

ESTIMATES FOR A CERTAIN SUBCLASS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. In this paper, a version of the boundary Schwarz Lemma for the holomorphic function belonging to $\mathcal{N}(\alpha)$ is investigated. For the function $f(z) = z + c_2z^2 + c_3z^3 + \dots$ which is defined in the unit disc where $f(z) \in \mathcal{N}(\alpha)$, we estimate the modulus of the angular derivative of the function $f(z)$ at the boundary point b with $f(b) = \frac{1}{b} \int_0^b f(t)dt$. The sharpness of these inequalities is also proved.

1. INTRODUCTION

The most classical version of the Schwarz Lemma examines the behavior of a bounded, holomorphic function mapping the origin to origin in the unit disc $D = \{z : |z| < 1\}$. It is actively involved in the theory of the geometric functions, the fixed point theory of the holomorphic mappings, hypergeometric theory and several areas of the analysis. It has also allowed the development of these areas. Moreover, it is possible to see its effectiveness in the proofs of many important theorems. One of them is Liouville Theorem saying that a bounded, entire function is constant. The Schwarz Lemma which has quite wide application area and is the direct application of the maximum modulus principle is given in the most basic form as follows ([5], p.329):

Let D be the unit disc in the complex plane \mathbb{C} . Let $f : D \rightarrow D$ be a holomorphic function with $f(0) = 0$. Under these conditions, $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation, which means $f(z) = ze^{i\theta}$, where θ is real.

In order to show our main results, we need the following lemma called Jack's Lemma [6].

Received by the editors October 16, 2017. Accepted September 09, 2018.

2010 *Mathematics Subject Classification*. 30C80, 32A10.

Key words and phrases. holomorphic function, Jack's lemma, angular derivative.

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Lemma 1.1 (Jack's Lemma). *Let $f(z)$ be a non-constant and holomorphic function in the unit disc D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 , then*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k,$$

where $k \geq 1$ is a real number.

Let \mathcal{A} denote the class of functions $f(z)$ which are holomorphic in the unit disc D such that

$$f(z) = z + c_2 z^2 + c_3 z^3 + \dots$$

Also, let $\mathcal{N}(\alpha)$ be the subclass of \mathcal{A} consisting of all functions $f(z)$ which satisfy

$$|\arg f'(z)| < \frac{\pi}{2} \left(\frac{5}{3} - \alpha_0 \right) \quad (z \in D),$$

where α_0 is the smallest positive root of the equation

$$\Pi(\alpha) = \frac{2}{\pi} \arctan \alpha + \frac{2}{\pi} \arctan \left(\frac{1}{2} \left(\alpha + \frac{2}{\pi} \arctan \alpha \right) \right) + 2\alpha - \frac{5}{6} = 0$$

and $0.266 < \alpha_0 < 0.267$. By applying the Maple program to the function $\Pi(\alpha)$, we approximately obtain $\alpha_0 = 0.2661770670432253\dots$ which is the root of this function.

Let $f(z) \in \mathcal{N}(\alpha)$ and consider the function

$$\phi(z) = \frac{\frac{zh'(z)}{h(z)} - 1}{\frac{zh'(z)}{h(z)} + 1},$$

where $h(z) = \frac{2}{z} \int_0^z f(t) dt$.

Clearly, $\phi(z)$ is holomorphic function in D and $\phi(0) = 0$.

Now let us show that the function $|\phi(z)|$ is less than 1 in the unit disc D . From the definition of $h(z)$, we take

$$2f(z) = h(z) + zh'(z)$$

and

$$2f'(z) = 2h'(z) + zh''(z).$$

Therefore, we obtain

$$2f'(z) = \frac{h(z)}{z} \left[\left(\frac{1 + \phi(z)}{1 - \phi(z)} \right)^2 + \frac{2z\phi'(z)}{(1 - \phi(z))^2} + \frac{1 + \phi(z)}{1 - \phi(z)} \right].$$

We suppose that there exists a point $z_0 \in D$ such that $\max_{|z| \leq |z_0|} |\phi(z)| = |\phi(z_0)| = 1$.

Thus, $\phi(z_0) = e^{i\vartheta}$ ($0 \leq \vartheta < 2\pi$).

From the Jack's lemma, we obtain

$$\frac{z_0 \phi'(z_0)}{\phi(z_0)} = k.$$

Using the last equality, we take by the elementary calculations

$$\begin{aligned} 2f'(z_0) &= \frac{h(z_0)}{z_0} \left[\left(\frac{1 + \phi(z_0)}{1 - \phi(z_0)} \right)^2 + \frac{2z\phi'(z_0)}{(1 - \phi(z_0))^2} + \frac{1 + \phi(z_0)}{1 - \phi(z_0)} \right] \\ &= \frac{h(z_0)}{z_0} \left[\left(\frac{1 + e^{i\vartheta}}{1 - e^{i\vartheta}} \right)^2 + \frac{2ke^{i\vartheta}}{(1 - e^{i\vartheta})^2} + \frac{1 + e^{i\vartheta}}{1 - e^{i\vartheta}} \right]. \end{aligned}$$

Since

$$\begin{aligned} \left(\frac{1 + e^{i\vartheta}}{1 - e^{i\vartheta}} \right)^2 + \frac{2ke^{i\vartheta}}{(1 - e^{i\vartheta})^2} + \frac{1 + e^{i\vartheta}}{1 - e^{i\vartheta}} &= \left(\frac{1 + \cos \vartheta + i \sin \vartheta}{1 - \cos \vartheta - i \sin \vartheta} \right)^2 \\ &\quad + \frac{2k(\cos \vartheta + i \sin \vartheta)}{(1 - \cos \vartheta - i \sin \vartheta)^2} + \frac{1 + \cos \vartheta + i \sin \vartheta}{1 - \cos \vartheta - i \sin \vartheta} \\ &= -\frac{(\sin \vartheta)^2}{(1 - \cos \vartheta)^2} - \frac{k}{1 - \cos \vartheta} + i \frac{\sin \vartheta}{1 - \cos \vartheta}, \end{aligned}$$

we obtain

$$\begin{aligned} \arg f'(z_0) &= \arg \left(\frac{h(z_0)}{z_0} \left[-\frac{(\sin \vartheta)^2}{(1 - \cos \vartheta)^2} - \frac{k}{1 - \cos \vartheta} + i \frac{\sin \vartheta}{1 - \cos \vartheta} \right] \right) \\ &= \arg \frac{h(z_0)}{z_0} + \arg \left((-1) \left(\frac{(\sin \vartheta)^2}{(1 - \cos \vartheta)^2} + \frac{k}{1 - \cos \vartheta} - i \frac{\sin \vartheta}{1 - \cos \vartheta} \right) \right) \\ &= \arg \frac{h(z_0)}{z_0} + \arg(-1) + \arg \left(\frac{(\sin \vartheta)^2}{(1 - \cos \vartheta)^2} + \frac{k}{1 - \cos \vartheta} - i \frac{\sin \vartheta}{1 - \cos \vartheta} \right) \\ &= \pi + \arg \left(\frac{(\sin \vartheta)^2 + k(1 - \cos \vartheta)}{(1 - \cos \vartheta)^2} - i \frac{\sin \vartheta(1 - \cos \vartheta)}{(1 - \cos \vartheta)^2} \right) + \arg \frac{h(z_0)}{z_0} \end{aligned}$$

and also since $\left| \arg \frac{h(z_0)}{z_0} \right| < \frac{\pi}{2} \alpha_0$ (see, [23]),

$$\begin{aligned} \arg f'(z_0) &\geq \pi + \arg \left(\frac{(\sin \vartheta)^2 + (1 - \cos \vartheta)}{(1 - \cos \vartheta)^2} - i \frac{\sin \vartheta(1 - \cos \vartheta)}{(1 - \cos \vartheta)^2} \right) - \frac{\pi}{2} \alpha_0 \\ &= \pi + \arg \left(\frac{1 - (\cos \vartheta)^2 + (1 - \cos \vartheta) - i \sin \vartheta(1 - \cos \vartheta)}{(1 - \cos \vartheta)^2} \right) - \frac{\pi}{2} \alpha_0 \end{aligned}$$

$$\begin{aligned}
&= \pi + \arg\left(\frac{2 + \cos \vartheta - i \sin \vartheta}{1 - \cos \vartheta}\right) - \frac{\pi}{2}\alpha_0 \\
&= \pi - \arctan \frac{|\sin \vartheta|}{2 + \cos \vartheta} - \frac{\pi}{2}\alpha_0 \\
&\geq \pi - \arctan \frac{\sqrt{3}}{3} - \frac{\pi}{2}\alpha_0 \\
&= \frac{5\pi}{6} - \frac{\pi}{2}\alpha_0 = \frac{\pi}{2}\left(\frac{5}{3} - \alpha_0\right).
\end{aligned}$$

But, this contradicts the condition $f(z) \in \mathcal{N}(\alpha)$. This means that there is no point $z_0 \in D$ such that $|\phi(z_0)| = 1$. Therefore, $|\phi(z)| < 1$ for $|z| < 1$. By the Schwarz Lemma, we obtain

$$(1.1) \quad |c_2| \leq 3.$$

The inequality (1.1) is sharp with equality for the function

$$f(z) = \frac{-z}{(z-1)^3}.$$

In this case, the following lemma is obtained.

Lemma 1.2. *If $f(z) \in \mathcal{N}(\alpha)$, then*

$$(1.2) \quad |c_2| \leq 3.$$

The result is sharp and the extremal function is

$$f(z) = \frac{-z}{(z-1)^3}.$$

Nunokowa and Oto (see, [23]) showed that $h(z)$ is a starlike function for the class $\mathcal{N}(\alpha)$ given above. In our study, the angular derivative of the function $f(z)$ in the class $\mathcal{N}(\alpha)$ is investigated at the boundary of the unit disc.

The boundary version of the Schwarz Lemma is basically given in the following way:

Let $f(z)$ be a holomorphic function in the unit disc D , $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. Assume that for some point $b \in \partial D$, f extends continuously to b , $|f(b)| = 1$ and $f'(b)$ exists. Therefore, the inequality $|f'(b)| \geq 1$, which is known as the boundary Schwarz Lemma as different from Schwarz Lemma, is obtained. The equality in $|f'(b)| \geq 1$ holds if and only if $f(z) = ze^{i\theta}$, θ real. This result of the Schwarz Lemma and its generalization are described as the Boundary Schwarz Lemma in the literature. This improvement was obtained by Helmut Unkelbach in [22] and rediscovered by R. Osserman in [16] 60 years later.

In the last 15 years, there have been tremendous studies on the boundary Schwarz Lemma (see, [1], [3], [4], [7], [8], [10], [11], [16], [17], [18], [20] and references therein). Some of them are about the estimates from below for the modulus of the derivative of the function at the boundary points which satisfy the condition $|f(b)| = 1$.

In [16], R. Osserman offered the following boundary refinement of the classical Schwarz Lemma. It is very much in the spirit of the sort of result.

Lemma 1.3. *Let $f : D \rightarrow D$ be holomorphic function with $f(0) = 0$. Assume that for some point $b \in \partial D$, f extends continuously to b , $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$(1.3) \quad |f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

Inequality (1.3) is sharp, with equality possible for each value of $|f'(0)|$.

Corollary 1.4. *Under the hypotheses of Lemma 3,*

$$(1.4) \quad |f'(b)| \geq 1$$

and

$$|f'(b)| > 1 \text{ unless } f(z) = ze^{i\theta}, \text{ where } \theta \text{ is real.}$$

Let us give the definitions needed for our results. A *Stolz angle* Δ at $b \in \partial D$ is the interior of any triangle in D symmetric to $[0, b]$ whose closure lies in D except for the vertex b . Basic for this paper is the notions of the angular limit and the angular derivative. Let $b \in \partial D$. We say that the angular limit $f(b)$ exists if

$$f(b) = \lim_{z \rightarrow b, z \in \Delta} f(z)$$

for every *Stolz angle* Δ at b and we say that the angular derivative $f'(b)$ exists if the angular limit $f(b)$ exists and

$$f'(b) = \lim_{z \rightarrow b, z \in \Delta} \frac{f(z) - f(b)}{z - b}$$

for every *Stolz angle* Δ at b .

The following lemma, which is known as the Julia-Wolff lemma, is needed in the sequel (see [19]).

Lemma 1.5 (Julia-Wolff lemma). *Let f be a holomorphic function in D , $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

D. M. Burns and S. G. Krantz [9] and D. Chelst [2] studied the uniqueness part of the Schwarz Lemma. For more general results and related estimates, see also ([12], [13], [14] and [15]).

X. Tang, T. Liu and J. Lu [11] established a new type of the classical boundary Schwarz Lemma for the holomorphic self-mappings of the unit polydisk D^n in \mathbb{C}^n . They extended the classical boundary Schwarz Lemma to high dimensions.

T. Liu, J. Wang and X. Tang [21] also established a new type of the classical boundary Schwarz Lemma for holomorphic self-mappings of the unit ball in \mathbb{C}^n . They, then, applied their new Schwarz Lemma to the problems studied on geometric function theory in several complex variables.

Also, M. Jeong [7] got some inequalities at a boundary point for a different form of holomorphic functions and showed the sharpness of these inequalities. Also, M. Jeong found a necessary and sufficient condition for a holomorphic map to have fixed points only on the boundary of the unit disc and compared its derivatives at fixed points to get some relations among them [8]. For historical background about the Schwarz Lemma and its applications on the boundary of the unit disc, we refer to (see [1], [20]).

2. MAIN RESULTS

In this section, for holomorphic function $f(z)$ belonging to the class of $\mathcal{N}(\alpha)$, it is estimated from below the modulus of the angular derivative of the function on the boundary point of the unit disc. It is proved that these result are sharp. Also, we derive improvements of Lemma 3 and Corollary 1 as the special cases of our main result.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\alpha)$. Suppose that for some $b \in \partial D$, f has an angular limit $f(b)$ at b , $f(b) = \frac{1}{b} \int_0^b f(t) dt$. Then*

$$(2.1) \quad |f'(b)| \geq \frac{1}{2} |f(b)|.$$

The inequality (2.1) is sharp with extremal function

$$f(z) = \frac{-z}{(z-1)^3}.$$

Proof. Let us consider the following function

$$\phi(z) = \frac{\frac{zh'(z)}{h(z)} - 1}{\frac{zh'(z)}{h(z)} + 1},$$

where $h(z) = \frac{2}{z} \int_0^z f(t)dt$.

Then $\phi(z)$ is holomorphic function in the unit disc D and $\phi(0) = 0$. By the Jack's lemma and since $f(z) \in \mathcal{N}(\alpha)$, we take $|\phi(z)| < 1$ for $|z| < 1$. Also, we have $|\phi(b)| = 1$ for $b \in \partial D$. It is clear that

$$\phi'(z) = \frac{2zh''(z)h(z) + 2h(z)h'(z) - 2zh'^2(z)}{(2h'(z) + h(z))^2}$$

and

$$\phi'(b) = \frac{2bh''(b)h(b) + 2h'(b)(h(b) - bh'(b))}{(2h'(b) + h(b))^2} = 2 \frac{f'(b)}{f(b)}$$

since

$$h(b) = \frac{2}{b} \int_0^b f(t)dt, \quad h'(b) = 0, \quad h''(b) = \frac{2}{b} f'(b).$$

Therefore, from (1.4), we get

$$1 \leq |\phi'(b)| = 2 \left| \frac{f'(b)}{f(b)} \right|$$

and

$$|f'(b)| \geq \frac{1}{2} |f(b)|.$$

Now we shall show that the inequality (2.1) is sharp. Let

$$f(z) = \frac{-z}{(z-1)^3}.$$

Then

$$f'(z) = \frac{2z+1}{(z-1)^4},$$

$$|f'(-1)| = \frac{1}{16}.$$

Since $\frac{1}{2} |f(-1)| = \frac{1}{16}$, (2.1) is satisfied with equality. Namely,

$$\frac{1}{2} |f(-1)| = \frac{1}{2} \left(- \int_0^{-1} f(t)dt \right) = \frac{1}{2} \left(\int_0^{-1} \frac{t}{(t-1)^3} dt \right) = \frac{1}{16}.$$

□

Theorem 2.2. Let $f(z) \in \mathcal{N}(\alpha)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b , $f(b) = \frac{1}{b} \int_0^b f(t) dt$. Then

$$(2.2) \quad |f'(b)| \geq \frac{3}{3 + |c_2|} |f(b)|.$$

The inequality (2.2) is sharp with extremal function

$$f(z) = \frac{-z}{(z-1)^3}.$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.3),

$$\frac{2}{1 + |\phi'(0)|} \leq |\phi'(b)| = 2 \left| \frac{f'(b)}{f(b)} \right|.$$

Since

$$\begin{aligned} \phi(z) &= \frac{z \left(-\frac{2}{z^2} \int_0^z f(t) dt + \frac{2}{z} f(z) \right) - \frac{2}{z} \int_0^z f(t) dt}{z \left(-\frac{2}{z^2} \int_0^z f(t) dt + \frac{2}{z} f(z) \right) + \frac{2}{z} \int_0^z f(t) dt} \\ &= \frac{-\frac{4}{z} \int_0^z f(t) dt + 2f(z)}{2f(z)} \\ &= \frac{-\frac{2}{z} \left(\frac{z^2}{2} + c_2 \frac{z^3}{3} + c_3 \frac{z^4}{4} \dots \right) + z + c_2 z^2 + c_3 z^3 + \dots}{z + c_2 z^2 + c_3 z^3 + \dots} \\ &= \frac{\frac{1}{3} c_2 z^2 + \frac{1}{2} c_3 z^3 + \dots}{z + c_2 z^2 + c_3 z^3 + \dots} \\ &= \frac{\frac{1}{3} c_2 z + \frac{1}{2} c_3 z^2 + \dots}{1 + c_2 z + c_3 z^2 + \dots}, \end{aligned}$$

it is clear that

$$|\phi'(0)| = \frac{1}{3} |c_2|.$$

Then

$$\frac{6}{3 + |c_2|} \leq 2 \left| \frac{f'(b)}{f(b)} \right|.$$

The last inequality shows that the inequality intended is obtained.

Now we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{-z}{(z-1)^3}.$$

Then

$$f'(z) = \frac{2z + 1}{(z - 1)^4},$$

$$|f'(-1)| = \frac{1}{16}.$$

Since $|c_2| = 3$, (2.2) is satisfied with equality. Namely,

$$f(z) = \frac{-z}{(z - 1)^3},$$

$$z + c_2 z^2 + c_3 z^3 + \dots = \frac{-z}{(z - 1)^3},$$

$$\begin{aligned} c_2 z + c_3 z^2 + \dots &= -\frac{1}{(z - 1)^3} - 1 = -\frac{1 + (z - 1)^3}{(z - 1)^3} \\ &= \frac{z^3 - 3z^2 + 3z}{(z - 1)^3} = z \left(\frac{z^2 - 3z + 3}{(z - 1)^3} \right) \end{aligned}$$

and

$$|c_2| = 3.$$

Hence,

$$\frac{3}{3 + |c_2|} |f(-1)| = \frac{3}{3 + 3} \frac{1}{18} = \frac{1}{16}.$$

□

The inequality (2.2) can be strengthened as below by taking into account c_2 which is second coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let $f(z) \in \mathcal{N}(\alpha)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b , $f(b) = \frac{1}{b} \int_0^b f(t) dt$. Then*

$$(2.3) \quad |f'(b)| \geq \frac{|f(b)|}{2} \left(1 + \frac{4(3 - |c_2|)^2}{2 + 3|3c_3 - 2c_2^2|} \right).$$

The equality in (2.3) occurs for the function

$$f(z) = \frac{-z}{(z - 1)^3}.$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$\psi(z) = \frac{\phi(z)}{B(z)},$$

where $B(z) = z$. The function $\psi(z)$ is holomorphic in D . According to the maximum modulus principle, we have $|\psi(z)| < 1$ for each $z \in D$. In particular,

$$(2.4) \quad |\psi(0)| = \frac{1}{3} |c_2| \leq 1$$

and

$$|\psi'(0)| = \frac{1}{2} \left| c_3 - \frac{2}{3} c_2^2 \right|.$$

Furthermore, it can be seen that

$$\frac{b\phi'(b)}{\phi(b)} = |\phi'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

Consider the function

$$F(z) = \frac{\psi(z) - \psi(0)}{1 - \overline{\psi(0)}\psi(z)}.$$

This function is holomorphic in D , $|F(z)| \leq 1$ for $|z| < 1$, $F(0) = 0$, and $|F(b)| = 1$ for $b \in \partial D$. From (1.3),

$$\begin{aligned} \frac{2}{1 + |F'(0)|} &\leq |F'(b)| = \frac{1 - |\psi(0)|^2}{|1 - \overline{\psi(0)}\psi(b)|^2} |\psi'(b)| \\ &\leq \frac{(1 - |\psi(0)|)(1 + |\psi(0)|)}{(1 - |\overline{\psi(0)}\psi(b)|)^2} |\psi'(b)| \\ &= \frac{(1 - |\psi(0)|)(1 + |\psi(0)|)}{(1 - |\psi(0)|)(1 - |\psi(0)|)} |\psi'(b)| \\ &= \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \left| \frac{\phi'(b)}{b} - \frac{\phi(b)}{b^2} \right| \\ &= \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \left| \frac{\phi'(b)}{B(b)} - \frac{\phi(b)B'(b)}{B(b)^2} \right| \\ &= \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \left| \frac{\phi(b)}{bB(b)} \right| \left| \frac{b\phi'(b)}{\phi(b)} - \frac{bB'(b)}{B(b)} \right| \\ &= \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \{ |\phi'(b)| - |B'(b)| \}. \end{aligned}$$

Since

$$F'(z) = \frac{1 - |\psi(0)|^2}{(1 - \overline{\psi(0)}\psi(z))^2} \psi'(z),$$

$$F'(0) = \frac{\psi'(0)}{1 - |\psi(0)|^2},$$

and

$$|F'(0)| = \frac{3}{2} \frac{|3c_3 - 2c_2^2|}{9 - |c_2|^2},$$

we obtain

$$\begin{aligned} \frac{2}{1 + \frac{3}{2} \frac{|3c_3 - 2c_2^2|}{9 - |c_2|^2}} &\leq \frac{1 + \frac{1}{3}|c_2|}{1 - \frac{1}{3}|c_2|} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\}, \\ \frac{4(9 - |c_2|^2)}{2 + 3|3c_3 - 2c_2^2|} &\leq 2 \frac{3 + |c_2|}{3 - |c_2|} \left| \frac{f'(b)}{f(b)} \right| - \frac{3 + |c_2|}{3 - |c_2|}, \\ \left| \frac{f'(b)}{f(b)} \right| &\geq \frac{1}{2} \frac{3 - |c_2|}{3 + |c_2|} \left(\frac{4(9 - |c_2|^2)}{2 + 3|3c_3 - 2c_2^2|} + \frac{3 + |c_2|}{3 - |c_2|} \right) \end{aligned}$$

and

$$|f'(b)| \geq \frac{f(b)}{2} \left(1 + \frac{4(3 - |c_2|)^2}{2 + 3|3c_3 - 2c_2^2|} \right).$$

The last inequality is intended inequality.

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{-z}{(z-1)^3}.$$

Then

$$|f'(-1)| = \frac{1}{16}.$$

Since $|c_2| = 3$, (2.3) is satisfied with equality. \square

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z) \in \mathcal{N}(\alpha)$ ($0 < \alpha \leq 1$), $f(z) - z$ has no zeros in D except $z = 0$ and $c_2 > 0$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b , $f(b) = \frac{1}{b} \int_0^b f(t) dt$. Then*

$$(2.5) \quad |f'(b)| \geq \frac{|f(b)|}{2} \left(1 - \frac{4c_2 (\ln \frac{1}{3} c_2)^2}{4c_2 \ln(\frac{1}{3} c_2) - 3|c_3 - \frac{2}{3} c_2^2|} \right)$$

The equality in (2.5) occurs for the function

$$f(z) = \frac{-z}{(z-1)^3}.$$

Proof. Let $c_2 > 0$. Let us consider the function $\psi(z)$ as in Theorem 2.3. Taking into account of the equality (2.4), we denote by $\ln \psi(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln \psi(0) = \ln \left(\frac{1}{3} c_2 \right) < 0.$$

Take the following auxiliary function

$$\Phi(z) = \frac{\ln \psi(z) - \ln \psi(0)}{\ln \psi(z) + \ln \psi(0)}.$$

It is obvious that $\Phi(z)$ is a holomorphic function in D , $\Phi(0) = 0$, $|\Phi(z)| \leq 1$ for $|z| < 1$, and also $|\psi(b)| = 1$ for $b \in \partial D$. So, we can apply (1.3) to the function $\Phi(z)$. Since

$$\Phi'(z) = 2 \ln \psi(0) \frac{\psi'(z)}{\psi(z) (\ln \psi(z) + \ln \psi(0))^2}$$

and

$$\Phi'(b) = 2 \ln \psi(0) \frac{\psi'(b)}{\psi(b) (\ln \psi(b) + \ln \psi(0))^2},$$

we obtain

$$\begin{aligned} \frac{2}{1 + |\Phi'(0)|} &\leq |\Phi'(b)| = \frac{2 |\ln \psi(0)|}{|\ln \psi(b) + \ln \psi(0)|^2} \left| \frac{\psi'(b)}{\psi(b)} \right|, \\ &= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \left| \frac{\phi'(b)}{B(b)} - \frac{\phi(b)B'(b)}{B(b)^2} \right| \\ &= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \left| \frac{\phi(b)}{b^2} \right| \left| \frac{b\phi'(b)}{\phi(b)} - \frac{bB'(b)}{B(b)} \right| \\ &= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \{ |\phi'(b)| - |B'(b)| \} \\ &\leq \frac{-2 \ln \psi(0)}{\ln^2 \psi(0)} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\} \\ &= \frac{-2}{\ln \psi(0)} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\}. \end{aligned}$$

Since

$$\begin{aligned} \Phi'(0) &= \frac{\psi'(0)}{2\psi(0) \ln \psi(0)}, \\ |\Phi'(0)| &= \frac{3 |c_3 - \frac{2}{3}c_2^2|}{4c_2 |\ln \frac{1}{3}c_2|} \end{aligned}$$

and thus, we get

$$\frac{2}{1 + \frac{3|c_3 - \frac{2}{3}c_2^2|}{4c_2 |\ln \frac{1}{3}c_2|}} \leq \frac{-2}{\ln \frac{1}{3}c_2} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\}.$$

By getting elementary arrangements, we obtain

$$\frac{4c_2 \left| \ln \frac{1}{3}c_2 \right|}{4c_2 \left| \ln \frac{1}{3}c_2 \right| + 3 \left| c_3 - \frac{2}{3}c_2^2 \right|} \leq \frac{-1}{\ln \frac{1}{3}c_2} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\},$$

$$- \frac{4c_2 \left(\ln \frac{1}{3}c_2 \right)^2}{4c_2 \left| \ln \frac{1}{3}c_2 \right| + 3 \left| c_3 - \frac{2}{3}c_2^2 \right|} \leq \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\}$$

and

$$\left| f'(b) \right| \geq \frac{\left| f(b) \right|}{2} \left(1 - \frac{4c_2 \left(\ln \frac{1}{3}c_2 \right)^2}{4c_2 \ln \frac{1}{3}c_2 - 3 \left| c_3 - \frac{2}{3}c_2^2 \right|} \right).$$

Since $|c_2| = 3$, (2.5) is satisfied with equality. \square

We note that the inequality (1.3) has been used in the proofs of Theorem 2.3 and Theorem 2.4. So, there are both c_2 and c_3 in the right side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker but more simpler inequality (not including c_3). It is formulated in the following theorem.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, then we have the inequality*

$$(2.6) \quad \left| f'(b) \right| \geq \frac{\left| f(b) \right|}{2} \left(1 - \frac{1}{2} \ln \left(\frac{1}{3}c_2 \right) \right).$$

In addition, the result is sharp and the extremal function is

$$f(z) = \frac{-z}{(z-1)^3}.$$

Proof. Applying the inequality (1.4) to $\Phi(z)$, we get

$$1 \leq \left| \Phi'(b) \right| \leq \frac{-2}{\ln \psi(0)} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\},$$

$$1 \leq \frac{-2}{\ln \frac{1}{3}c_2} \left\{ 2 \left| \frac{f'(b)}{f(b)} \right| - 1 \right\},$$

$$\left| f'(b) \right| \geq \frac{1}{2} \left| f(b) \right| \left(1 - \frac{1}{2} \ln \left(\frac{1}{3}c_2 \right) \right).$$

Thus, the inequality (2.6) is obtained with an obvious equality case. \square

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