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THE HILBERT FUNCTION OF THE ARTINIAN QUOTIENT OF CODIMENSION 3

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Abstract. We investigate all kinds of the Hilbert function of the Artinian quotient of the coordinate ring of a linear star configuration in $\mathbb{P}^2$ of type 3 (or 3-general points in $\mathbb{P}^2$). As an application, we prove that such an Artinian quotient has the SLP.

1. Introduction

Let $R = \Bbbk[x_0, x_1, \dots, x_n]$ be an $(n+1)$-variable polynomial ring over a field $\Bbbk$ of characteristic 0 and I be a homogeneous ideal of R. A standard graded $\Bbbk$-algebra $A = R/I = \oplus_{i\geq 0} A_i$ has the *weak Lefschetz property* (WLP) if there is a linear form ℓ such that the multiplication by $\times\ell : A_i \to A_{i+1}$ has maximal rank for every $i \geq 0$, and A has the *strong Lefschetz property* (SLP) if $\times\ell^d : A_i \to A_{i+d}$ has maximal rank for every $i \geq 0$ and $d \geq 1$. In this case, ℓ is called a *strong Lefschetz element* of A. If $d = 1$, then ℓ is a *weak Lefschetz element* of A.

The Hilbert function of $A = R/I$, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, is defined by

$$\mathbf{H}_A(t) = \dim_{\Bbbk} R_t - \dim_{\Bbbk} I_t.$$

If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by $\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}(R/I_{\mathbb{X}}, t)$. This function contains a great deal of information about the geometry of this subscheme.

In 2006, Geramita, Migliore, and Sabourin [3] introduced the notion of a star configuration set of points in $\mathbb{P}^2$, called a linear star configuration in $\mathbb{P}^2$ in this article (see Definition 2.1 in Section 2). In [1], the authors found the graded minimal free resolution of a star configuration in $\mathbb{P}^n$ of codimention 2 before the general case.

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In 2014 [7], Park and Shin gave a general definition of a star configuration in $\mathbb{P}^n$ of codimension r, and found the minimal graded free resolution of a general star configuration in $\mathbb{P}^n$. Recently, in the series of papers [4, 5, 6], the authors worked on the WLP and the SLP for the quotient of the ideals of two star configurations in $\mathbb{P}^n$. In particular, in [5] the authors found an Artinian quotient having the SLP based on the union of two $\Bbbk$-configurations in $\mathbb{P}^2$, which are contained in a basic configuration (see [2] for the definition of a basic configuration). In this case, the Artinian ring is Gorenstein, and thus the Hilbert function of this Artinian ring is symmetric. It is also known [6] that if $\mathbb{X}_1, \ldots, \mathbb{X}_r$ are linear star configurations in $\mathbb{P}^n$ of any type (codimension n), then an Artinian quotient $R/(I_{\mathbb{X}_1} + \cdots + I_{\mathbb{X}_r})$ has the WLP. In [6], the authors found the Artinian star configuration quotient having the SLP and proved that the Artinian quotient of a coordinate ring of $(n+1)$-general points in $\mathbb{P}^n$ has the SLP.

In this paper, we focus on the following question.

Question 1.1. Let $\mathbb{X}$ and $\mathbb{Y}$ be star configurations in $\mathbb{P}^2$.

What is the Hilbert function of the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ of a coordinate ring of a linear star configuration in $\mathbb{P}^2$ of type 3 and a star configuration in $\mathbb{P}^2$ of type t with $t \geq 3$?

In Theorem 3.1, we have a complete answer to Question 1.1. Moreover, we also show that such an Artinian quotient has the SLP. Surprisingly, there are only three kinds of the Hilbert function of the Artinian quotient of a coordinate ring of 3-general points and a star configuration in $\mathbb{P}^2$ (see Theorem 3.1 and Remark 3.2). As an application, we prove that such an Artinian quotient has the SLP. But it is unknown how the Hilbert function of the Artinian quotient of the coordinate ring changes if we replace a star configuration by a finite set of points in $\mathbb{P}^2$.

2. Special Configurations

We find some Artinian star configuration quotients having the SLP. We first recall the definitions of a star configuration in $\mathbb{P}^n$ and a $\Bbbk$-configuration in $\mathbb{P}^2$, and then introduce some related results.

Definition 2.1. Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \dots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \dots, F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type (r, s). In particular, if $F_1, \dots, F_s$ are general linear forms in R, then we call $\mathbb{X}$ *a linear star-configuration in* $\mathbb{P}^n$ *of type* (r, s).

Notice that any n-forms $F_{i_1}, \dots, F_{i_n}$ among s-general forms $F_1, \dots, F_s$ in R define $d_{i_1} \cdots d_{i_n}$ points in $\mathbb{P}^n$ for each $1 \le i_1 < \cdots < i_n \le s$. Thus the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_n \le s} (F_{i_1}, \dots, F_{i_n})$$

defines a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ with

$$\deg(\mathbb{X}) = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

In this case, we call $\mathbb{X}$ *a star configuration in* $\mathbb{P}^n$ *of type* s instead of *type* (n, s).

Theorem 2.2 ([7, Theorem 2.3])**.** *Let* $\mathbb{X}$ *be a star-configuration in* $\mathbb{P}^n$ *of type* (r, s) *defined by general forms* $F_1, \dots, F_s$ *in* $R = \Bbbk[x_0, x_1, \dots, x_n]$ *with* $1 \le r \le \min\{s, n\}$. *Then*

$$I_{\mathbb{X}} = \bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \dots, F_{i_r}) = \sum_{1 \le j_1 < \cdots < j_{r-1} \le s} \left(\frac{\prod_{\ell=1}^{s} F_\ell}{F_{j_1} \cdots F_{j_{r-1}}} \right).$$

Definition 2.3. A $\Bbbk$-configuration in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfies the following conditions: there exist integers $1 \le d_1 < \cdots < d_m$, subsets $\mathbb{X}_1, \dots, \mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1, \dots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that

(a) $\mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i$,
(b) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \dots, m$, and
(c) $\mathbb{L}_i$ $(1 < i \le m)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In this case, the $\Bbbk$-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \dots, d_m)$.

We first recall the result in [6].

Proposition 2.4 ([6, Proposition 5.3])**.** *Let* $\mathbb{X}$ *be a set of* $(n+1)$*-general points in* $\mathbb{P}^n$, *and let* A *be the Artinian quotient of a coordinate ring of* $\mathbb{X}$ *having Hilbert function of the form*

$$\mathbf{H}_A \; : \; 1 \quad n+1 \quad \cdots \quad n+1 \quad h_s \quad \cdots \quad h_t,$$

where $2 \le s \le t$. *Then* A *has the SLP.*

Proposition 2.5 ([5, Proposition 2.5])**.** *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^2$ of type 3, and $\mathbb{Y}$ be a linear star configuration in $\mathbb{P}^2$ of type t with $t \geq 3$. Then the Artinian star configuration quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP having Hilbert function*

$$\mathbf{H}_A \ : \ 1 \quad 3 \quad \cdots \quad \overset{(t-2)\text{-}nd}{3} \quad 0.$$

Remark 2.6. In particular, if $\mathbb{X}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,2)$ (i.e., a star configuration in $\mathbb{P}^2$ of type 3), then any Artinian quotient of a coordinate ring of $\mathbb{X}$ has the SLP.

3. The Hilbert Function of the Artinian Quotient of 3-general Points in $\mathbb{P}^2$

In this section, we find the Hilbert function of the Artinian quotient of coordinate rings of a linear star configuration in $\mathbb{P}^2$ of type 3 and a general star configuration in $\mathbb{P}^2$ of type t with $t \geq 3$. Using [7, Theorem 3.4] or [1, Theorem 3.3], we can prove the following theorem, but we introduce an easier proof here without those two theorems.

Theorem 3.1. *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^2$ of type 3 and $\mathbb{Y}$ be a star configuration in $\mathbb{P}^2$ of type t with $t \geq 3$ defined by forms of degree $d_1 \geq d_2 \geq \cdots \geq d_t$ with $d_1 > 1$. Define $d = \sum_{i=2}^{t} d_i$. Then the Artinian star configuration quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP with Hilbert function*

$$\mathbf{H}_A \ : \ 1 \quad 3 \quad \cdots \quad 3 \quad \overset{d\text{-th}}{h_d} \quad 0,$$

where

$$h_d = \begin{cases} 0, & \text{for } d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t \text{ with } s \geq 3, \\ 1, & \text{for } d_1 = d_2 > d_3 \geq \cdots \geq d_t, \quad \text{and} \\ 2, & \text{for } d_1 > d_2 \geq \cdots \geq d_t. \end{cases}$$

Proof. We first find the Hilbert function of A. Note that $I_{\mathbb{Y}}$ has no minimal generators in degree $d-1$, and thus, $I_{\mathbb{X}\cup\mathbb{Y}}$ has no minimal generators in degree $d-1$, as well. Hence

$$\mathbf{H}_{\mathbb{Y}}(d-1) = \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d-1) = \binom{2+(d-1)}{2}.$$

Using the exact sequence

$$0 \to R/I_{\mathbb{X}\cup\mathbb{Y}} \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0, \tag{3.1}$$

we have that $\mathbf{H}_A(d-1) = 3$. We now find $\mathbf{H}_A(d)$ and $\mathbf{H}_A(d+1)$.

(a) Let $d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t$ with $s \geq 3$. First, since $d_1 \geq \cdots \geq d_t$, we see that, by Theorem 2.2, the initial degree of $I_{\mathbb{Y}}$ is d. Hence

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{2+d}{2} - s, \quad \text{and so,} \quad \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \binom{2+d}{2} - s + 3.$$

By equation (3.1), $\mathbf{H}_A(d) = 0$. Hence the Hilbert function of A is

$$\mathbf{H}_A \ : \ 1 \ \ 3 \ \ \cdots \ \ 3 \ \ \overset{d\text{-th}}{0}\ .$$

(b) Let $d_1 = d_2 > d_3 \geq \cdots \geq d_t$. Recall that $I_{\mathbb{Y}}$ has two minimal generators in degree d, so

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{2+d}{2} - 2.$$

Since $\mathbb{X}$ is a set of 3-general points in $\mathbb{P}^2$, we get that

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \binom{2+d}{2}.$$

By equation (3.1), $\mathbf{H}_A(d) = 1$. Suppose $d_1 = d_2 = d_3 + 1 = \cdots = d_s + 1$ and $d_s > d_{s+1} \geq \cdots \geq d_t$ with $s \geq 3$. We assume that $\mathbb{Y}$ is defined by t-forms $M_1, \ldots, M_t$ of degrees $d_1, \ldots, d_t$. Define

$$F = \frac{\prod_{i=1}^t M_i}{M_1}, \quad G = \frac{\prod_{i=1}^t M_i}{M_2}, \quad \text{and} \quad H_j = \frac{\prod_{i=1}^t M_i}{M_j} \quad \text{for} \quad 3 \leq j \leq s.$$

Notice that, by Theorem 2.2, $I_{\mathbb{Y}}$ has two minimal generators F, G in degree d and $(s-2)$-more minimal generators H_j in degree $d+1$ for such j, and that $M_1, \ldots, M_t$ are general forms, i.e., M_1 and M_i have no common component for such i. Now assume that for some linear form L_1, L_2 and $\alpha_j \in \mathbb{k}$,

$$L_1 F + L_2 G + \sum_{j=3}^{s} \alpha_j H_j = 0.$$

Then $M_1 \mid L_1 F = L_1 M_2 \cdots M_t$, which is impossible. In other words,

$$\mathbf{H}_{\mathbb{Y}}(d+1) = \binom{2+(d+1)}{2} - 6 - (s-2),$$

and thus,

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d+1) = \binom{2+(d+1)}{2} - 3 - (s-2).$$

By equation (3.1), $\mathbf{H}_A(d+1) = 0$, and thus, the Hilbert function of A is

$$\mathbf{H}_A \ : \ 1 \ \ 3 \ \ \cdots \ \ 3 \ \ \overset{d\text{-th}}{1} \ \ 0.$$

(c) Let $d_1 > d_2 \geq \cdots \geq d_t$. Recall that $I_{\mathbb{Y}}$ has one minimal generator in degree d, and thus

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{2+d}{2} - 1.$$

Since $\mathbb{X}$ is a set of 3-general points in $\mathbb{P}^2$,

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \binom{2+d}{2}.$$

By equation (3.1), $\mathbf{H}_A(d) = 2$. Assume $d_1 = d_2 + 1 = \cdots = d_s + 1$ and $d_s > d_{s+1} \geq \cdots \geq d_t$ with $s \geq 2$. With notation as above, we define

$$F = \frac{\prod_{i=1}^t M_i}{M_1} \quad \text{and} \quad G_j = \frac{\prod_{i=1}^t M_i}{M_j} \quad \text{for} \quad 2 \leq j \leq s.$$

Recall that, by Theorem 2.2, $I_{\mathbb{Y}}$ has one minimal generator F in degree d and $(s-1)$-more minimal generators G_j in degree $d+1$ for such j. Now assume that for some linear form L and $\alpha_j \in \Bbbk$,

$$LF + \sum_{j=2}^{s} \alpha_j G_j = 0.$$

Then $M_1 \mid LF = L_1 M_2 \cdots M_t$, which is impossible. In other words,

$$\mathbf{H}_{\mathbb{Y}}(d+1) = \binom{2+(d+1)}{2} - 3 - (s-1),$$

and thus,

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d+1) = \binom{2+(d+1)}{2} - (s-1).$$

By equation (3.1), $\mathbf{H}_A(d+1) = 0$, and hence, the Hilbert function of A is

$$\mathbf{H}_A \ : \ 1 \ \ 3 \ \ \cdots \ \ 3 \ \ \overset{d\text{-th}}{2} \ \ 0.$$

Therefore, by Proposition 2.4, A has the SLP. This completes the proof. □

Remark 3.2. (a) In Theorem 3.1, if $d_1 = \cdots = d_t = 1$, then by Proposition 2.5, the Hilbert function of A is

$$\mathbf{H}_A \ : \ 1 \ \ 3 \ \ \cdots \ \ 3 \ \ \overset{d\text{-th}}{0},$$

which is the case (a) in the proof of Theorem 3.1.

(b) As we have seen in Theorem 3.1, it is impossible to obtain the Artinian quotient of 3-general points $\mathbb{P}^2$ and a star configuration in $\mathbb{P}^2$ having Hilbert function one of the following.

$$\begin{array}{cccccccccc} 1 & 3 & \cdots & 3 & 2 & 2 & \cdots & 2 & \cdots & 2 & 0, \\ 1 & 3 & \cdots & 3 & 2 & 2 & \cdots & 1 & \cdots & 1 & 0, \\ 1 & 3 & \cdots & 3 & 1 & 1 & \cdots & 1 & \cdots & 1 & 0. \end{array}$$

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