

**STRONG INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5)
FUNCTION BETWEEN TWO COMPARABLE
REAL-VALUED FUNCTIONS**

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ABSTRACT. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ -kernel of sets are F_σ -sets.

1. INTRODUCTION

A generalized class of closed sets was considered by Maki in 1986 [18]. He investigated the sets that can be represented as union of closed sets and called them V -sets. Complements of V -sets, i.e., sets that are intersection of open sets are called Λ -sets [18].

Recall that a real-valued function f defined on a topological space X is called A -continuous [23] if the preimage of every open subset of \mathbb{R} belongs to A , where A is a collection of subsets of X . Most of the definitions of function used throughout this paper are consequences of the definition of A -continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 8, 9, 10, 12, 13, 22].

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Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ -kernel of sets are F_σ -sets.

A real-valued function f defined on a topological space X is called *contra-Baire-1* (*Baire-.5*) if the preimage of every open subset of \mathbb{R} is a G_δ -set in X [24].

If g and f are real-valued functions defined on a space X , we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X .

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *B-.5-property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P . If P_1 and P_2 are *B-.5-properties*, the following terminology is used: (i) A space X has the *weak B-.5-insertion property for (P_1, P_2)* if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a Baire-.5 function h such that $g \leq h \leq f$. (ii) A space X has the *strong B-.5-insertion property for (P_1, P_2)* if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a Baire-.5 function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X , then $g(x) < h(x) < f(x)$.

In this paper, for a topological space that F_σ -kernel of sets are F_σ -sets, a sufficient condition for the weak *B-.5-insertion property* is given. Also for a space with the weak *B-.5-insertion property*, we give necessary and sufficient conditions for the space to have the strong *B-.5-insertion property*. Several insertion theorems are obtained as corollaries of these results.

2. THE MAIN RESULT AND APPLICATIONS

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ) . We define the subsets A^Λ and A^V as follows:

$$A^\Lambda = \cap \{O : O \supseteq A, O \in (X, \tau)\} \text{ and } A^V = \cup \{F : F \subseteq A, F^c \in (X, \tau)\}.$$

In [7, 17, 21], A^Λ is called the *kernel* of A .

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$$G_\delta(A) = \cup\{O : O \subseteq A, O \text{ is } G_\delta\text{-set}\} \text{ and}$$

$$F_\sigma(A) = \cap\{F : F \supseteq A, F \text{ is } F_\sigma\text{-set}\}.$$

$F_\sigma(A)$ is called the F_σ -kernel of A .

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho \nu$ implies $x \rho \nu$ and $u \rho x$ implies $u \rho y$ for any u and ν in S .

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.

2) If $A \subseteq B$, then $A \bar{\rho} B$.

3) If $A \rho B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$. The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ , then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ .

We now give the following main results:

Theorem 2.1. Let g and f be real-valued functions on the topological space X , that F_σ -kernel sets in X are F_σ -sets, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers \mathbb{Q} into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of \mathbb{Q} with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By [15, Lemmas 1 and 2] it follows that there exists a function H mapping \mathbb{Q} into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any x in X , let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is a G_δ -set in X , i.e., h is a Baire-.5 function on X . \square

The above proof used the technique of [14, Theorem 1].

Definition 2.5. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a G_δ -set for any real number t .

The abbreviations *usc*, *lsc*, *cusB.5* and *clsB.5* are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1 ([14, 15]). A space X has the weak c -insertion property for (*usc*, *lsc*) if and only if X is normal.

Before stating the consequences, we suppose that X is a topological space that F_σ -kernel of sets are F_σ -sets.

Corollary 2.1. For each pair of disjoint F_σ -sets F_1, F_2 , there are two G_δ -sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak $B - .5$ -insertion property for (*cusB - .5*, *clsB - .5*).

Proof. Let g and f be real-valued functions defined on the X , such that f is *lsB₁*, g is *usB₁*, and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2

are any elements of \mathbb{Q} with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a F_σ -set and since $\{x \in X : g(x) < t_2\}$ is a G_δ -set, it follows that $F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let F_1 and F_2 be disjoint F_σ -sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is $clsB - .5$, g is $cusB - .5$, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then G_1 and G_2 are disjoint G_δ -sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. \square

Remark 2 ([25]). A space X has the weak c -insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 2.2. *For every G of G_δ -set, $F_\sigma(G)$ is a G_δ -set if and only if X has the weak $B - .5$ -insertion property for $(clsB - .5, cusB - .5)$.*

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 2.1. *The following conditions on the space X are equivalent:*

- (i) *For every G of G_δ -set we have $F_\sigma(G)$ is a G_δ -set.*
- (ii) *For each pair of disjoint G_δ -sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.*

The proof of Lemma 2.1 is a direct consequence of the definition F_σ -kernel of sets.

We now give the proof of Corollary 2.2.

Proof. Let g and f be real-valued functions defined on the X , such that f is $clsB - .5$, g is $cusB - .5$, and $f \leq g$. If a binary relation ρ is defined by $A \rho B$ in case $F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$ for some G_δ -set G in X , then by hypothesis and Lemma 2.1 ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of \mathbb{Q} with $t_1 < t_2$, then

$$\begin{aligned} A(g, t_1) &= \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} \\ &= A(f, t_2); \end{aligned}$$

since $\{x \in X : g(x) < t_1\}$ is a G_δ -set and since $\{x \in X : f(x) \leq t_2\}$ is a F_σ -set, by hypothesis it follows that $A(g, t_1) \rho A(f, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let G_1 and G_2 be disjoint G_δ -sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is $clsB - .5$, g is $cusB - .5$, and $f \leq g$.

Thus there exists Baire-.5 function h such that $f \leq h \leq g$. Set $F_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $F_2 = \{x \in X : h(x) \geq 2/3\}$ then F_1 and F_2 are disjoint F_σ -sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset$. \square

If a space has the strong $B - .5$ -insertion property for (P_1, P_2) , then it has the weak $B - .5$ -insertion property for (P_1, P_2) . The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $B - .5$ -insertion property to satisfy the strong $B - .5$ -insertion property.

Theorem 2.2. *Let P_1 and P_2 be $B - .5$ -property and X be a space that satisfies the weak $B - .5$ -insertion property for (P_1, P_2) . Also assume that g and f are functions on X such that $g \leq f$, g has property P_1 and f has property P_2 . The space X has the strong $B - .5$ -insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n , F_n and $A(f - g, 2^{-n})$ are completely separated by Baire-.5 functions, and (ii) $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.*

Proof. [20, Theorem 3.1]. \square

Theorem 2.3. *Let P_1 and P_2 be $B - .5$ -properties and assume that the space X satisfied the weak $B - .5$ -insertion property for (P_1, P_2) . The space X satisfies the strong $B - .5$ -insertion property for (P_1, P_2) if and only if X satisfies the strong $B - .5$ -insertion property for $(P_1, B - .5)$ and for $(B - .5, P_2)$.*

Proof. [20, Theorem 3.2]. \square

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 2.2. *The following conditions on the space X are equivalent:*

- (i) *Every two disjoint F_σ -sets of X can be separated by G_δ -sets of X .*
- (ii) *If F is a F_σ -set of X which is contained in a G_δ -set G , then there exists a G_δ -set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.*

Proof. (i) \Rightarrow (ii) Suppose that $F \subseteq G$, where F and G are F_σ -set and G_δ -set of X , respectively. Hence, G^c is a F_σ -set and $F \cap G^c = \emptyset$.

By (i) there exist two disjoint G_δ -sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a F_σ -set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) \Rightarrow (i) Suppose that F_1, F_2 be two disjoint F_σ -sets of X .

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a G_δ -set. Hence by (ii) there exists a G_δ -set H such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.

But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a G_δ -set of X . Hence $F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \emptyset$. This means that condition (i) holds. \square

Lemma 2.3. *Suppose that X is a topological space such that we can separate every two disjoint F_σ -sets by G_δ -sets. If F_1 and F_2 be two disjoint F_σ -sets of X , then there exists a Baire-.5 function $h : X \rightarrow [0, 1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 be two disjoint F_σ -sets of X . Since $F_1 \cap F_2 = \emptyset$, $F_1 \subseteq F_2^c$. In particular, since F_2^c is a G_δ -set of X containing F_1 , by Lemma 2.2, there exists a G_δ -set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a G_δ -set and contains F_1 , and F_2^c is a G_δ -set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 2.2, there exist G_δ -sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain G_δ -sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into $[0,1]$. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X . For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are G_δ -sets of X . Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a G_δ -set. Consequently h is a Baire-.5 function. \square

Lemma 2.4. *Suppose that X is a topological space such that we can separate every two disjoint F_σ -sets by G_δ -sets. If F_1 and F_2 are two disjoint F_σ -sets of X and F_1 is a countable intersection of G_δ -sets, then there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h(F_2) = \{1\}$.*

Proof. Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$, where G_n is a G_δ -set of X . We can suppose that $G_n \cap F_2 = \emptyset$, otherwise we can substitute G_n by $G_n \setminus F_2$. By Lemma 2.3, for every $n \in \mathbb{N}$, there exists a Baire-.5 function h_n on X into $[0,1]$ such that $h_n(F_1) = \{0\}$ and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a Baire-.5 function from X to $[0,1]$. Since for every $n \in \mathbb{N}, F_2 \subseteq X \setminus G_n$, therefore $h_n(F_2) = \{1\}$ and consequently $h(F_2) = \{1\}$. Since $h_n(F_1) = \{0\}$, $h(F_1) = \{0\}$. It suffices to show that if $x \notin F_1$, then $h(x) \neq 0$.

Now if $x \notin F_1$, since $F_1 = \bigcap_{n=1}^{\infty} G_n$, there exists $n_0 \in \mathbb{N}$ such that $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$. \square

Lemma 2.5. *Suppose that X is a topological space such that we can separate every two disjoint F_σ -sets by G_δ -sets. The following conditions are equivalent:*

- (i) *For every two disjoint F_σ -sets F_1 and F_2 , there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.*
- (ii) *Every F_σ -set is a countable intersection of G_δ -set.*
- (iii) *Every G_δ -set is a countable union of F_σ -set.*

Proof. (i) \Rightarrow (ii). Suppose that F is a F_σ -sets. Since \emptyset is a F_σ -set, by (i) there exists a Baire-.5 function h on X into $[0,1]$ such that $h^{-1}(0) = F$. Set $G_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, G_n is a G_δ -set and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) \Rightarrow (i). Suppose that F_1 and F_2 are two disjoint F_σ -sets. By Lemma 2.4, there exists a Baire-.5 function f on X into $[0, 1]$ such that $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two F_σ -sets and $(G \cup F) \cap F_2 = \emptyset$. By Lemma 2.4, there exists a Baire-.5 function g on X into $[\frac{1}{2}, 1]$ such that $g^{-1}(1) = F_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. h is well-defined and a Baire-.5 function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0, 1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) \Leftrightarrow (iii) By De Morgan law and noting that the complement of every F_σ -set is a G_δ -set and complement of every G_δ -set is a F_σ -set, the equivalence is holds. \square

Remark 3 ([19]). A space X has the strong c -insertion property for (usc, lsc) if and only if X is perfectly normal.

Corollary 2.3. *For every two disjoint F_σ -sets F_1 and F_2 , there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ if and only if X has the strong $B - .5$ -insertion property for $(cusB - .5, clsB - .5)$.*

Proof. Since for every two disjoint F_σ -sets F_1 and F_2 , there exists a Baire-.5 function h on X into $[0, 1]$ such that $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then G_1 and G_2 are two disjoint G_δ -sets that contain F_1 and F_2 , respectively. This means that, we can separate every two disjoint F_σ -sets by G_δ -sets. Hence by Corollary 2.1, X has the weak $B - .5$ -insertion property for $(cusB - .5, clsB - .5)$. Now, assume that g and f are functions on X such that $g \leq f$, g is $cusB - .5$ and f is $clsB - .5$. Since $f - g$ is $clsB - .5$, the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a F_σ -set. By Lemma 2.5, we can choose a sequence $\{F_n\}$ of F_σ -sets such that $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}$, F_n and $A(f - g, 2^{-n})$ are disjoint. By Lemma 2.3, F_n and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, X has the strong $B - .5$ -insertion property for $(cusB - .5, clsB - .5)$.

On the other hand, suppose that F_1 and F_2 be two disjoint F_σ -sets. Since $F_1 \cap F_2 = \emptyset$, $F_2 \subseteq F_1^c$. Set $g = \chi_{F_2}$ and $f = \chi_{F_1^c}$. Then f is $clsB - .5$ and g is $cusB - .5$ and furthermore $g \leq f$. By hypothesis, there exists a Baire-.5 function h on

X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$ we have $g(x) < h(x) < f(x)$. By definitions of f and g , we have $h^{-1}(1) = F_2 \cap F_1^c = F_2$ and $h^{-1}(0) = F_1 \cap F_2^c = F_1$. \square

Remark 4 ([2]). A space X has the strong c -insertion property for (lsc, usc) if and only if each open subset of X is closed.

Corollary 2.4. *Every G_δ -set is a F_σ -set if and only if X has the strong $B - .5$ -insertion property for $(clsB - .5, cusB - .5)$.*

Proof. By hypothesis, for every G of G_δ -set, we have $F_\sigma(G) = G$ is a G_δ -set. Hence by Corollary 2.2, X has the weak $B - .5$ -insertion property for $(clsB - .5, cusB - .5)$. Now, assume that g and f are functions on X such that $g \leq f$, g is $clsB - .5$ and f is $B - .5$. Set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) < 2^{-n}\}$. Then, since $f - g$ is $cusB - .5$, we can say that $A(f - g, 2^{-n})$ is a G_δ -set. By hypothesis, $A(f - g, 2^{-n})$ is a F_σ -set. Set $G_n = X \setminus A(f - g, 2^{-n})$. Then G_n is a G_δ -set. This means that G_n and $A(f - g, 2^{-n})$ are disjoint G_δ -sets and also are two disjoint F_σ -sets. Therefore G_n and $A(f - g, 2^{-n})$ can be completely separated by Baire-.5 functions. Now, we have $\bigcup_{n=1}^{\infty} G_n = \{x \in X : (f - g)(x) > 0\}$. By Theorem 2.2, X has the strong $B - .5$ -insertion property for $(clsB - .5, B - .5)$. By an analogous argument, we can prove that X has the strong $B - .5$ -insertion property for $(B - .5, cusB - .5)$. Hence, by Theorem 2.3, X has the strong $B - .5$ -insertion property for $(clsB - .5, cusB - .5)$.

On the other hand, suppose that X has the strong $B - .5$ -insertion property for $(clsB - .5, cusB - .5)$. Also, suppose that G is a G_δ -set. Set $f = 1$ and $g = \chi_G$. Then f is $cusB - .5$, g is $clsB - .5$ and $g \leq f$. By hypothesis, there exists a Baire-.5 function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$, we have $g(x) < h(x) < f(x)$. It is clear that $h(G) = \{1\}$ and for $x \in X \setminus G$ we have $0 < h(x) < 1$. Since h is a Baire-.5 function, $\{x \in X : h(x) \geq 1\} = G$ is a F_σ -set, i.e., G is a F_σ -set. \square

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