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FIBONACCI SEQUENCES ON MV-ALGEBRAS

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Abstract. In this paper, we introduce the concept of Fibonacci sequences on MV-algebras and study them accurately. Also, by introducing the concepts of periodic sequences and power-associative MV-algebras, other properties are also obtained. The relation between MV-algebras and Fibonacci sequences is investigated.

1. Introduction

The Fibonacci sequence is a beautiful mathematical concept, making surprise appearances in everything from seashell patterns to the Parthenon. The Fibonacci sequence is an integer sequence defined by a simple linear recurrence relation. The sequence appears in many settings in mathematics and in other sciences. In particular, the shape of many naturally occurring biological organisms is governed by the Fibonacci sequence and its close relative, the golden ratio. Fibonacci -number has been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. Surveys and connections of the type just mentioned are provided for a very minimal set of examples of such texts in [1] and [5]. Given the usual Fibonacci-sequence in [1, 5] and other sequences of this type, one is naturally interested in considering what may happen in more general circumstances. Thus, one may consider what happens if one replaces (positive) integers by the modulo integer n or what happens in even more general circumstances. Han considered several properties of the Fibonacci sequence in arbitrary groupoids in [6]. Kim, Neggers and So in [8] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed it in particular for the case where the groupoid contained idempotents and pre-idempotents.

BCI/BCK-algebras were first introduced in mathematics in 1966 by Imai and Iseki,

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as a generalization of the concept of set-theoretic difference and propositional calculi [7]. The notion of the MV-algebra, originally introduced by Chang, is an attempt at developing a theory of algebraic systems that would correspond to the $\aleph_0$-valued propositional calculus. The axioms for this calculus are known as the Lukasiewicz axioms. In 1959, Chang proved the completeness theorem which stated the real unit interval [0,1] as a standard model of this logic and also constructed an MV-algebra from an arbitrary totally ordered abelian group. Moreover, he showed that every linearly ordered MV-algebra is isomorphic to an MV-algebra constructed from a group [3]. With the new definition we propose, we aim to examine the relationship between this sequence and algebraic structures such as MV-algebras and show the relations and concepts which exist in MV-algebras in Fibonacci sequences and vice versa. Also, the results can be useful based on the relationship between Fibonacci sequences and sequences such as Lucas series or applied algebras such as Clifford algebras which are used in many domains, including geometry, theoretical physics, and digital image processing.

In this paper, we introduce the notion of the Fibonacci sequences on MV-algebras and study it where the MV-algebras have idempotent, infinitesimal and archimedean elements. We make a new generalization of the Fibonacci sequences and derive various identities involving the Fibonacci sequences on MV-algebra. One direction is concerned with structures obtained by adding operations to the MV-algebra structure, or even combining the MV-algebras with other structures in order to obtain more expressive models and powerful logical systems. We obtain several relations on the MV-algebras which are derived from the generalized Fibonacci sequences and make some connections between the Fibonacci sequences and MV-algebras via bounded commutative BCK-algebras. We find some results regarding MV-algebras and the results is an elegant expression illustrating the connection between the Fibonacci sequences and Lukasiewicz many valued logic.

2. PRELIMINARIES

Definition 2.1 ([9])**.** An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if the following conditions are fulfilled for all $x, y, z \in X$:
BCI-1 $(((x * y) * (x * z)) * (z * y) = 0)$,
BCI-2 $((x * (x * y)) * y = 0)$,
BCI-3 $(x * x = 0)$,

BCI-4 $(x * y = 0$ and $y * x = 0$ imply $x = y)$.
If a BCI-algebra X satisfies the following identity:
BCK-5 $(\forall x \in X)$ $(0 * x = 0)$, then X is called a BCK-algebra.

The partial order on a BCI/BCK-algebra is defined such that $x \leq y$ if and only if $x * y = 0$.

A BCI/BCK-algebra X is said to be *commutative* if $x * (x * y) = y * (y * x)$, for all $x, y \in X$.
A bounded commutative BCK-algebra is an algebra $A = (A, *, 0, 1)$ of type $(2, 0, 0)$ satisfying the following identities:
(1) $(x * y) * z = (x * z) * y$,
(2) $x * (x * y) = y * (y * x)$,
(3) $x * x = 0$,
(4) $x * 0 = x$,
(5) $x * 1 = 0$.
Bounded commutative BCK-algebras were introduced in [12]. Mundici in [10] showed that MV-algebras and bounded commutative BCK-algebras are categorically equivalent.

Definition 2.2 ([3])**.** An MV-algebra A is an abelian monoid $(A, 0, \oplus)$ equipped with an operation $*$ such that $x^{**} = x$, $x \oplus 0^* = 0^*$ and, finally $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Definition 2.3 ([2, 11])**.** An ideal of an MV-algebra A is a non-empty subset I of A satisfying the following conditions:
(I_1) $If\ x \in I, y \in A\ \ and\ \ y \leq x, then\ \ y \in I,$
(I_2) $If\ x, y \in I, then\ \ x \oplus y \in I.$

Remark 2.4 ([2, 11])**.** If I is an ideal of A, then $0 \in I$, $x, y \in I \Rightarrow x \vee y \in I$, $x \oplus y \in I \Leftrightarrow x \vee y \in I$. If $M \subseteq A$ is a nonempty set, then $(M] = \{x \in A : x \leq x_1 \oplus ... \oplus x_n\ for\ some\ x_1, ..., x_n \in M\}$. We denote by Id(A) the set of ideals of an MV-algebra A. If I is an ideal of $A = (A, \oplus, *, 0)$ and $x \in A$, the congruence class of x with respect to $\sim_I$ will be denoted by x/I, i.e., $x/I = \{y \in A : x \sim_I y\}$, one can easy to see that $x \in I$ if and only if $x/I = 0/I$. We shall denote the quotient set $A/\sim_I$ by A/I. Since $\sim_I$ is a congruence on A, the MV-algebra operations on A/I given by

$$x/I \oplus y/I = (x \oplus y)/I \ \text{ and } \ (x/I)^* = x^*/I,$$

are well defined. Hence, the system $(A/I, \oplus, *, 0/I)$ becomes an MV-algebra, called the quotient algebra of A by ideal I.

Theorem 2.5 ([2]). *For any bounded commutative BCK-algebra $(A, *, 0, 1)$, upon defining $x^* =_{def} 1*x$ and $x \oplus y =_{def} 1*((1*x)*y)$, then $(A, \oplus, *, 0)$ is an MV-algebra, and $x \ominus y = x * y$.*

Remark 2.6 ([4]). For any infinitesimal element $a > 0$, the sequence $(0 \leq a \leq 2a \leq 3a \leq ... \leq na \leq ...)$ is strictly increasing.

Definition 2.7 ([3, Chang's MV-algebra C]). Let $\{c, 0, 1, +, -\}$ be a set of found symbols. For any $n \in N$ we define the following abbreviations:

$$nc := \begin{cases} 0 \text{ if } n = 0, \\ c \text{ if } n = 1, \\ c + (n-1)c \text{ if } n > 1, \end{cases} \qquad 1 - nc := \begin{cases} 0 \text{ if } n = 0, \\ 1 - c \text{ if } n = 1, \\ 1 - (n-1)c - c \text{ if } n > 1. \end{cases}$$

We consider $C = \{nc : n \in N\} \cup \{1 - nc : n \in N\}$ and define the MV-algebra operations as follows:

($\oplus$1) $if\ x = nc\ and\ y = mc,\ then\ x \oplus y := (m+n)c,$
($\oplus$2) $if\ x = 1 - nc\ and\ y = 1 - mc,\ then\ x \oplus y := 1,$
($\oplus$3) $if\ x = nc\ and\ y = 1 - mc\ and\ m \leq n,\ then\ x \oplus y := 1,$
($\oplus$4) $if\ x = nc\ and\ y = 1 - mc\ and\ n < m,\ then\ x \oplus y := 1 - (m-n)c,$
($\oplus$5) $if\ x = 1 - mc\ and\ y = nc\ and\ m \leq n,\ then\ x \oplus y := 1,$
($\oplus$6) $if\ x = 1 - mc\ and\ y = nc\ and\ n < m,\ then\ x \oplus y := 1 - (m-n)c,$
(*1) $if\ x = nc,\ then\ x^* := 1 - nc,$
(*2) $if\ x = 1 - nc,\ then\ x^* := nc.$

Then, the structure $(C, \oplus, *, 0)$ is an MV-algebra, which is called the Chang's MV-algebra.

3. Fibonacci Sequences Generated by MV-algebras

In this section, we introduce the notion of the Fibonacci sequences on MV-algebras and provide some examples in particular for different modes in the MV-algebras.

The Fibonacci sequence is a series of numbers where a number is found by adding up the two numbers before it. Starting with 0 and 1, the sequence goes 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, and so forth. Written as a rule, the expression is $X_n = X_{n-1} + X_{n-2}$. In what follows, let A denote an MV-algebra, unless otherwise specified.

Definition 3.1. If $a, b \in A$, we construct a sequence as follows:

$$[a, b] := \{a, b, u_0, u_1, u_2, ..., u_k, ...\},$$

where $u_0 := a \oplus b$, $u_1 = b \oplus u_0$, $u_2 = u_0 \oplus u_1$, and $u_{k+2} = u_k \oplus u_{k+1}$.
A sequence $[a, b]$ is called a *Fibonacci sequence* on MV-algebra.

Example 3.2. If we consider an MV-algebra $([0, 1], \oplus, *, 0)$ and for all $x, y \in [0, 1]$, we define $x \oplus y = \min\{1, x + y\}$ and $x^* = 1 - x$, then the Fibonacci sequence can be denoted as follows: $[0, 1] := \{0, 1, u_0, u_1, u_2, ...\}$, where $u_0 = 0 \oplus 1 = \min\{1, 0+1\} = 1$, $u_1 = 1 \oplus 1 = \min\{1, 1+1\} = \min\{1, 2\} = 1$, $u_2 = 1 \oplus 1 = \min\{1, 1+1\} = \min\{1, 2\} = 1$, ... Then $[0, 1] = \{0, 1, 1, 1, ...\}$ and $[0, \frac{1}{2}] = \{0, \frac{1}{2}, \frac{1}{2}, 1, 1, ...\}$.

Example 3.3. For each integer $n \geq 2$, the n-element set $L_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$, MV-sub algebras of [0,1] and its Fibonacci sequence $[\frac{1}{n-1}, \frac{n-2}{n-1}]$ can be denoted as follows:
$[\frac{1}{n-1}, \frac{n-2}{n-1}] := \{\frac{1}{n-1}, \frac{n-2}{n-1}, u_0, u_1, u_2, ...\}$,
where $u_0 = \frac{1}{n-1} \oplus \frac{n-2}{n-1} = \min\{1, \frac{1}{n-1} + \frac{n-2}{n-1}\} = \min\{1, 1\} = 1$,
$u_1 = \frac{n-2}{n-1} \oplus 1 = \min\{1, \frac{n-2}{n-1} + 1\} = \min\{1, \frac{2n-3}{n-1}\} = 1, ...$. Then
$[\frac{1}{n-1}, \frac{n-2}{n-1}] := \{\frac{1}{n-1}, \frac{n-2}{n-1}, 1, 1, 1, ...\}$, and so on

$$[\frac{1}{5}, \frac{1}{8}] = \{\frac{1}{5}, \frac{1}{8}, \frac{13}{40}, \frac{18}{40}, \frac{31}{40}, 1, 1, 1, ...\}.$$

Using Definition 2.1 and Theorem 2.5, we can provide examples of classes of bounded commutative BCK-algebras, which is an MV-algebra.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

Then X is an MV-algebra. It is easy to see that Fibonacci sequence $[1, 2]$ can be denoted as follows: $[1, 2] := \{1, 2, u_0, u_1, u_2, ...\}$, then by Theorem 2.5 we have: $u_0 = 1 \oplus 2 = 1 * ((1 * 1) * 2) = 1 * (0 * 2) = 1 * 0 = 1$, $u_1 = 2 \oplus 1 = 1 * ((1 * 2) * 1) = 1 * (0 * 1) = 1 * 0 = 1$, $u_3 = 1 \oplus 1 = 1 * ((1 * 1) * 1) = 1 * (0 * 1) = 1 * 0 = 1$, $u_4 = 1 \oplus 1 = 1$, ... Hence $[1, 2] := \{1, 2, 1, 1, 1, ...\}$, and $[2, 1] := \{2, 1, u_0, u_1, u_2, ...\}$.

By Theorem 2.5, we have: $[2,1] := \{2,1,1,1,...\}$, and $[0,3] := \{0,3,1,1,1,...\}$.

Example 3.5. Let $X = \{0, a, b, c, d, 1\}$ be a set with the following table:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	a	0	0
b	b	b	0	0	0	0
c	c	c	b	0	b	0
d	d	b	a	a	0	0
1	1	c	d	a	b	0

Then X is an MV-algebra. If $a, b \in X$, then $[a,b] := \{a, b, u_0, u_1, u_2, ...\}$. By Theorem 2.5 we have:
$u_0 = a \oplus b = 1*((1*a)*b) = 1*(c*b) = 1*b = d$, $u_1 = b \oplus d = 1*((1*b)*d) = 1*(d*d) = 1*0 = 1$, $u_3 = d \oplus 1 = 1*((1*d)*1) = 1*(b*1) = 1*0 = 1$, $u_4 = 1 \oplus 1 = 1, \ldots$. Hence $[a,b] := \{a,b,d,1,1,1,...\}$, and $[b,a] := \{b,a,d,d,1,1,1,...\}$, $[d,1] := \{d,1,1,1,...\}$, $[c,b] := \{c,b,c,c,c,...\}$, $[b,c] := \{b,c,c,c,...\}$.

Example 3.6. Let C be Chang's MV-algebra. Then the Fibonacci sequence $[nc, 1-mc]$ can be denoted as follows: $[nc, 1-mc] := \{nc, 1-mc, u_0, u_1, u_2, ...\}$, then we have:
(a) $[nc, 1-mc] := \{nc, 1-mc, 1, 1, 1, ...\}$,
(b) $[nc, 1-mc] := \{nc, 1-mc, 1-(m-n)c, (1-mc) \oplus (1-(m-n)c), 1-(m-n)c \oplus ((1-mc) \oplus (1-(m-n)c)), ...\}$.
If $n = 1$, then we have:
(a) $[c, 1-mc] := \{c, 1-mc, 1, 1, 1, ...\}$,
(b) $[c, 1-mc] := \{c, 1-mc, 1-(m-1)c, 1, 1, ...\}$,
and $[nc, mc] := \{nc, mc, u_0, u_1, u_2, ...\}$, so we have:
$[nc, mc] := \{nc, mc, (m+n)c, (2m+n)c, (3m+2n)c, (5m+3n)c, ..., (km+pn)c, ...\}$
and $[0, 1-mc] := \{0, 1-mc, u_0, u_1, u_2, ...\}$, thus we have:
$[0, 1-mc] := \{0, 1-mc, 1-mc, 2(1-mc), 3(1-mc), ..., k(1-mc), ...\}$,
and $[1-mc, nc] := \{1-mc, nc, u_0, u_1, u_2, ...\}$, hence we have:
$[1-mc, nc] := \{1-mc, nc, 1, 1, 1, ...\}$, and $[1-mc, nc] := \{1-mc, nc, 1-(m-n)c, 1-(m-2n)c, 1, 1, ...\}$.

For any MV-algebra A we shall denote by $B(A)$ the set of all complemented elements of $L(A)$, the elements of $B(A)$ are called the Boolean elements of A.
We can provide examples of MV-algebras with some properties, in our case, Boolean MV-algebras and ideal in MV-algebras.

Example 3.7. We give an example of a finite MV-algebra which is not a chain. The set $L_{3\times 2} = \{0, a, b, c, d, 1\} \approx L_3 \times L_2 = \{0, 1, 2\} \times \{0, 1\}$ with $0 < a, b < c < 1$, $0 < b < d < 1$. We have in $L_{3\times 2}$ the following table:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

It is easy to see that $B(A) = \{0, a, d, 1\}$. If $c \in A$, i.e., c is not Boolean and $d \in B(A)$, then the Fibonacci sequence $[c, d]$ can be denoted as follows:
$[c, d] := \{c, d, u_0, u_1, u_2, ...\} = \{c, d, 1, 1, 1, ...\}$ and $[d, c] := \{d, c, 1, 1, 1, ...\}$.
If $\{a, d, 0, 1\} \in B(A)$, then we have:
$[a, d] := \{a, d, 1, 1, 1, ...\}$ and $[0, d] := \{0, d, d, d, ...\}$, $[d, 0] := \{d, 0, d, d, ...\}$.

Example 3.8. Consider the MV-algebra C from Example 3.6, then
Id(A)=$\{I_1 = \{0\}, I_2 = \{0, c, 2c, 3c, ...\}, I_3 = C\}$. For $c \in I_2$, we have: $[c, 2c] := \{c, 2c, u_0, u_1, u_2, ...\}$, then we have:
$[c, 2c] := \{c, 2c, 3c, 5c, 8c, ...\}$ and $[2c, c] := \{2c, c, 3c, 4c, 7c, 11c, ...\}$.

Remark 3.9. In Example 3.8, consider $I = I_1$ and $a, b \in C$, then the Fibonacci sequence $[a/I, b/I]$ can denoted as follows:
$[a/I, b/I] := \{a/I, b/I, u_0, u_1, u_2, ...\}$. Thus $[a/\{0\}, b/\{0\}] :=\{a/\{0\}, b/\{0\}, (a \oplus b)/\{0\}, (b \oplus (a \oplus b))/\{0\}, ((a \oplus b) \oplus (b \oplus (a \oplus b)))/\{0\}, ...\} = \{a, b, (a \oplus b), (b \oplus (a \oplus b)), ((a \oplus b) \oplus (b \oplus (a \oplus b))), ...\} = [a, b]$ and $[a, b]/\{0\} := \{a, b, (a \oplus b), (b \oplus (a \oplus b)), ((a \oplus b) \oplus (b \oplus (a \oplus b))), ...\}/\{0\} = [a, b]$. Hence $[a, b]/I = [a/I, b/I]$.
If $I \in Id(A)$ and $I \neq I_1$, then the Fibonacci sequence $[a/I, b/I]$ can denoted as follows: $[a/I, b/I] := \{a/I, b/I, (a \oplus b)/I, (a \oplus 2b)/I, (2a \oplus 3b)/I, ...\}$ and $[a, b]/I := \{a, b, (a \oplus b), (a \oplus 2b), (2a \oplus 3b), ...\}/I$. Hence $[a, b]/I \neq [a/I, b/I]$.

4. On Derivation of Fibonacci Sequences

In this section, using the Boolean algebra, power-associative and periodic notions, we obtain several relations on MV-algebras, which are derived from the Fibonacci sequences.
An element a of an MV-algebra A is called an *idempotent* or *Boolean* if $a \oplus a = a$, if a and b are idempotents, then $a \oplus b$ and $a \odot b$ are also idempotents.

Remark 4.1. For $1 \neq a, 0 \in A$ we have:
$[a,0] := \{a, 0, u_0, u_1, u_2, ..., u_k, ...\}$, then $[a,0] := \{a, 0, a, a, 2a, 3a, ..., ka, ...\}$ and $[0,a] := \{0, a, a, 2a, 3a, 4a, ..., (k+1)a, ...\}$.

Recall that an element a in A is said to be *infinitesimal* if for each integer $n \geq 0$, $na \leq \neg a$, equivalently, if and only if $na \ominus \neg a = na \odot a = 0$. An element a in A is said to be *archimedean* if there is an integer $n \geq 0$, such that $(n+1)a \ominus na = 0$, equivalently, if and only if the sequence $(a \leq 2a \leq 3a \leq ... \leq na \leq ...)$ is stationary. Note that the only archimedean infinitesimal element is 0.
From Remark 4.1 and Remark 2.6 we deduce that:

Corollary 4.2. *For any element $1 \neq a$, the Fibonacci sequences $[a,0]$ and $[0,a]$ are strictly increasing.*

Proposition 4.3. *For $1 \neq a, 0 \in A$, the following are equivalent:*
(i) A is Boolean algebra,
(ii) Fibonacci sequences $[a,0]$ and $[0,a]$ are stationary,
(iii) a is archimedean.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (i)$. Since a is archimedean, there is an integer $n \geq 0$ such that $(n+1)a \ominus na = 0$, hence $na \in B(A)$. Thus A is Boolean algebra. □

Definition 4.4. A is said to be *power-associative* if, for any $a, b \in A$ there exists $k \in Z$ such that the Fibonacci sequence $[a,b]$ has $u_{k-2} = u_{k-1}$ for some $u \in A$.

Example 4.5. Example 3.2 and Example 3.3 are power-associative.

Remark 4.6. If A is a Boolean algebra, then for any $a, b \in A$, the MV-algebra A is power-associative. Since for any $a, b \in A$, $a \oplus a = a$. Hence we have:
$[a,b] := \{a, b, u_0, u_1, u_2, ..., u_k, ...\}$, so $u_0 = a \oplus b$, $u_1 = b \oplus (a \oplus b) = (b \oplus b) \oplus a = b \oplus a = a \oplus b$, $u_2 = (a \oplus b) \oplus (a \oplus b) = a \oplus b, ...$. Hence $[a,b] := \{a, b, a \oplus b, a \oplus b, a \oplus b, ...\}$.

Theorem 4.7. *Let A be power-associative and $a, b \in A$. Then $[a,b]$ contains a subsequence $\{u_k\}$ such that $u_{k+n} = (F_{n+3})u$ for some $u \in A$, where F_n is the usual Fibonacci number.*

Proof. Given $a, b \in A$, since A is power-associative, $[a,b]$ contains an element u such that $u_{k-2} = u_{k-1} = u$. It follows that $u_k = u_{k-1} \oplus u_{k-2} = u \oplus u = 2u$. This shows that $u_{k+1} = u_k \oplus u_{k-1} = 2u \oplus u = 3u$. In this fashion, we have $u_{k+2} = 5u$, $u_{k+3} = 8u = (F_6)u$, ..., $u_{k+n} = (F_{n+3})u$. $\square$

Corollary 4.8. *Every power-associative MV-algebra A is not a Boolean algebra.*

Proof. Since A is power-associative, [a,b] contains an element u such that $u_{k-2} = u_{k-1} = u$. It follows that $u_k = u_{k-1} \oplus u_{k-2} = u \oplus u = 2u$. This shows that $u_{k+1} = u_k \oplus u_{k-1} = 2u \oplus u = 3u$. In the sequel, we have $u_{k+2} = 5u$, $u_{k+3} = 8u, \ldots$. Hence $[a, b] := \{a, b, u_0, u_1, ..., u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, ...\} = \{a, b, a \oplus b, b \oplus (a \oplus b), ..., u, u, 2u, 3u, 5u, ...\}$, that is, a and b are not Boolean. Thus A is not Boolean algebra. $\square$

Remark 4.9. Let $a, b \in A$ be archimedean elements of A. Then the Fibonacci sequence $[a, b]$ is of the form $[a, b] := \{a, b, a \oplus b, a \oplus b, a \oplus b, ...\}$. Hence A is power-associative.

We recall that a lattice -ordered group (l-group) is a structure $(G, +, 0, \leq)$ such that $(G, +, 0)$ is a group, $(G, \leq)$ is a lattice and the following property is satisfied:

$$\text{for any x, y, a,b} \in G, x \leq y \Rightarrow a + x + b \leq a + y + b.$$

In the sequel, an lu-group will be a pair (G, u) where G is an l-group and u is a strong unit of G. $u > 0$ is a strong unit for G (that is, for all $x \in G$ there is some natural number $n \geq 1$ such that $-nu \leq x \leq nu$). A strong unit u of G is an archimedean element of G, i.e., an element $u \in G$ such that for each $x \in G$ there is an integer $n \geq 0$ with $-nu \leq x \leq nu$.

Let $G = (G, +, 0, -, \vee, \wedge)$ be an abelian l-group and $0 \in G$. For any $x, y \in [0, u] = \{x \in G; 0 \leq x \leq u\}$ set $x \oplus y = (x + y) \wedge u$ and $x^* = u - x$. Put $\Gamma(G, u) = ([0, u], \oplus, *, 0)$. Then $\Gamma(G, u)$ is an MV-algebra.

Proposition 4.10. *Let $\Gamma(G, u) = ([0, u], \oplus, *, 0)$ be an MV-algebra. Then for any $a, b \in \Gamma(G, u)$ the Fibonacci sequence $[a, b]$ is of the form $[a, b] := \{a, b, u, u, u, ...\}$.*

Proof. If (G, u) is an abelian lu-group, then for any $x \geq 0$ in G there are $x_1, x_2, ..., x_n \in [0, u]$ such that $x = x_1 + x_2 + ... + x_n$. Hence, any abelian lu-group is generated by its unit interval $[0, u]$. Then $[a, b] := \{a, b, u_0, u_1, u_2, ..., u_k, ...\}$, so using $x \oplus y = (x+y) \wedge u$ we have:

$u_0 = a \oplus b = (a + b) \wedge u = (x) \wedge u = u$, $u_1 = b \oplus u = (b + u) \wedge u = u$, $u_2 = u \oplus u = (u + u) \wedge u = u, \ldots$. Hence $[a, b] := \{a, b, u, u, u, ...\}$. $\square$

Remark 4.11. A sequence $a = (a_1, a_2, ...)$ of elements of A is said to be *good* if and only if for each i=$\{1, 2, ...\}$, $a_i \oplus a_{i+1} = a_i$ and there is an integer n such that $a_r = 0$ for all $r > n$. Then for $a_1, a_2 \in A$, we have: $[a_1, a_2] := \{a_1, a_2, a_1, a_2, ...\}$.

Proposition 4.12. *If I is an ideal of A and $a, b \in I$ such that $a \leq b$, then a Fibonacci sequence $[a, b]$ is of the form $\{a, b, b, b, ...\}$ and hence A is power-associative.*

Proof. By Remark 2.4 and Definition 4.4 we have: $a \oplus b = a \vee b$ and because $a \leq b$, $a \vee b = b$. Thus $u_0 = a \oplus b = b$ and $u_1 = b \oplus b = b \vee b = b, \ ...$. Hence $[a, b] := \{a, b, b, b, ...\}$. □

An MV-algebra A is locally finite if every non-zero element of A has finite order.

Proposition 4.13. *Let A be a locally finite MV-algebra and $a, b \in A$. Then a Fibonacci sequence $[a, b]$ is of the form $\{a, b, a \oplus b, 1, 1, 1, ...\}$ and the converse is true.*

Proof. Since the MV-algebra A is locally finite, there is a natural number m such that ma=1. It follows that $[a, b] = \{a, b, a \oplus b, b \oplus (a \oplus b) = 2b \oplus a = 1 \oplus a = 1, (a \oplus b) \oplus 1 = 1, 1 \oplus 1 = 1, ...\}$.
Conversely, suppose that $[a, b] = \{a, b, a \oplus b, 1, 1, 1, ...\}$, by Definition 3.1 we have: $[a, b] := \{a, b, u_0, u_1, u_2, ..., u_k, ...\} = \{a, b, a \oplus b, b \oplus (a \oplus b), (a \oplus b) \oplus (b \oplus (a \oplus b)), ...\}$, hence $b \oplus (a \oplus b) = 1, (a \oplus b) \oplus (b \oplus (a \oplus b)) = 1$, i.e., $2b \oplus a = 1$ and $2a \oplus 3b = 1, ...$ It follows that A is a locally finite MV-algebra. □

Definition 4.14. A Fibonacci sequence $[a, b]$ is said to be *periodic sequence* if, for any $a, b \in A$ we have $[a, b] := \{a, b, u_0, u_1, u_0, u_1, ...\}$.

Example 4.15. The Fibonacci sequence $[a_1, a_2]$ in Remark 4.11 is periodic sequence $[a_1, a_2] := \{a_1, a_2, a_1, a_2, ...\}$.

Theorem 4.16. *For $a, b \in A$, the followings are equivalent:*
(i) A is a Boolean algebra,
(ii) Fibonacci sequence $[a, b]$ is periodic.

Proof. $(i) \Rightarrow (ii)$. The proof is similar to Remark 4.6.
$(ii) \Rightarrow (i)$. For a Fibonacci sequence [a,b] we have:
$[a, b] = [a, b, a \oplus b, b \oplus (a \oplus b), (a \oplus b) \oplus (b \oplus (a \oplus b)), (b \oplus (a \oplus b)) \oplus ((a \oplus b) \oplus (b \oplus (a \oplus b))), ...]$.
Since the Fibonacci sequence $[a, b]$ is periodic, for $k \in \{0, 1, 2, ...\}$, we have:

$$\{u_0 = u_2, u_1 = u_3, u_2 = u_4, ..., u_k = u_{k+2}, ...\}.$$

Because $u_0 = u_2$ and $u_2 = u_4$, we have $a \oplus b = (a \oplus b) \oplus (b \oplus (a \oplus b))$ and $(a \oplus b) \oplus (b \oplus (a \oplus b))=((a \oplus b) \oplus (b \oplus (a \oplus b))) \oplus ((b \oplus (a \oplus b)) \oplus ((a \oplus b) \oplus (b \oplus (a \oplus b))))$. It follows that $a \oplus b = (a \oplus b) \oplus (a \oplus b)$. In particular by taking b=0 in $u_0 = u_2$ and $u_2 = u_4$ we obtain $a = a \oplus a$, that is, A is a Boolean algebra. $\square$

An MV-algebra A is said to be *pre-idempotent* if $x \oplus y$ is an idempotent in A for any $x, y \in A$. Note that if A is an idempotent MV-algebra, then it is a pre-idempotent MV-algebra as well.

Proposition 4.17. *Let A be an pre-idempotent MV-algebra. Then the Fibonacci sequence $[a, b]$ is of the form $[a, b] := \{a, b, a \oplus b, a \oplus b, a \oplus b, ...\}$ and the converse is true.*

Proof. If A is a pre-idempotent MV-algebra, then for a Fibonacci sequence [a,b] we have:
$[a, b] := \{a, b, u_0, u_1, u_2, u_3, ...\} = \{a, b, a \oplus b, b \oplus (a \oplus b), (a \oplus b) \oplus (b \oplus (a \oplus b)), ...\}$, hence $u_0 = a \oplus b$, $u_1 = b \oplus (a \oplus b) = a \oplus 2b = a \oplus b$, $u_2 = (a \oplus b) \oplus (b \oplus (a \oplus b)) = ((a \oplus b) \oplus (a \oplus b)) \oplus b = (a \oplus b) \oplus b = a \oplus 2b = a \oplus b$, $u_3 = (a \oplus b) \oplus (a \oplus b) = a \oplus b, ...$ So $[a, b] := \{a, b, a \oplus b, a \oplus b, a \oplus b, ...\}$.
Conversely, suppose that $[a, b] = \{a, b, a \oplus b, a \oplus b, a \oplus b, a \oplus b, ...\}$, by Definition 3.1 we have: $[a, b] := \{a, b, u_0, u_1, u_2, u_3...\} = \{a, b, a \oplus b, b \oplus (a \oplus b), (a \oplus b) \oplus (b \oplus (a \oplus b)), (b \oplus (a \oplus b)) \oplus ((a \oplus b) \oplus (b \oplus (a \oplus b))), ...\}$, hence $b \oplus (a \oplus b) = a \oplus b, (a \oplus b) \oplus (b \oplus (a \oplus b)) = a \oplus b, (b \oplus (a \oplus b)) \oplus ((a \oplus b) \oplus (b \oplus (a \oplus b))) = a \oplus b$, i.e., $2b \oplus a = a \oplus b$, $3b \oplus 2a = a \oplus b$, $5b \oplus 3a = a \oplus b$ .... Since $3b \oplus 2a = a \oplus b$ and $2b \oplus a = a \oplus b$ we deduce that $(2b \oplus a) \oplus (a \oplus b) = a \oplus b$, so $(a \oplus b) \oplus (a \oplus b) = (a \oplus b)$. Since $5b \oplus 3a = a \oplus b$ and $3b \oplus 2a = a \oplus b$ we obtain $(3b \oplus 2a) \oplus (2b \oplus a) = a \oplus b$, thus $(a \oplus b) \oplus (a \oplus b) = (a \oplus b)$. It follows that A is a pre-idempotent MV-algebra. $\square$

Theorem 4.18. *Let $u \in A$ be such that $[a, b] := \{a, b, u, u, u, ...\}$ for any $a, b \in A$. Then A is a Boolean algebra.*

Theorem 4.19. *Let $u, q \in A$ be such that $[a, b] := \{a, b, u, u, u, ...\}$ and $[b, a] := \{b, a, q, q, q, ...\}$ for any $a, b \in A$. Then*
(i) if $u = q = 1$, then A is a standard MV-algebra,
(ii) if $u = q = a \oplus b$, then A is a Boolean algebra,
(iii) if $u = q = a$ or b, then I is an ideal in MV-algebra $(A/I, \oplus, *, 0/I)$ for $a, b \in A/I$.

Proof. By Remark 4.6 and Proposition 4.12 it is trivial. □

By Theorem 4.19 and Proposition 4.10 we have the following corollary:

Corollary 4.20. (*i*) *Let* $A = (A, \oplus, *, 0)$ *be an linearly ordered* MV*-algebra and* $a, b \in A$*. Then the Fibonacci sequence* $[a, b]$ *is of the form* $[a, b] := \{a, b, u, u, u, ...\}$*,*
(*ii*) *Let* $A = ([0, 1], \oplus, *, 0)$ *be an* MV*-algebra and* $a, b \in A$*. Then the Fibonacci sequence* $[a, b]$ *is of the form* $[a, b] := \{a, b, u, u, u, ...\}$*.*

Conclusions

We first introduced the notion of the Fibonacci sequences on MV-algebras, investigated their properties and proved some relations between the Fibonacci sequences in the MV-algebra. We observed that the Fibonacci sequences can be used for the study of the MV-algebras, to provide orders which we can built algebras with some properties. Moreover, using the notion of ideal and Boolean algebra in the MV-algebras, we obtained several relations on the MV-algebras which were derived from the Fibonacci sequences. Furthermore, we obtained the interesting Fibonacci sequences for the MV-algebras and investigated the special cases of this sequence.

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