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GENERALIZED HYERS-ULAM STABILITY OF A QUADRATIC-CUBIC FUNCTIONAL EQUATION IN MODULAR SPACES

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Abstract. In this paper, I prove the stability problem for a quadratic-cubic functional equation

$$f(x+ky)-k^2f(x+y)-k^2f(x-y)+f(x-ky)$$
$$+f(kx)-\frac{k^3-3k^2+4}{2}f(x)+\frac{k^3-k^2}{2}f(-x)=0$$

in modular spaces by applying the direct method.

1. Introduction

In 1940, Ulam [19] first posed a stability problem in group homomorphisms. In the next year, Hyers [7] gave a clear answer to this problem for additive mappings between Banach spaces. Since then, many mathematicians came to deal with this problem (cf. [1, 6, 11, 15]).

The definitions and terminologies used in this paper were introduced by Nakano [14] and Musielak and Orlicz [13].

Definition 1.1 Let X be a real vector space.
(a) A functional $\rho : X \to [0,\infty]$ is called a *modular* if for arbitrary $x, y \in X$,

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta > 0$,

(b) We say that ρ is a *convex modular* if the last condition (iii) is replaced by

(iii′) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta > 0$.

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A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$.

Definition 1.2 Let $\{x_n\}$ and x be in X_ρ.

(i) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is *ρ-convergent* to x and write $x_n \to x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$, with $x_n \in X_\rho$, is called *ρ-Cauchy* if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(iii) A subset S of X_ρ is called *ρ-complete* if and only if every ρ-Cauchy sequence is ρ-convergent to an element of S.

Recently, Sadeghi [16] and K. Wongkum etc. [21] investigated the generalized Hyers-Ulam stability of a generalized Jensen functional equation and a quadratic functional equation for mappings from linear spaces into modular spaces, respectively.

A solution of the functional equation

$$f(x+y) - f(x-y) - 2f(x) - 2f(y) = 0$$

is called a *quadratic mapping* ([5, 17]) and a solution of the functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0.$$

is called a *cubic mapping.* A mapping f is called a *quadratic-cubic mapping* if f is represented by sum of a quadratic mapping and a cubic mapping. A functional equation is called a *quadratic-cubic functional equation* provided that each solution of that equation is a quadratic-cubic mapping and every quadratic-cubic mapping is a solution of that equation. Many mathematicians investigated the stability problem for several types of quadratic-cubic functional equations [3, 4, 9, 10, 12, 18, 20]. Now, consider the following functional equation

$$\begin{aligned} &f(x+ky) - k^2 f(x+y) - k^2 f(x-y) + f(x-ky) \\ &+ f(kx) - \frac{k^3 - 3k^2 + 4}{2} f(x) + \frac{k^3 - k^2}{2} f(-x) = 0, \end{aligned} \tag{1.1}$$

where f is a mapping from a real vector space to a ρ-complete modular space and k is a fixed real number such that $|k| > \sqrt{2}$. In this paper, we show that the functional equation (1.1) is a quadratic-cubic functional equation if k is a rational number and we prove the stability of that equation by applying the direct method in [7]. More precisely, starting from the given mapping f that approximately satisfies the functional equation (1.1), we explicitly construct an exact solution F of that equation, which approximates the mapping f, given by

$$F(x) = \lim_{n\to\infty} \frac{(k^n+1)f(k^n x) + (k^n-1)f(-k^n x)}{2k^{3n}}.$$

2. Main Results

Throughout this section, let V and W be real vector spaces and let ρ be a convex modular on a real vector space Y. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned}
f_o(x) :=& \frac{f(x) - f - x)}{2}, \quad f_e(x) := \frac{f(x) + f - x)}{2},\\
Qf(x,y) :=& f(x+y) + f(x-y) - 2f(x) - 2f(y),\\
Cf(x,y) :=& f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),\\
D_k f(x,y) :=& f(x+ky) - k^2 f(x+y) - k^2 f(x-y) + f(x-ky)\\
&+ f(kx) - \frac{k^3 - 3k^2 + 4}{2} f(x) + \frac{k^3 - k^2}{2} f(-x)
\end{aligned}$$

for all $x, y \in V$. Notice that the solutions of the functional equations $Qf \equiv 0$ and $Cf \equiv 0$ are called a quadratic mapping and a cubic mapping, respectively.

We need the following particular case of Baker's theorem [2] to prove Theorem 3.2.

Theorem 2.1 (Theorem 1 in [2])**.** *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \le j < l \le m$. If $f_l : V \to B$ for $0 \le l \le m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m - 1$.

We easily obtain following theorem from Baker's Theorem.

Theorem 2.2. *If a mapping $f : V \to W$ satisfies either the functional equation $D_k f(x,y) = 0$ for all $x, y \in V$, then f is a "generalized" polynomial mapping of "degree" at most 3.*

Suppose that $f, g : V \to W$ are generalized polynomial mapping of degree at most 3. It is well known that if the equalities $f(kx) = k^2 f(x)$ and $g(kx) = k^3 g(x)$ hold for all $x \in V$ and any nonzero fixed rational number k such that $|k| \neq 1$, then f and g are a quadratic mapping and a cubic mapping, respectively.

In the next theorem we will show that the functional equation $D_k f \equiv 0$ is a quadratic-cubic functional equation when k be a nonzero fixed rational number such that $|k| \neq 1$.

Theorem 2.3. *Let k be a nonzero fixed rational number such that $|k| \neq 1$. A mapping f satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$ if and only if f_e is quadratic and f_o is cubic.*

Proof. If a mapping f satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$, then the equalities $f_o(kx) = k^3 f_o(x)$ and $f_e(kx) = k^2 f_e(x)$ follow from the equalities

$$f_o(kx) - k^3 f_o(x) = \tfrac{D_k f(x,0) - D_k f(-x,0)}{2}, \quad f_e(kx) - k^2 f_e(x) = \tfrac{D_k f(x,0) + D_k f(-x,0)}{2}$$

for all $x \in V$. Since f_o and f_e are generalized polynomial mappings of degree at most 3, f_o is a cubic mapping and f_e is a quadratic mapping.

Conversely, assume that f_o is a cubic mapping and f_e is a quadratic mapping, i.e., f is a quadratic-cubic mapping. Notice that f_o satisfies the equality $f_o(kx) = k^3 f_o(x)$ and $f_o(x) = -f_o(-x)$, f_e satisfies $f_e(kx) - k^2 f_e(x)$ and $f_e(x) = f_e(-x)$ for all $x \in V$ and all $k \in \mathbb{Q}$, and $f(x) = f_o(x) + f_e(x)$.

The equalities $D_2 f_o(x, y) = 0$ and $D_3 f_o(x, y) = 0$ follow from the equalities

$$\begin{aligned} D_2 f_o(x, y) =& C f_o(x, y) - C f_o(x - y, y), \\ D_3 f_o(x, y) =& D_2 f_o(x + y, y) + D_2 f_o(x - y, y) + 4 D_2 f_o(x, y) \end{aligned}$$

for all $x, y \in V$. If the equality $D_j f_o(x, y) = 0$ holds for all $j \in \mathbb{N}$ when $2 \leq j \leq n-1$, then the equality $D_n f_o(x, y) = 0$ follows from the equality

$$D_n f_o(x, y) = D_{n-1} f_o(x + y, y) + D_{n-1} f_o(x - y, y) - D_{n-2} f_o(x, y) + (n-1)^2 D_2 f_o(x, y)$$

for all $x, y \in V$. Using mathematical induction, we obtain

$$D_n f_o(x, y) = 0$$

for all $x, y \in V$ and any $n \in \mathbb{N}$. Since the equality $D_n f_e(x, y) = 0$ follows from the equality

$$D_n f_e(x, y) = Q f_e(x, ny) - n^2 Q f_e(x, y)$$

for all $x, y \in V$, we have

$$D_n f(x,y) = 0$$

for all $x, y \in V$ and any $n \in \mathbb{N}$. Using the equalities

$$\begin{aligned} D_k f_o(x,y) =& f_o(x+ky) - k^2 f_o(x+y) - k^2 f_o(x-y) + f_o(x-ky) + 2(k^2-1) f_o(x), \\ D_k f_e(x,y) =& f_e(x+ky) - k^2 f_e(x+y) - k^2 f_e(x-y) + f_e(x-ky) + 2(k^2-1) f_e(x) \end{aligned}$$

for all $x, y \in X$ and any $k \in \mathbb{Q}$, we get

$$D_k f(x,y) = f(x+ky) - k^2 f(x+y) - k^2 f(x-y) + f(x-ky) + 2(k^2-1) f(x)$$

for all $x, y \in X$ and any $k \in \mathbb{Q}$. Therefore, if $k \in \mathbb{Q}$ is represented by either $k = \frac{n}{m}$ or $k = \frac{-n}{m}$ for some $n, m \in \mathbb{N}$, then the desired equalities $D_k f(x,y) = 0$ follows from the equalities

$$\begin{aligned} D_{\frac{n}{m}} f(x,y) =& D_n f\left(x, \frac{y}{m}\right) - \frac{n^2}{m^2} D_m f\left(x, \frac{y}{m}\right), \\ D_{\frac{-n}{m}} f(x,y) =& D_{\frac{n}{m}} f_e(x,y) \end{aligned}$$

for all $x, y \in X$ and $n, m \in \mathbb{N}$. □

The following properties given in the paper [8] are necessary to prove main theorem.

Remark. Let ρ be a convex modular on X. If $0 < \alpha < \beta$ and $\alpha_i > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then properties $\rho(\alpha x) \leq \rho(\beta x)$ and

$$\rho\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \rho(x_i)$$

hold for all $x, x_1, \ldots, x_n \in X$.

Now we will prove the generalized Hyers-Ulam stability of the functional equation $D_k f(x,y) = 0$.

Theorem 2.4. *Let V be a real vector space, Y_ρ be a ρ-complete modular space and k be a fixed real number such that $|k| > \sqrt{2}$. Suppose $f : V \to Y_\rho$ satisfies an inequality of the form*

$$\rho(D_k f(x,y)) \leq \varphi(x,y) \tag{2.1}$$

for all $x, y \in V$, where $\varphi : V^2 \to [0, \infty)$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\varphi(k^i x, k^i y)}{k^{2i}} < \infty \tag{2.2}$$

for all $x, y \in V$. Then there exists a unique solution $F : V \to Y_\rho$ of the functional equation (1.1) *such that*

$$\rho(f(x) - F(x)) \le \sum_{i=0}^{\infty} \left(\frac{|k^{i+1} + 1|}{2|k|^{3i+3}} \varphi(k^i x, 0) + \frac{|k^{i+1} - 1|}{2|k|^{3i+3}} \varphi(-k^i x, 0) \right) \tag{2.3}$$

for all $x \in V$.

Proof. Notice that the inequality $\sum_{i=0}^{\infty} \big(\frac{|k^{i+1}+1|}{|k|^{3i+3}} + \frac{|k^{i+1}-1|}{|k|^{3i+3}} \big) < 1$ holds for $|k| > \sqrt{2}$.

Let $J_n f : V \to Y_\rho$ be the mappings defined by

$$J_n f(x) := \frac{(k^n + 1) f(k^n x) + (k^n - 1) f(-k^n x)}{2k^{3n}}$$

for all $x \in V$ and any $n \in \mathbb{N}$. Then the equality

$$J_{m+n} f(x) = J_m J_n f(x)$$

follows from the equality

$$\begin{aligned}
&\frac{(k^{m+n} + 1) f(k^{n+m} x) + (k^{m+n} - 1) f(-k^{n+m} x)}{2k^{3n+3m}} \\
&\qquad = \frac{(k^m + 1)(k^n + 1) f(k^{n+m} x) + (k^m + 1)(k^n - 1) f(-k^{n+m} x)}{2k^{3n+3m}} \\
&\qquad\quad + \frac{(k^m - 1)(k^n + 1) f(-k^{n+m} x) + (k^m - 1)(k^n - 1) f(k^{n+m} x)}{2k^{3n+3m}}
\end{aligned}$$

for all $x \in V$ and any $n, m \in \mathbb{N} \cup \{0\}$. Since the inequality

$$\sum_{i=0}^{\infty} \Big(\frac{|k^{i+1} + 1|}{2|k|^{3i+3}} + \frac{|k^{i+1} - 1|}{2|k|^{3i+3}} \Big) < 1$$

and the equality

$$J_i f(x) - J_{i+1} f(x) = \frac{-(k^{i+1} + 1) D_k f(k^i x, 0)}{2k^{3i+3}} - \frac{(k^{i+1} - 1) D_k f(-k^i x, 0)}{2k^{3i+3}}$$

holds for all $x \in V$ and any $i \in \mathbb{N}$, we have

$$\begin{aligned}
&\rho\big(J_n f(x) - J_{n+m} f(x)\big) \\
&= \rho\Bigg(\sum_{i=n}^{n+m-1} \big(J_i f(x) - J_{i+1} f(x) \big) \Bigg) \\
&\le \rho\Bigg(\sum_{i=n}^{n+m-1} \Bigg(\frac{-(k^{i+1} + 1) D_k f(k^i x, 0)}{2k^{3i+3}} - \frac{(k^{i+1} - 1) D_k f(-k^i x, 0)}{2k^{3i+3}} \Bigg) \Bigg) \qquad (2.4)
\end{aligned}$$

$$\leq \sum_{i=n}^{n+m-1} \left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}} \rho\big(D_k f(k^i x, 0)\big) + \frac{|k^{i+1}-1|}{2|k|^{3i+3}} \rho\big(D_k f(-k^i x, 0)\big) \right)$$
$$\leq \sum_{i=n}^{n+m-1} \left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}} \varphi(k^i x, 0) + \frac{|k^{i+1}-1|}{2|k|^{3i+3}} \varphi(-k^i x, 0) \right)$$

for all $x \in V$ and any $n, m \in \mathbf{N} \cup \{0\}$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in V$. Since Y_ρ is complete, the sequence $\{J_n f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y_\rho$ by

$$F(x) := \lim_{n\to\infty} \frac{(k^n+1)f(k^n x) + (k^n-1)f(-k^n x)}{2k^{3n}}$$

for all $x \in V$. From the definition of F, we have the properties $\lim_{n\to\infty} \rho(J_n f(x) - F(x)) = 0$ and

$$F(x) = \lim_{n\to\infty} J_{m+n} f(x) = \lim_{n\to\infty} J_m J_n f(x) = J_m \lim_{n\to\infty} J_n f(x) = J_m F(x)$$

for all $x \in V$ and any $m \in \mathbb{N}$. If we choose $m \in \mathbb{N}$ such that

$$\sum_{i=0}^{\infty} \left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}} + \frac{|k^{i+1}-1|}{2|k|^{3i+3}} \right) + \frac{|k^m+1|}{2|k|^{3m}} + \frac{|k|^m-1}{2|k|^{3m}} < 1,$$

then we obtain

$$\begin{aligned}
&\rho(f(x) - F(x)) \\
&= \rho(f(x) - J_m F(x)) \\
&= \rho\Big(\sum_{i=0}^{m+n-1} (J_i f(x) - J_{i+1} f(x)) + J_m J_n f(x) - J_m F(x)\Big) \\
&\leq \rho\Bigg(\sum_{i=0}^{n+m-1} \left(\frac{-(k^{i+1}+1) D_k f(k^i x, 0)}{2k^{3i+3}} - \frac{(k^{i+1}-1) D_k f(-k^i x, 0)}{2k^{3i+3}} \right) \\
&\quad + \frac{(k^m+1)(J_n f - F)(k^m x) + (k^m-1)(J_n f - F)(-k^m x)}{2k^{3m}} \Bigg) \\
&\leq \sum_{i=0}^{n+m-1} \left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}} \varphi((k^i x, 0) + \frac{|k^{i+1}-1|}{2|k|^{3i+3}} \varphi((-k^i x, 0) \right. \\
&\quad \left. + \frac{|k^m+1| \rho(J_n f(k^m x) - F(k^m x)) + |k^m-1| \rho(J_n f(-k^m x) - F(-k^m x))}{2|k|^{3m}} \right) \\
&\to \sum_{i=0}^{\infty} \left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}} \varphi(k^i x, 0) + \frac{|k^{i+1}-1|}{2|k|^{3i+3}} \varphi(-k^i x, 0) \right), \text{ as } n \to \infty
\end{aligned}$$

for all $x \in V$, i.e., the inequality (2.3) holds for all $x \in V$. From the definition of F and the properties of F, we get

$$\begin{aligned}
&\rho\Big(\frac{D_kF(x,y)}{|k|^3+4|k|^2+6}\Big)\\
&=\rho\Big(\frac{1}{|k|^3+4|k|^2+6}\Big((F-J_nf)(x+ky)-k^2(F-J_nf)(x+y)\\
&\quad -k^2(F-J_nf)(x-y)+(F-J_nf)(x-ky)+(F-J_nf)(kx)\\
&\quad -\frac{k^3-3k^2+2}{2}(F-J_nf)(x)+\frac{k^3-k^2}{2}(F-J_nf)(-x)\\
&\quad +\frac{(k^n+1)D_kf(k^nx,k^ny)}{2k^{3n}}+\frac{(k^n-1)D_kf(-k^nx,-k^ny)}{2k^{3n}}\Big)\Big)\\
&\le\frac{1}{|k|^3+4|k|^2+6}\Big(\rho\big((F-J_nf)(x+ky)\big)+k^2\rho\big((F-J_nf)(x+y)\big)\\
&\quad +k^2\rho\big((F-J_nf)(x-y)\big)+\rho\big((F-J_nf)(x-ky)\big)+\rho\big((F-J_nf)(kx)\big)\\
&\quad +\frac{|k|^3+3k^2+2}{2}\rho\big((F-J_nf)(x)\big)+\frac{|k|^3+k^2}{2}\rho\big((F-J_nf)(-x)\big)\\
&\quad +\frac{|k^n+1|\rho(k^nx,k^ny)}{2|k|^{3n}}+\frac{|k^n-1|\rho(k^nx,k^ny)}{2|k|^{3n}}\Big)\Big)\\
&\to\ 0,\ \text{as } n\to\infty
\end{aligned}$$

for all $x, y \in V$. Hence we obtain the equality $\frac{D_kF(x,y)}{|k|^3+4|k|^2+6} = 0$ for all $x, y \in V$, i.e., F is a solution of the functional equation (1.1).

To prove the uniqueness of F, assume that $F' : V \to Y_\rho$ is another solution of the functional equation (1.1) which satisfies the inequality in (2.3). Notice that the property $F'(x) = J_nF'(x)$ is obtained from $J_0F'(x) - J_nF'(x) = \sum_{i=0}^{n-1} \frac{-(k^{i+1}+1)D_kF'(k^ix,0)}{2k^{3i+3}} - \frac{(k^{i+1}-1)D_kF'(-k^ix,0)}{2k^{3i+3}}$ for all $x \in V$ and any $n \in \mathbb{N}$. From the relation

$$\begin{aligned}
&\rho\Big(J_nf(x)-F'(x)\Big)\\
&=\rho\Big(J_nf(x)-J_nF'(x)\Big)\\
&\le\rho\big(\frac{(k^n+1)(f-F')(k^nx)+(k^n-1)(f-F')(-k^nx)}{2k^{3n}}\big)\\
&\le\sum_{i=0}^{\infty}\Big(\frac{|k^n+1||k^{i+1}+1|}{2|k|^{3i+3n+3}}\varphi(k^{i+n}x,0)+\frac{|k^n+1||k^{i+1}-1|}{2|k|^{3i+3n+3}}\varphi(-k^{i+n}x,0)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{\infty}\left(\frac{|k^n-1||k^{i+1}+1|}{2|k|^{3i+3n+3}}\varphi(-k^{i+n}x,0)+\frac{|k^n-1||k^{i+1}-1|}{2|k|^{3i+3n+3}}\varphi(k^{i+n}x,0)\right)\\
&\le\sum_{i=0}^{\infty}\left(\frac{|k^{n+i+1}+1|}{2|k|^{3i+3n+3}}\varphi(k^{i+n}x,0)+\frac{|k^{n+i+1}-1|}{2|k|^{3i+3n+3}}\varphi(-k^{i+n}x,0)\right)\\
&\le\sum_{i=n}^{\infty}\left(\frac{|k^{i+1}+1|}{2|k|^{3i+3}}\varphi((k^{i}x,0)+\frac{|k^{i+1}-1|}{2|k|^{3i+3}}\varphi((-k^{i}x,0)\right)\\
&\to\ 0,\ \text{as } n\to\infty
\end{aligned}
$$

for all $x \in V$, we get the equality $F'(x) = \lim_{n\to\infty} J_n f(x)$ for all $x \in V$, i.e., $F'(x) = F(x)$ for all $x \in V$. □

We can easily prove the following corollary by using Theorem 2.4.

Collorary 2.5. *Let X be a real normed space and let p, θ be nonnegative real constants such that $p < 2$. If a mapping $f : X \to Y_\rho$ satisfies the inequality*

$$\rho(D_k f(x, y)) \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique solution $F : X \to Y_\rho$ of the functional equation (1.1) *such that*

$$\rho(f(x) - F(x)) \le \frac{\theta}{k^2 - |k|^p}\|x\|^p \tag{2.5}$$

for all $x \in X$.

Proof. If we put $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, then φ satisfies the inequality (2.5). □

References

1. T. Aoki: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2** (1950), 64-66.
2. J. Baker: A general functional equation and its stability. Proc. Natl. Acad. Sci. **133** (2005), no. 6, 1657-1664.
3. I.-S. Chang & Y.-S. Jung: Stability of a functional equation deriving from cubic and quadratic functions. J. Math. Anal. Appl. **283** (2003), 491-500.
4. Y.-J. Cho, M. Eshaghi Gordji & S. Zolfaghari: Solutions and Stability of Generalized Mixed Type QC Functional Equations in Random Normed Spaces. J. Inequal. Appl. **2010** (2010), Art. ID 403101.

5. S. Czerwik: On the stability of the quadratic mapping in the normed space. Abh. Math. Sem. Hamburg **62** (1992), 59-64.
6. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. and Appl. **184** (1994), 431-436.
7. D.H. Hyers: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.
8. S.-S. Jin & Y.-H. Lee: Generalized Hyers-Ulam stability of a 3-dimensional quadratic functional equation in modular spaces. Int. J. Math. Anal. (Ruse) **10** (2016), 953-963.
9. C.-J. Lee & Y.-H. Lee: On the stability of a mixed type quadratic and cubic functional equation. J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. **19** (2012), 383-396.
10. Y.-H. Lee: On the Hyers-Ulam-Rassias stability of a quadratic and cubic functional equation. Int. J. Math. Anal. (Ruse) **12** (2018), 577-583.
11. Y.-H. Lee & K.-W. Jun: On the stability of approximately additive mappings. Proc. Amer. Math. Soc. **128** (2000), 1361-1369.
12. Y.-H. Lee & S.-M. Jung: Fuzzy stability of the cubic and quadratic functional equation. Appl. Math. Sci. (Ruse) **10** (2016), 2671-2686.
13. J. Musielak & W. Orlicz: On modular spaces. Studia Mathematica **18** (1959), 591-597.
14. H. Nakano: Modular Semi-Ordered Spaces. Tokyo, Japan, 1959.
15. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc., **72** (1978), 297-300.
16. G. Sadeghi: A fixed point approach to stability of functional equations in modular spaces. Bull. the Malays. Math. Sci. Soc. **37** (2014) 333-344.
17. F. Skof: Proprietà locali e approssimazione di operatori. Rend. Sem. Mat. Fis. Milano **53** (1983), 113-129.
18. W. Towanlong & P. Nakmahachalasint: A mixed-type quadratic and cubic functional equation and its stability. Thai J. Math. **8** (2012), no. 4, 61-71.
19. S.M. Ulam: Problems in Modern Mathematics. Wiley, New York, 1964.
20. Z. Wang & W.X. Zhang: Fuzzy stability of quadratic-cubic functional equations. Acta Math. Sin. (Engl. Ser.) **27** (2011), 2191-2204.
21. K. Wongkum, P. Chaipunya & Poom Kumam: On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without Δ_2-conditions. J. Funct. Spaces (2015), Article ID 461719, 6 pages.

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