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MEASURES OF COMPARATIVE GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS ON THE BASIS OF THEIR RELATIVE (p,q)-TH TYPE AND RELATIVE (p,q)-TH WEAK TYPE

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Abstract. The main aim of this paper is to establish some comparative growth properties of composite entire functions on the basis of their relative (p,q)-th order, relative (p,q)-th lower order, relative (p,q)-th type, relative (p,q)-th weak type of entire function with respect to another entire function where p and q are any two positive integers.

1. Introduction and Definitions

Let us consider that the reader is familiar with the fundamental results and the standard notations of the theory of entire functions which are available in [14].For $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. However for any entire function f defined in the open complex plane $\mathbb{C}$, the maximum modulus function $M_f(r)$ is defined as $M_f(r) = \max_{|z|=r} |f(z)|$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0,\infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. However, for another entire function g, $M_g(r)$ is defined and the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The maximum term $\mu_f(r)$ of f can be defined in the following way:

$$\mu_f(r) = \max_{n\geq 0}\left(|a_n| r^n\right).$$

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In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. So from another angle of view $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ is also called the growth of f with respect to g where $\mu_g(r)$ denotes the maximum term of entire g.

However, the order $\rho(f)$ and lower order $\lambda(f)$ of an entire function f which are generally used in computational purpose are defined in terms of the growth of f with respect to the $\exp z$ function as

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r}$$

and

$$\lambda(f) = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r}.$$

Extending this notion, Juneja et. al. [6] defined the (p,q)-th order (respectively (p,q)-th lower order) of an entire function f for any two positive integers p, q with $p \geq q$ which is as follows:

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \left(\text{respectively } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}\right).$$

These definitions extend the generalized order $\rho^{[l]}(f)$ and generalized lower order $\lambda^{[l]}(f)$ of an entire function f considered in [10] for each integer $l \geq 2$ since these correspond to the particular case $\rho^{[l]}(f) = \rho^{(l,1)}(f)$ and $\lambda^{[l]}(f) = \lambda^{(l,1)}(f)$. Clearly, $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$.

In this connection we just recall the following definition due to Juneja et. al. [6]:

Definition 1 ([6])**.** An entire function f is said to have index-pair (p,q) where p and q are any two positive integers with $p \geq q$, if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

An entire function f of index-pair (p,q) is said to be of regular (p,q) growth if its (p,q)-th order coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q) growth.

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \quad \{cf.\ [12]\ \}$$

it is easy to see that

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r}.$$

In order to compare the growth of entire functions having the same (p,q)-th order, Juneja, Kapoor and Bajpai [7] also introduced the concepts of (p,q)-th type and (p,q)-th lower type in the following manner :

Definition 2 ([7])**.** The (p,q)-th type and the (p,q)-th lower type of entire function f having finite positive (p,q)-th order $\rho^{(p,q)}(f)$ $(b < \rho^{(p,q)}(f) < \infty)$ (p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$) are defined as:

$$\sigma^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}}.$$

It is obvious that $0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty$.

Likewise, to compare the growth of entire functions having the same (p,q)-th lower order, one can also introduced the concepts of (p,q)-th weak type in the following manner :

Definition 3. The (p,q)-th weak type $\tau^{(p,q)}(f)$ and the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function f having finite positive (p,q)-th tower order $\lambda^{(p,q)}(f)$ $(b < \lambda^{(p,q)}(f) < \infty)$ are defined as:

$$\tau^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} \text{ and } \overline{\tau}^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}},$$

where p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$.

It is obvious that $0 \leq \tau^{(p,q)}(f) \leq \overline{\tau}^{(p,q)}(f) \leq \infty$.

L. Bernal [1, 2] introduced the concept of relative order between two entire functions to avoid comparing growth just with $\exp z$. In the case of relative order, it was then natural for Lahiri and Banerjee [8] to define the relative (p,q)-th order of entire functions as follows.

Definition 4 ([8])**.** Let p and q be any two positive integers with $p > q$. The relative (p,q)-th order of f with respect to g is defined by

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

Improveing the Definition 4 by ignoring the restriction $p \geq q$, Sánchez Ruiz et al. [9] gave a more natural definition of relative (p,q)-th order and relative (p,q)-th lower order of an entire function in the light of index-pair which are as follows:

Definition 5 ([9])**.** Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

If f and g have got index-pair $(m,1)$ and (m,k), respectively, then Definition 5 reduces to generalized relative order of f with respect to g. If the entire functions f and g have the same index-pair $(p,1)$ where p is any positive integer, we get the definition of relative order introduced by Bernal [1, 2] and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho^{[m]}(f)$ and $\rho_g^{(p,q)}(f) = \rho^{(m,q)}(f)$. Further if f is an entire function with index-pair $(2,1)$ and $g = \exp z$, then Definition 5 becomes the classical one given in [13].

An entire function f for which relative (p,q)-th order and relative (p,q)-th lower order with respect to another entire function g are the same is called a function of regular relative (p,q) growth with respect to g. Otherwise, f is said to be irregular relative (p,q) growth with respect to g.

In terms of maximum terms of entire functions, Definition 5 can be reformulated as:

Definition 6. For any positive integer p and q, the growth indicators $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g are defined as:

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} r}.$$

In fact, the equivalence of Definition 5, Definition 6 has been established in [3].

In order to refine the above growth scale, now we intend to introduce the definitions of an another growth indicators, such as relative (p, q)-th type and relative (p, q)-th lower type of entire function with respect to another entire function in the light of their index-pair which are as follows:

Definition 7. Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q and m are all positive integers such that $m \geq p$ and $m \geq q$. The relative (p, q)-th type and relative (p, q)-th lower type of entire function f with respect to the entire function g having finite positive relative (p, q) th order $\rho_g^{(p,q)}(f)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ are defined as:

$$\sigma_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}$$

and

$$\overline{\sigma}_g^{(p,q)}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}.$$

Analogously, to determine the relative growth of two entire functions having same nonzero finite relative (p, q)-th lower order with respect to another entire function, one can introduced the definition of relative (p, q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g of finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 8. Let f and g be any two entire functions having finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ where p and q are any two positive integers. Then the relative (p, q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of entire function f with respect to the entire function g are defined as:

$$\tau_g^{(p,q)}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}}$$

and

$$\overline{\tau}_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}}.$$

For entire functions, the notions of their growth indicators such as order, type, weak type are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders, relative type and relative weak type of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders relative type and relative weak type are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([11]). *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}\left(r\right) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g\left(R\right)\right).$$

Lemma 2 ([11]). *If f and g are any two entire functions with $g\left(0\right) = 0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - \left|g\left(0\right)\right|\right).$$

Lemma 3 ([5]). *If f be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta\mu_f(r).$$

Lemma 4 ([2]). *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

Lemma 5 ([4]). *If f and g are any two entire functions then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{16}M_g\left(\frac{r}{2}\right)\right) \leq M_{f\circ g}(r) \leq M_f\left(M_g\left(r\right)\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 9. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, taking $R = \beta r$, $(\beta > 1)$ in Lemma 1 and in view of Lemma 3 it follows for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \log^{[p]} \mu_h^{-1}\left(\mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)\right)$$

$$i.e., \ \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \mu_g(\beta r) + O(1).$$

Since $q = m-1$ and $\mu_g(r) \leq M_g(r)$ $\{cf.$ [12] $\}$, we get from above for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(\beta r) + O(1)$$

(1) $$i.e., \ \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)\left(\sigma^{(m,n)}(g) + \varepsilon\right)\left(\log^{[n-1]}(\beta r)\right)^{\rho^{(m,n)}(g)} + O(1).$$

Now from the definition of $\lambda_h^{(p,q)}(f)$ in terms of maximum terms, we obtain for all sufficiently large values of r that

(2) $$\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}.$$

Therefore from (1) and (2), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\left(\rho_h^{(p,q)}(f) + \varepsilon\right)\left(\sigma^{(m,n)}(g) + \varepsilon\right)\left(\log^{[n-1]}(\beta r)\right)^{\rho^{(m,n)}(g)} + O(1)}{\left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}}$$

$$i.e., \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Thus the theorem is established. □

Remark 10. In Theorem 9, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then Theorem 9 remains valid with " limit inferior" replaced by " limit superior".

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 9.

Theorem 11. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then Then for any $\beta > 1$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]} \beta r\right)^{\rho_g(m,n)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

Remark 12. In Theorem 11, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then Theorem 11 remains valid with " limit inferior" replaced by " limit superior".

Remark 13. We remark that in Theorem 11, if we will replace the condition " $\rho_h^{(p,q)}(f) < \infty$" by " $\lambda_h^{(p,q)}(f) < \infty$" , then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}. \tag{3}$$

Remark 14. In Remark 13, if we will replace the conditions " $\lambda_h^{(p,n)}(g) > 0$ and $\lambda_h^{(p,q)}(f) < \infty$" by "$\rho_h^{(p,n)}(g) > 0$ and $\rho_h^{(p,q)}(f) < \infty$" respectively, then is need to go the same replacement in right part of (3).

Using the concept of the growth indicator $\overline{\tau}^{(m,n)}(g)$ of an entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 9 and Theorem 11 respectively.

Theorem 15. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\tau}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Remark 16. We remark that in Theorem 15, if we will replace the condition " $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$" , then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \sigma^{(m,n)}(g).$$

Theorem 17. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\tau}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

Further using the notion of (p, q)-th weak type we may also state the following two theorems without proof because it can be carried out in the line of Theorem 15 and Theorem 17 respectively.

Theorem 18. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\tau^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Remark 19. We remark that in Theorem 18, if we will replace the condition "$0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$" , then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \overline{\tau}^{(m,n)}(g).$$

Theorem 20. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\tau^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

Remark 21. We remark that in Theorem 20, if we will replace the condition " $\rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$" by " $\lambda_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$" , then

$$\varliminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]} \beta r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\tau}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}. \tag{4}$$

Remark 22. In Remark 21, if we will replace the conditions " $\lambda_h^{(p,n)}(g) > 0$ and $\lambda_h^{(p,q)}(f) < \infty$" by "$\rho_h^{(p,n)}(g) > 0$ and $\rho_h^{(p,q)}(f) < \infty$" respectively, then is need to go the same replacement in right part of (4).

Remark 23. The same results of above theorems for $\beta = 1$ and in terms of maximum modulus of entire functions can also be deduced with the help of the second part of Lemma 5.

Theorem 24. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}^{(m,n)}(g) > 0$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Proof. In view of Lemma 2, we get for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right) \quad \{cf.\ [12]\ \} \ .$$

As $\mu_h^{-1}(r)$ is an increasing function of r, in view of above and Lemma 3 we get for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq \log^{[p]} \mu_h^{-1}\left(\mu_f\left(\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right)\right)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\left\{\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right\}.$$

Since $M_f(r)\leq\frac{\beta}{\beta-1}\mu_f(\beta r)$ for any $\beta>1$ $\{cf.$ [12] $\}$, we obtain from above for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\left\{\frac{1}{48}\left(\frac{\beta-1}{\beta}\right)M_g\left(\frac{r}{4\beta}\right)\right\}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[m-1]}M_g\left(\frac{r}{4\beta}\right)+O(1)$$

$$(5)\quad i.e.,\ \log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)\geq$$

$$\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g)-\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}+O(1).$$

Now from the definition of $\lambda_h^{(p,q)}(f)$ in terms of maximum terms, we obtain for all sufficiently large values of r that

$$(6)\quad \log^{[p]}\mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)\leq$$

$$\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}.$$

Therefore from (5) and (6), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p]}\mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)}\geq$$

$$\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g)-\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}}$$

$$i.e.,\ \underline{\lim}_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p]}\mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)}\geq\frac{\overline{\sigma}^{(m,n)}(g)\cdot\lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Thus the theorem is established. □

Remark 25. In Theorem 24, if we will replace "$\overline{\sigma}^{(m,n)}(g)$" by "$\sigma^{(m,n)}(g)$", then Theorem 24 remains valid with " limit superior" replaced by " limit inferior".

Remark 26. We remark that in Theorem 24, if we will replace the condition " $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$" , then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_f\left(\exp^{[q]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \overline{\sigma}^{(m,n)}(g).$$

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 24.

Theorem 27. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f)$, $\rho_h^{(p,n)}(g) < \infty$ and $\overline{\sigma}^{(m,n)}(g) > 0$ where p, q, m, n are all positive integers with $m \geq n$ and $q = m - 1$. Then for any $\beta > 1$*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}.$$

Remark 28. In Theorem 27, if we will replace "$\overline{\sigma}^{(m,n)}(g)$" by "$\sigma^{(m,n)}(g)$", then Theorem 27 remains valid with " limit inferior" replaced by " limit superior".

Remark 29. We remark that in Theorem 27, if we will replace the condition " $\rho_h^{(p,n)}(g) < \infty$" by " $\lambda_h^{(p,n)}(g) < \infty$" , then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}\left(\mu_g\left(\exp^{[n]}\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}. \tag{7}$$

Remark 30. In Remark 29, if we will replace the conditions " $0 < \lambda_h^{(p,q)}(f)$ and $\lambda_h^{(p,n)}(g) < \infty$" by "$0 < \rho_h^{(p,q)}(f)$ and $\rho_h^{(p,n)}(g) < \infty$" respectively, then is need to go the same replacement in right part of (7).

Using the concept of (m,n)-th weak type of an entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 24 and Theorem 27 respectively.

Theorem 31. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) > 0$ where p, q, m, n are all positive integers with $m \geq n$*

and $q = m - 1$. *Then for any* $\beta > 1$

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1} \left(\mu_f \left(\exp^{[q]} \left(\log^{[n-1]} \left(\frac{r}{4\beta}\right)\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \frac{\tau^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 24 and Theorem 31 respectively.

Remark 32. In Theorem 31, if we will replace "$\tau^{(m,n)}(g)$" by "$\overline{\tau}^{(m,n)}(g)$", then Theorem 31 remains valid with " limit superior" replaced by " limit inferior".

Remark 33. We remark that in Theorem 31, if we will replace the condition " $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$" , then

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1} \left(\mu_f \left(\exp^{[q]} \left(\log^{[n-1]} \left(\frac{r}{4\beta}\right)\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \tau^{(m,n)}(g).$$

Theorem 34. *Let* f, g *and* h *be any three entire functions such that* $0 < \lambda_h^{(p,q)}(f)$, $\rho_h^{(p,n)}(g) < \infty$ *and* $\tau^{(m,n)}(g) > 0$ *where* p, q, m, n *are all positive integers with* $m \geq n$ *and* $q = m - 1$. *Then for any* $\beta > 1$

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1} \left(\mu_g \left(\exp^{[n]} \left(\log^{[n-1]} \left(\frac{r}{4\beta}\right)\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \frac{\tau^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}.$$

Remark 35. In Theorem 34, if we will replace "$\tau^{(m,n)}(g)$" by "$\overline{\tau}^{(m,n)}(g)$", then Theorem 34 remains valid with " limit superior" replaced by " limit inferior".

Remark 36. We remark that in Theorem 34, if we will replace the condition " $0 < \lambda_h^{(p,q)}(f)$" by " $0 < \rho_h^{(p,q)}(f)$" , then

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \left(\mu_{f\circ g}(r)\right)}{\log^{[p]} \mu_h^{-1} \left(\mu_g \left(\exp^{[n]} \left(\log^{[n-1]} \left(\frac{r}{4\beta}\right)\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \frac{\tau^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}. \tag{8}$$

Remark 37. In Remark 36, if we will replace the conditions " $0 < \rho_h^{(p,q)}(f)$ and $0 < \rho_h^{(p,n)}(g)$" by "$0 < \lambda_h^{(p,q)}(f)$ and $0 < \lambda_h^{(p,n)}(g)$", then is need to go the same replacement in right part of (8).

Remark 38. The same results of above theorems for $\beta = \frac{1}{2}$ and in terms of maximum modulus of entire functions can also be deduced with the help of the first part of Lemma 5.

Theorem 39. *Let f, g and h be any three entire functions such that $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m - 1$. Then for any $\beta \geq 6$,*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}. \tag{9}$$

Proof. In view of the condition $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$,we obtain from (1) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)\left(\sigma^{(m,n)}(g) + \varepsilon\right)\left[\log^{[n-1]}(\beta r)\right]^{\rho_h^{(p,q)}(f)} + O(1). \tag{10}$$

Now taking $R = \alpha r$ in the inequalities $\mu_g(r) \leq M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ [12] $\}$, for $0 \leq r < R$ we obtain that

$$M_g^{-1}(r) \leq \mu_g^{-1}(r).$$

Since $M_g^{-1}(r)$ and $\mu_g^{-1}(r)$ are increasing functions of r, then for any $\alpha > 1$ it follows from the above and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{\alpha}{\alpha-1}\mu_f(\alpha r)$ $\{cf.$ [12] $\}$ that

$$M_g^{-1}(M_f(r)) \leq \mu_g^{-1}\left(\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right).$$

Therefore in view of Lemma 3, we obtain from above that

$$M_g^{-1}(M_f(r)) \leq \mu_g^{-1}\left(\mu_f\left(\frac{(2\alpha-1)\alpha}{(\alpha-1)} \cdot r\right)\right). \tag{11}$$

Now if we consider $\beta = \frac{(2\alpha-1)\alpha}{(\alpha-1)}$, then we get from (11) for all sufficiently large values of r that

$$\mu_h^{-1}(\mu_f(\beta r)) \geq M_h^{-1}(M_f(r)).$$

Therefore using the definition of $\sigma_h^{(p,q)}(f)$, we get from above for a sequence of values of r tending to infinity that

$$\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\beta^2 r\right)\right) \geq \log^{[p-1]} M_h^{-1}(M_f(\beta r)) \geq \left(\sigma_h^{(p,q)}(f) - \varepsilon\right)\left(\log^{[n-1]}(\beta r)\right)^{\rho_h^{(p,q)}(f)}. \tag{12}$$

Now from (10) and (12), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\beta^2 r\right)\right)} \leq$$

$$\frac{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\sigma^{(m,n)}(g)+\varepsilon\right)\left(\log^{[n-1]}(\beta r)\right)^{\rho_h^{(p,q)}(f)}+O(1)}{\left(\sigma_h^{(p,q)}(f)-\varepsilon\right)\left(\log^{[n-1]}\beta r\right)^{\rho_h^{(p,q)}(f)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\beta^2 r\right)\right)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

□

Remark 40. In Theorem 39, if we will replace the conditions " $\sigma^{(m,n)}(g)<\infty$" and "$0<\sigma_h^{(p,q)}(f)<\infty$" by "$\overline{\sigma}^{(m,n)}(g)<\infty$" and "$0<\overline{\sigma}_h^{(p,q)}(f)<\infty$", then is need to go the same replacement in right part of (9). Also if we replace the conditions $0<\rho_h^{(p,q)}(f)<\infty$ and $0<\sigma_h^{(p,q)}(f)<\infty$ of Theorem 39 by $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ respectively, then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\beta^2 r\right)\right)} \leq \frac{\lambda_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ of Theorem 39 by $0<\sigma_h^{(p,q)}(f)<\infty$, then Theorem 39 remains valid with " limit superior" replaced by " limit inferior".

Now we state the following three theorems without itheir proofs as those can easily be carried out in the line of Theorem 39.

Theorem 41. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, $\lambda_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g)<\infty$ and $0<\overline{\tau}_h^{(p,q)}(f)<\infty$ where p,q,m,n are all positive integers with $q=n=m-1$. Then for any $\beta\geq 6$,*

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\beta^2 r\right)\right)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}. \tag{13}$$

Remark 42. In Theorem 41, if we will replace the conditions " $\overline{\tau}^{(m,n)}(g)<\infty$" and "$0<\overline{\tau}_h^{(p,q)}(f)<\infty$" by "$\tau^{(m,n)}(g)<\infty$" and "$0<\tau_h^{(p,q)}(f)<\infty$", then is need to go the same replacement in right part of (13). Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of Theorem 41 by $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ respectively, then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of Theorem 41 by $0 < \tau_h^{(p,q)}(f) < \infty$, then Theorem 41 remains valid with " limit superior" replaced by " limit inferior".

Theorem 43. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m - 1$. Then for any $\beta \geq 6$,*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}. \tag{14}$$

Remark 44. In Theorem 43, if we will replace the conditions " $\sigma^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$" by "$\overline{\sigma}^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of (14). Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of Theorem 43 by $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ respectively, then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\lambda_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of Theorem 43 by $0 < \tau_h^{(p,q)}(f) < \infty$, then Theorem 43 remains valid with " limit superior" replaced by " limit inferior".

Theorem 45. *Let f, g and h be any three entire functions such that $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m - 1$. Then for any $\beta \geq 6$,*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}. \tag{15}$$

Remark 46. In Theorem 45, if we will replace the conditions " $\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$" by "$\tau^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of (15). Also if we replace the conditions

$0 < \rho_h^{(p,q)}(f) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ of Theorem 45 by $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ respectively, then

$$\varliminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}(\mu_f(\beta^2 r))} \leq \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \sigma_h^{(p,q)}(f) < \infty$ of Theorem 45 by $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$, then Theorem 45 remains valid with " limit superior" replaced by " limit inferior".

Remark 47. The same results of Theorem 39 to Theorem 45 for $\beta = 1$ and in terms of maximum modulus of entire functions can also be deduced with the help of the second part of Lemma 5.

Theorem 48. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\overline{\sigma}^{(m,n)}(g) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m - 1$. Then for any $\beta \geq 3$,*

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\sigma}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}. \tag{16}$$

Proof. In view of the condition $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, we obtain from (5) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g) - \varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho_h^{(p,q)}(f)} + O(1). \tag{17}$$

Taking $R = \alpha r$ in the inequalities $\mu_g(r) \leq M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ [12] $\}$, for $0 \leq r < R$ we obtain that

$$\mu_g^{-1}(r) \leq \alpha M_g^{-1}\left(\frac{\alpha r}{(\alpha - 1)}\right).$$

Since $M_g^{-1}(r)$ and $\mu_g^{-1}(r)$ are increasing functions of r, then for any $\alpha > 1$ it follows from the above and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{\alpha}{\alpha-1}\mu_f(\alpha r)$ $\{cf.$ [12] $\}$ that

$$\mu_g^{-1}(\mu_f(r)) \leq \alpha M_g^{-1}\left(\frac{\alpha}{(\alpha - 1)} M_f(r)\right).$$

Therefore, in view of Lemma 4 it follows from above that

$$\mu_g^{-1}(\mu_f(r)) \leq \alpha M_g^{-1}\left(M_f\left(\left(\frac{2\alpha - 1}{\alpha - 1}\right) \cdot r\right)\right). \tag{18}$$

Now if we consider $\beta = \frac{2\alpha-1}{\alpha-1}$, then we get from (18) for all sufficiently large values of r and any $\alpha > 1$ that

$$\mu_h^{-1}\left(\mu_f\left(r\right)\right) \leq \alpha M_h^{-1}\left(M_f\left(\beta r\right)\right).$$

Therefore in view of definition of $\overline{\sigma}_h^{(p,q)}\left(f\right)$, we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right) \leq \log^{[p-1]}M_h^{-1}\left(M_f\left(\frac{r}{4\beta}\right)\right) + O(1) \leq \left(\overline{\sigma}_h^{(p,q)}\left(f\right)+\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho_h^{(p,q)}(f)} + O(1). \tag{19}$$

Now from (17) and (19),it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}\left(f\right)-\varepsilon\right)\left(\overline{\sigma}^{(m,n)}\left(g\right)-\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho_h^{(p,q)}(f)} + O(1)}{\left(\overline{\sigma}_h^{(p,q)}\left(f\right)+\varepsilon\right)\left(\log^{[n-1]}\left(\frac{r}{4\beta}\right)\right)^{\rho_h^{(p,q)}(f)} + O(1)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\lambda_h^{(p,q)}\left(f\right)\cdot\overline{\sigma}^{(m,n)}\left(g\right)}{\overline{\sigma}_h^{(p,q)}\left(f\right)}.$$

□

Remark 49. In Theorem 48, if we will replace the conditions " $\overline{\sigma}^{(m,n)}\left(g\right) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}\left(f\right) < \infty$" by "$\sigma^{(m,n)}\left(g\right) < \infty$" and "$0 < \sigma_h^{(p,q)}\left(f\right) < \infty$", then is need to go the same replacement in right part of (16). Also if we replace the conditions $0 < \lambda_h^{(p,q)}\left(f\right) \leq \rho_h^{(p,q)}\left(f\right) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}\left(f\right) < \infty$ of Theorem 48 by $0 < \rho_h^{(p,q)}\left(f\right) < \infty$ and $0 < \sigma_h^{(p,q)}\left(f\right) < \infty$ respectively, then

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[p-1]}\mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\rho_h^{(p,q)}\left(f\right)\cdot\overline{\sigma}^{(m,n)}\left(g\right)}{\sigma_h^{(p,q)}\left(f\right)}.$$

Further if we replace the condition $0 < \overline{\sigma}_h^{(p,q)}\left(f\right) < \infty$ of Theorem 48 by $0 < \sigma_h^{(p,q)}\left(f\right) < \infty$, then Theorem 48 remains valid with " limit inferior" replaced by " limit superior".

Now we state the following three theorems without itheir proofs as those can easily be carried out in the line of Theorem 48.

Theorem 50. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\tau^{(m,n)}(g) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m-1$. Then for any $\beta \geq 3$,*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\lambda_h^{(p,q)}(f) \cdot \tau^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}. \tag{20}$$

Remark 51. In Theorem 50, if we will replace the conditions " $\tau^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$" by "$\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of (20). Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ of Theorem 50 by $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ respectively, then

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\rho_h^{(p,q)}(f) \cdot \tau^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \tau_h^{(p,q)}(f) < \infty$ of Theorem 50 by $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$, then Theorem 50 remains valid with " limit inferior" replaced by " limit superior".

Theorem 52. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\overline{\sigma}^{(m,n)}(g) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m-1$. Then for any $\beta \geq 3$,*

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\sigma}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}. \tag{21}$$

Remark 53. In Theorem 52, if we will replace the conditions " $\overline{\sigma}^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$" by "$\sigma^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of (21). Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ of Theorem 52 by $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ respectively, then

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\rho_h^{(p,q)}(f) \cdot \overline{\sigma}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \tau_h^{(p,q)}(f) < \infty$ of Theorem 52 by $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$, then Theorem 52 remains valid with " limit inferior" replaced by " limit superior".

Theorem 54. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\tau^{(m,n)}(g) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $q = n = m - 1$. Then for any $\beta \geq 3$,*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\lambda_h^{(p,q)}(f) \cdot \tau^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}. \tag{22}$$

Remark 55. In Theorem 54, if we will replace the conditions " $\tau^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$" by "$\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of (22). Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ of Theorem 54 by $0 < \rho_h^{(p,q)}(f) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ respectively, then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]} \mu_h^{-1}\left(\mu_f\left(\frac{r}{4\beta^2}\right)\right)} \geq \frac{\rho_h^{(p,q)}(f) \cdot \tau^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ of Theorem 54 by $0 < \sigma_h^{(p,q)}(f) < \infty$, then Theorem 54 remains valid with " limit inferior" replaced by " limit superior".

Remark 56. The same results of Theorem 48 to Theorem 54 for $\beta = \frac{1}{2}$ and in terms of maximum modulus of entire functions can also be deduced with the help of the second part of Lemma 5.

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