g_1\cdot g_2}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1)$. If $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\tau_{g_1\cdot g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1)$.

Case IV. Suppose $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ and f_1 satisfy the Property (A). Therefore in view of (49) and (51) we obtain for a sequence of values of σ tending to infinity that

$$M_{g_1 \cdot g_2}(\sigma) \leq M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right) \times M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_2}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}\right)\right). \tag{79}$$

Now in view of $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, we get that

$$\lim_{r \to +\infty} \frac{\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}}{\left(\frac{\log^{[p-1]}\sigma}{\left(\tau_{g_2}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}} = \infty .$$

As $M_{f_1}(\sigma)$ is an increasing function of σ, therefore it follows from (79) for a sequence of values of σ tending to infinity that

$$M_{g_1 \cdot g_2}(\sigma) < \left(M_{f_1}\left(\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma}{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\right)\right)\right)^2 . \tag{80}$$

Now using the similar technique as explored in the proof of Case III of Theorem 28, we obtain from (80) that $\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$. Similarly if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_2}^{(p,q)}(f_1)$. Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 20 and the above cases. □

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