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ON THE STABILITY OF RECIPROCAL-NEGATIVE FERMAT'S EQUATION IN QUASI-β-NORMED SPACES

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Abstract. In this paper we introduce the reciprocal-negative Fermat's equation induced by the famous equation in the Fermat's Last Theorem, establish the general solution in the simplest cases and the differential solution to the equation, and investigate, then, the generalized Hyers-Ulam stability in a quasi-β-normed space with both the direct estimation method and the fixed point approach.

1. Introduction

One of the interesting questions concerning the stability problems of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation? Such an idea was suggested in 1940 by Ulam [19] as follows: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [9] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. The famous Hyers stability result that appeared in [9] was generalized in the stability involving a sum of powers of norms by Aoki [1]. In 1978, Rassias [16] provided a generalization of Hyers Theorem which

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allows the Cauchy difference to be unbounded. For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [3, 4, 6, 7, 10, 13, 12]).

Let β be a real number with $0 < \beta \le 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will consider the definition and some preliminary results of a quasi-β-norm on a linear space.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the followings:

(1) $||x|| \ge 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^\beta \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \ge 1$ such that $||x+y|| \le K(||x||+||y||)$ for all $x,\, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $||\cdot||$ is called a $(\beta,\, p)$-norm $(0 < p \le 1)$ if $||x+y||^p \le ||x||^p + ||y||^p$, for all $x,\, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space; see [2] and [18].

In number theory, Fermat's Last Theorem states that no three positive integers a, b, and c satisfy the equation $c^n = a^n + b^n$ for any integer value of n greater than 2. The equation $\frac{1}{c^n} = \frac{1}{a^n} + \frac{1}{b^n}$ can be considered the reciprocal-negative Fermat's equation. This equation induces that $\frac{1}{c^n} = \frac{a^n+b^n}{a^n\, b^n}$, that is, we have

$$c^n = \frac{a^n\, b^n}{a^n + b^n}$$

for any integer value of n greater than 2. In particular, if we have the case of $n = 1$ then the above equation is the harmonic mean of a and b out of the well-known three Pythagorean means; arithmetic mean, geometric mean, and harmonic mean in geometry.

In 2010, Ravi and Kumar [17] investigated the generalized Hyers-Ulam stability of the the reciprocal functional equation $f(x+y) = \dfrac{f(x)f(y)}{f(x)+f(y)}$. Also see [11] for a fixed point approach. With the motivation of the Pythagorean means Narasimman, Ravi, and Pinelas [15] in 2015 introduced the Pythagorean mean functional equation $f(\sqrt{x^2+y^2}) = \dfrac{f(x)f(y)}{f(x)+f(y)}$ for all positive numbers x and y and studied the generalized Hyers-Ulam stability of the equation providing counter-examples for singular cases.

In order to give our results in Section 4 it is convenient to state the definition of a generalized metric on a set X and a result on a fixed point theorem of the alternative by Diaz and Margolis [5].
Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.2. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive n_0 such that*

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ for all $y \in Y$.

In this paper, we consider the following the functional equation:

$$f\left(\sqrt[n]{x^n + y^n}\right) = \frac{f(x)f(y)}{f(x) + f(y)} \tag{1.1}$$

for fixed positive integers n and for all $x, y \in X$. Due to the reciprocal-negative Fermat's equation, we call the mapping f the reciprocal-negative Fermat's function. In Section 2 we establish the general solution of the reciprocal-negative Fermat's equation (4.1) in the simplest case and give the differential solution to the equation (4.1). In Section 3 we prove the generalized Hyers-Ulam stability of the reciprocal-negative Fermat's equation (4.1) in a quasi-β-normed space. Lastly, we'll investigate the generalized stability of the equation (4.1) with a fixed theorem approach.

2. General Solution of the Reciprocal-negative Fermat's Functional Equation

In this section we show the general solution of the reciprocal-negative Fermat's equation (4.1) in the simple case by the limiting process argument and also present the differential solution to the equation (4.1) following the work by Ger [8]

Theorem 2.1 (Simple Case). *Let $n \in \mathbb{N}$ be an odd integer. The only nonzero solution $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ (or $f : (0, \infty) \longrightarrow \mathbb{R}$ with the case of an even integer $n \in \mathbb{N}$), admitting a finite limit of the quotient $\frac{f(x)}{\frac{1}{x^n}}$ at zero, of the equation (4.1) is of the form $\frac{c}{x^n}$ for a constant $c \in \mathbb{R}$.*

Proof. Substituting $y = x$ in (4.1) we have the equality $f(\sqrt[n]{2}x) = \frac{1}{2} f(x)$ for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$) for an even number n.
Let $g(x) = \frac{f(x)}{\frac{1}{x}}$ for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). Then we have the limit

$$\lim_{x \to 0} \frac{g(x)}{\dfrac{1}{x^{n-1}}} = c$$

for some nonzero $c \in \mathbb{R}$ and using the property of $f(x)$ we obtain

$$g\left(\sqrt[n]{2}x\right) = \frac{1}{\sqrt[n]{2^{n-1}}} g(x)$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). By the mathematical induction for every positive integer k, we also have

$$g\left(\frac{x}{(\sqrt[n]{2})^k}\right) = (\sqrt[n]{2^{n-1}})^k g(x) \tag{2.1}$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). Therefore we conclude, by the above equality (2.1), that

$$\frac{g(x)}{\dfrac{1}{x^{n-1}}} = \frac{(\sqrt[n]{2^{n-1}})^k g(x)}{(\sqrt[n]{2^{n-1}})^k \dfrac{1}{x^{n-1}}} = \frac{g\left(\dfrac{x}{(\sqrt[n]{2})^k}\right)}{\left(\dfrac{(\sqrt[n]{2})^k}{x}\right)^{n-1}} \longrightarrow c \tag{2.2}$$

as $n \to \infty$. By the definition of $g(x)$ we get the solution

$$f(x) = \frac{1}{x} g(x) = \frac{1}{x}\left(\frac{c}{x^{n-1}}\right) = \frac{c}{x^n}$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$) and this completes the proof. □

The following theorem gives the differentiable solution of the reciprocal-negative Fermat's functional equation (4.1) applying the work of Ger [8]. For simplicity we will consider the case of an odd integer $n \in \mathbb{N}$.

Theorem 2.2 (Differential Solution)**.** *Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be continuously differentiable function with the derivative $f'(x) \neq 0$ for all $x \in (0, \infty)$. Then f is a solution to the reciprocal-negative Fermat's equation if and only if there exists a nonzero constant $c \in \mathbb{R}$ such that $f(x) = \frac{c}{x^n}$ for all $x \in (0, \infty)$.*

Proof. Differentiating the equation (4.1) with respect to x on both sides we have

$$f'(\sqrt[n]{x^n + y^n}) \left(\frac{x}{\sqrt[n]{x^n + y^n}} \right)^{n-1} = \frac{f'(x)(f(y))^2}{(f(x) + f(y))^2} \tag{2.3}$$

for all $x, y \in (0, \infty)$. Letting $y = x$ in the equation (4.1) and the equation (2.3), respectively, we just have

$$f(\sqrt[n]{2}x) = \frac{1}{2} f(x) \tag{2.4}$$

and

$$f'(\sqrt[n]{2}x) = \frac{1}{2^{\frac{n+1}{n}}} f'(x) \tag{2.5}$$

for all $x \in (0, \infty)$. Putting $y = \sqrt[n]{x}$ in (2.3) again and using (2.4) and (2.5) we obtain

$$f'(\sqrt[n]{3}x) = \frac{1}{3^{\frac{n+1}{n}}} f'(x) \tag{2.6}$$

for all $x \in (0, \infty)$. The equations (2.5) and (2.6) give

$$f'((\sqrt[n]{2})^l (\sqrt[n]{3})^m x) = \frac{1}{(2^{\frac{n+1}{n}})^l (3^{\frac{n+1}{n}})^m} f'(x) \tag{2.7}$$

for all integers l and m. It is well-known that the set $\{(2^{\frac{n+1}{n}})^l (3^{\frac{n+1}{n}})^m : n, m \in \mathbb{Z}\}$ is dense in $(0, \infty)$. Since the function f' is continuous we derive the following differential equation

$$f'(\lambda) = f'(1) \frac{1}{\lambda^{n+1}} \tag{2.8}$$

for $\lambda \in (0, \infty)$. Therefore, the solution $f(x) = \frac{c}{x^n} + d$ for some constants c and d for $x \in (0, \infty)$. It is obvious that the constant d should be zero since $f(\sqrt[n]{2}x) = \frac{1}{2} f(x)$ and it completes the proof.

□

3. Stability of a Reciprocal-negative Fermat's Functional Equation

Throughout this section, let X be a linear space and let Y be a quasi-β-Banach space with a quasi-β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (4.1). For a given mapping $f : X \to Y$ and a fixed positive integer n, let

$$D_n f(x,y) := f\Big(\sqrt[n]{x^n+y^n}\Big) - \frac{f(x)f(y)}{f(x)+f(y)}$$

for all $x, y \in X$ and $\mathbb{R}^+ := [0,\infty)$

Theorem 3.1. *Suppose that there exists a function $\phi : X \times X \to \mathbb{R}^+$ for which a mapping $f : X \to Y$ satisfies*

$$||D_n f(x,y)||_Y \le \phi(x,y) \tag{3.1}$$

and the series $\sum_{j=0}^{\infty}(2^\beta K)^j \phi((\sqrt[n]{2})^j x, (\sqrt[n]{2})^j y)$ converges for all $x, y \in X$. Then there exists a unique reciprocal-negative Fermat's function $R : X \to Y$ which satisfies the equation (4.1) and the inequality

$$||f(x) - R(x)||_Y \le 2^\beta K \sum_{j=0}^{\infty}(2^\beta K)^j \phi((\sqrt[n]{2})^j x, (\sqrt[n]{2})^j x)\,, \tag{3.2}$$

for all $x \in X$.

Proof. Letting $x = y$ in the equation (3.1), we have

$$||D_n f(x,x)||_Y = ||\frac{1}{2}f(x) - f(\sqrt[n]{2}x)||_Y \le \phi(x,\, x)$$

that is,

$$||f(x) - 2f(\sqrt[n]{2}x)||_Y \le 2^\beta \phi(x,\, x) \tag{3.3}$$

for all $x \in X$. Let m be a positive integer. Putting $x = (\sqrt[n]{2})^m x$ and multiplying $2^{m\beta}$ in the inequality (3.3), we get

$$||2^m f((\sqrt[n]{2})^m x) - 2^{m+1} f((\sqrt[n]{2})^{m+1} x)||_Y \le 2^\beta \cdot 2^{m\beta} \phi((\sqrt[n]{2})^m x,\, (\sqrt[n]{2})^m x) \tag{3.4}$$

for all $x \in X$. According to the mathematical induction, we have the following inequality:

$$||f(x) - 2^m f((\sqrt[n]{2})^m x)||_Y \le (2^\beta K) \sum_{j=0}^{m-1}(2^\beta K)^j \phi((\sqrt[n]{2})^j x,\, (\sqrt[n]{2})^j x) \tag{3.5}$$

for all positive integers m and for all $x \in X$. Also, for all positive integers s and t with $s > t$, we have

$$(3.6) \qquad ||2^t f((\sqrt[n]{2})^t x) - 2^s f((\sqrt[n]{2})^s x)||_Y \leq (2^\beta K) \sum_{j=t}^{s-1} (2^\beta K)^j \phi((\sqrt[n]{2})^j x,\, (\sqrt[n]{2})^j x)$$

for all $x \in X$. Since the series $\sum_{j=0}^{\infty} (2^\beta K)^j \phi((\sqrt[n]{2})^j x, (\sqrt[n]{2})^j y)$ converges, we may conclude that the right-hand side of the inequality (3.6) tends to 0 as $t \to \infty$. Hence $\{2^m f((\sqrt[n]{2})^m x)\}$ is a Cauchy sequence in the quasi-β-Banach space Y . Thus we may define

$$R(x) = \lim_{m \to \infty} 2^m f((\sqrt[n]{2})^m x)$$

for all $x \in X$. Now, we claim that $R(x)$ is a reciprocal-negative Fermat's equation. Let m be a positive integer. Letting $x = (\sqrt[n]{2})^m x$ and $y = (\sqrt[n]{2})^m y$ and multiplying $2^{m\beta}$ in the inequality (3.1), we get

$$\begin{aligned} & 2^{m\beta} ||D_n f((\sqrt[n]{2})^m x,\, (\sqrt[n]{2})^m y)||_Y \\ = \;& 2^{m\beta} ||f((\sqrt[n]{2})^m \sqrt[n]{x^n + y^n}) - \frac{f((\sqrt[n]{2})^m x) f((\sqrt[n]{2})^m y)}{f((\sqrt[n]{2})^m x) + f((\sqrt[n]{2})^m y)}||_Y \\ \leq \;& (2^\beta K)^m \phi((\sqrt[n]{2})^m x,\, (\sqrt[n]{2})^m y) \end{aligned}$$

for all $x\,, y \in X$. On taking $m \to \infty$, the definition of R implies that R satisfies the equation (4.1) for all $x,\, y \in X$, that is, R is the reciprocal-negative Fermat's equation. Also, the inequality (3.5) implies the inequality (3.2).

Now, it remains to show the uniqueness of the reciprocal-negative Fermat's equation R. Assume that there exists $r : X \to Y$ satisfying (4.1) and (3.2). Then

$$\begin{aligned} ||R(x) - r(x)||_Y &= ||2^{-m} R((\sqrt[n]{2})^{-m} x) - 2^{-m} r((\sqrt[n]{2})^{-m} x)||_Y \\ &= 2^{-m\beta} ||R((\sqrt[n]{2})^{-m} x) - r((\sqrt[n]{2})^{-m} x)||_Y \\ &\leq K\Big(||2^{-m} R((\sqrt[n]{2})^{-m} x) - f(x)||_Y \\ &\quad + ||2^{-m} r((\sqrt[n]{2})^{-m} x) - f(x)||_Y\Big) \\ &\leq 2^{\beta+1} K^2 \sum_{j=0}^{m-1} (2^\beta K)^j \phi((\sqrt[n]{2})^j x, (\sqrt[n]{2})^j x) \end{aligned}$$

for all $x \in X$. By letting $m \to \infty$, we immediately have the uniqueness of the reciprocal-negative Fermat's mapping R , as desired. □

Now we have another equivalent version of Theorem 3.1 by scaling down the approximate $f(x)$ in (3.1) as follows:

Theorem 3.2. *Suppose that there exists a function $\phi : X \times X \to \mathbb{R}^+$ for which a mapping $f : X \to Y$ satisfies*

$$||D_n f(x,y)||_Y \leq \phi(x,y) \tag{3.7}$$

and the series $\sum_{j=0}^{\infty}(K/2^\beta)^j \phi((\sqrt[n]{2})^{-j}x, (\sqrt[n]{2})^{-j}y)$ converges for all $x, y \in X$. Then there exists a unique reciprocal-negative Fermat's function $R : X \to Y$ which satisfies the equation (4.1) and the inequality

$$||f(x) - R(x)||_Y \leq K \sum_{j=0}^{\infty}(K/2^\beta)^j \phi((\sqrt[n]{2})^{-j}x, (\sqrt[n]{2})^{-j}x)\,, \tag{3.8}$$

for all $x \in X$.

Proof. The proof is obtained by starting with the replacement $x = y = \dfrac{x}{\sqrt[n]{2}}$ in (3.7) and following the same arguments as in Theorem 3.1. □

As an immediate consequence of Theorem 3.2 we have the following Hyers-Ulam-Rassias type stability of the functional equation (4.1).

Corollary 3.3. *Let X be a quasi-β normed space with a norm $||\cdot||$ and and choose a constant $p > \left(\dfrac{n}{\beta}\right)\left(\dfrac{\ln K}{\ln 2} - n\right)$. Suppose that $f : X \to Y$ satisfies*

$$||D_n f(x,y)||_Y \leq c(||x||^p + ||y||^p) \tag{3.9}$$

for all $x, y \in X$ with a nonnegative constant c. Then there exists a unique function $R : X \to Y$ such that

$$||f(x) - R(x)||_Y \leq 2cK\left(\frac{2^{\frac{\beta p}{n}+\beta}}{2^{\frac{\beta p}{n}+\beta} - K}\right)||x||^p \tag{3.10}$$

for all $x \in X$.

Proof. Taking $\phi(x,y) = c(||x||^p + ||y||^p)$ in Theorem 3.2 completes the proof. □

Remark 3.4. By the symmetric property of stability of the reciprocal-negative Fermat's equation (4.1) from Theorem 3.1 and 3.2 we note that it is not hard to get the corresponding result to Corollary 3.3 as a consequence of Theorem 3.1, i.e.,

$$||f(x) - R(x)||_Y \leq 2cK\left(\frac{2^{-\frac{\beta p}{n}-\beta}}{2^{-\frac{\beta p}{n}-\beta} - K}\right)||x||^p \tag{3.11}$$

for $p > \left(\dfrac{n}{\beta}\right)\left(\dfrac{-\ln K}{\ln 2} - n\right)$.

Not only the Hyers-Ulam-Raissias stability type as Corollary 3.3, but it is also possible to consider the Ulam-Gavruta-Rasssias stability and Hyers-Ulam-J.M.Rassias stability of the reciprocal-negative Fermat's equation (4.1) with $c_1(||x||^{\frac{p}{2}}||y||^{\frac{p}{2}})$ and $c_2(||x||^{2\alpha} + ||y||^{2\alpha} + ||x||^{\alpha}||y||^{\alpha})$ for the control function ϕ in Theorems 3.1 and 3.2, respectively. Obtaining all results of these stability types is very similar to the arguments as in the proof of Corollary 3.3.

4. Generalized Stability of a Reciprocal-Negative Fermat's Functional Equations: A Fixed Point Theorem of the Alternative Approach

In this section we will investigate the generalized Hyers-Ulam stability of the reciprocal-negative Fermat's functional equation which is introduced earlier in previous sections

$$f\Big(\sqrt[n]{x^n + y^n}\Big) = \frac{f(x)f(y)}{f(x) + f(y)} \tag{4.1}$$

for fixed positive integer n and for all $x, y \in X$ by the approach of the fixed point of the alternative. As we used the notations in the previous sections we assume that X is a linear space and $(Y, ||\cdot||)$ is a quasi-β-Banach space in this section. A set $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

Theorem 4.1. *Suppose that a function $\phi : X \times X \longrightarrow \mathbb{R}_+$ is given and there exists a constant L with $0 < L < 1$ such that*

$$\phi(x, y) \leq 2^{-1}L\phi\Big(x/\sqrt[n]{2}, y/\sqrt[n]{2}\Big) \tag{4.2}$$

for all $x, y \in X$ and

$$\sum_{j=0}^{\infty}(2^{\beta}K)^j\phi\Big((\sqrt[n]{2})^j x, (\sqrt[n]{2})^j y\Big) < \infty. \tag{4.3}$$

Furthermore, let $f : X \longrightarrow Y$ be a mapping such that

$$||D_n f(x, y)|| = \left\|f\Big(\sqrt[n]{x^n + y^n}\Big) - \frac{f(x)f(y)}{f(x) + f(y)}\right\| \leq \phi(x, y) \tag{4.4}$$

for all $x, y \in X$.

Then there exists the unique reciprocal-negative Fermat's function $C : X \longrightarrow Y$ satisfying (4.1) and

$$||f(x) - C(x)|| \leq \frac{2^{\beta}}{1 - L}\phi(x, x) \tag{4.5}$$

for all $x \in X$.

Proof. We consider the set $\Omega = \{g | g : X \longrightarrow Y\}$ and then define a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{\lambda \in [0,\infty] : \|g(x) - h(x)\| \leq \lambda\phi(x,x) \,\text{for all}\, x \in X\}$$

with $\inf \emptyset = \infty$. Then (Ω, d) is a complete generalized metric space; see Lemma 2.1 in [14]. Now we define a mapping $T : \Omega \longrightarrow \Omega$ by

$$T(g)(x) = 2g(\sqrt[n]{2}x) \tag{4.6}$$

for all $x \in X$. We, then, will show that T is strictly contractive on Ω.
Given $g, h \in \Omega$, let $\lambda \in [0,\infty]$ be a constant with $d(g,h) \leq \lambda$. Then we have $\|g(x) - h(x)\| \leq \lambda\phi(x,x)$ for all $x \in X$.
We also have

$$\begin{aligned}&\|T(g)(x) - T(h)(x)\| = \|2g(\sqrt[n]{2}x) - 2h(\sqrt[n]{2}x)\|\\ &\leq 2^\beta\lambda\phi(\sqrt[n]{2}x, \sqrt[n]{2}x) \leq 2^\beta\lambda(1/2)\phi(x,x)\\ &\leq \lambda L\phi(x,x)\end{aligned}$$

for all $x \in X$, which implies

$$d(T(g), T(h)) \leq L\lambda.$$

Therefore we conclude that

$$d(T(g), T(h)) \leq Ld(g,h)$$

for any $g, h \in \Omega$. Since L is a constant with $0 < L < 1$, T is strictly contractive as claimed.
Letting $x = y$ in (4.4) we should have

$$\|\frac{1}{2}f(x) - f(\sqrt[n]{2})\| \leq \phi(x,x) \tag{4.7}$$

or

$$\|f(x) - 2f(\sqrt[n]{2})\| \leq 2^\beta\phi(x,x) \tag{4.8}$$

for all $x \in X$. Hence we just have

$$d(T(f), f) \leq 2^\beta < \infty. \tag{4.9}$$

By the Alternative of Fixed Point as we introduced in Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of T such that $d(T^m(f), C) \to 0$ as $m \to \infty$, that is,

$$C(x) = \lim_{m\to\infty} T^m(f)(x)$$

for all $x \in X$. Then we will show that C is the reciprocal-negative Fermat's function. It would not be hard if we recall the approximation inequality (4.4) for f where we let $x = \sqrt[n]{2}x$ and $y = \sqrt[n]{2}y$, respectably, as follows:

$$\begin{aligned}
&\left\| C(\sqrt[n]{x^n + y^n}) - \frac{C(x)C(y)}{C(x)+C(y)} \right\| \\
&= \lim_{m\to\infty} 2^{m\beta} \left\| f(\sqrt[n]{2}\sqrt[n]{x^n+y^n}) - \frac{f(\sqrt[n]{x})f(\sqrt[n]{y})}{f(\sqrt[n]{x})+f(\sqrt[n]{y})} \right\| \\
&\leq \lim_{m\to\infty} (2^\beta K)^m \phi(\sqrt[n]{x}, \sqrt[n]{y}) = 0
\end{aligned}$$

for all $x, y \in X$, which implies that C is a reciprocal-negative Fermat's function. By the Alternative of Fixed Point theorem and the inequality (4.9) we get

$$d(f, C) \leq \left(\frac{1}{1-L}\right) d(f, T(f)) \leq \frac{2^\beta}{1-L}.$$

Hence the inequality (4.10) is true for all $x \in X$.
By the uniqueness of the fixed point of T, the function C should be unique, which completes the proof. □

Let us give the classical Cauchy difference type stability of the reciprocal-negative Fermat's equation (4.1) from Theorem 4.1. For the following result we assume that X is a normed vector space with $||\cdot||$ and that as we did $(Y, ||\cdot||)$ is a quasi-β-Banach space with $K = 1$.

Corollary 4.2. *Let $\epsilon \geq 0$ and p be a real number with $p < -n$ for a fixed positive integer n. Suppose $f : X \longrightarrow Y$ is a function and it satisfies*

$$||D_n f(x,y)|| = \left\| f\left(\sqrt[n]{x^n + y^n}\right) - \frac{f(x)f(y)}{f(x)+f(y)} \right\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then there exists the unique reciprocal-negative Fermat's function $C : X \longrightarrow Y$ satisfying (4.1) and

$$||f(x) - C(x)|| \leq \epsilon \left(\frac{2^{\beta+1}}{1-L}\right) ||x||^p \tag{4.10}$$

for all $x \in X$.

Proof. This proof follows from Theorem 4.1 by taking $\phi(x, y) = \epsilon(||x||^p + ||y||^p)$ for all $x, y \in X$ with $L = 2^{1+p/n}$. □

Remark 4.3. As an application of the reciprocal-negative Fermat's equation (4.1) we consider a parallel circuit having two resistors. It is well-know from physics that the inverse of total resistance r of the circuit is sum of the inverses of the individual resistances r_1 and r_2,

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

or

$$r = \frac{r_1 r_2}{r_1 + r_2}$$

If we take $r_1 = \dfrac{1}{x^n}$ and $r_2 = \dfrac{1}{y^n}$ then we have

$$r = \frac{\frac{1}{x^n}\frac{1}{y^n}}{\frac{1}{x^n} + \frac{1}{y^n}}. \tag{4.11}$$

Since the electric conductance is reciprocal to the resistance we have the total conductance g of the circuit should be $g = x^n + y^n$. From the equation (4.11) we conclude that

$$\frac{1}{g} = \frac{\frac{1}{x^n}\frac{1}{y^n}}{\frac{1}{x^n} + \frac{1}{y^n}},$$

that is,

$$\frac{1}{x^n + y^n} = \frac{\frac{1}{x^n}\frac{1}{y^n}}{\frac{1}{x^n} + \frac{1}{y^n}},$$

which is the reciprocal-negative Fermat's equation (4.1) when, in particular, $f(x) = \dfrac{c}{x^n}$ for some constant c.

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