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# $(L, *, \odot)$-QUASIUNIFORM CONVERGENCE SPACES INDUCED BY OPERATORS

Jung Mi Ko [a] and Yong Chan Kim [b,*]

Abstract. In this paper, we introduce quasiuniform convergence structure induced by operators on ecl-premonoid $(L, *, \odot)$. Moreover, we obtain $(L, *, \odot)$-quasiuniform convergence structure induced by two $(L, *, \odot)$-quasiuniform convergence structures and gives their examples.

## 1. Introduction

Gäher [2,3] introduced the notions of fuzzy filters in a frame $L$. Höhle and Sostak [4] introduced the concept of $L$-filters for a complete quasimonoidal lattice $L$. For the case that the lattice is a stsc quantale, $L$-filters were introduced in [12]. Jäger [5-6] developed stratified $L$-convergence structures based on the concepts of $L$-filters where $L$ is a complete Heyting algebra. Yao [15] extended stratified $L$-convergence structures to complete residuated lattices and investigated between stratified $L$-convergence structures and $L$-fuzzy topological spaces. As an extension of Yao [15], Fang [7-11] introduced $L$-ordered convergence structures and (pre, quasi,semi) uniform convergence spaces on $L$-filters and investigated their relations. Ko and Kim [13] introduced the $(L, *, \odot)$-quasiuniform convergence spaces as an extension of Fang's uniform convergence spaces on ecl-premonoid in Orpen's sense [14].

In this paper, we introduce quasiuniform convergence structure induced by operators on ecl-premonoid $(L, *, \odot)$ and gives their examples. Moreover, we obtain $(L, *, \odot)$-quasiuniform convergence structure induced by two $(L, *, \odot)$-quasiuniform convergence structures.

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*Corresponding author.

## 2. PRELIMINARIES

**Definition 2.1** ([14])**.** A complete lattice $(L, \leq, \perp, \top)$ is called a GL-monoid $(L, \leq, *, \perp, \top)$ with a binary operation $* : L \times L \to L$ satisfying the following conditions:

(G1) $a * \top = a$, for all $a \in L$,

(G2) $a * b = b * a$, for all $a, b \in L$,

(G3) $a * (b * c) = (a * b) * c$, for all $a, b \in L$,

(G4) if $a \leq b$, there exists $c \in L$ such that $b * c = a$,

(G5) $a * \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a * b_i)$.

We can define an implication operator:

$$a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}.$$

**Remark 2.2** ([1, 4, 14])**.** (1) A continuous t-norm $([0,1], \leq, *)$ is a GL-monoid.

(2) A frame $(L, \leq, \wedge)$ is a GL-monoid.

**Definition 2.3** ([1, 4, 14])**.** A complete lattice $(L, \leq, \perp, \top)$ is called a *cl-premonoid* $(L, \leq, \odot)$ with a binary operation $\odot : L \times L \to L$ satisfying the following conditions:

(CL1) $a \leq a \odot \top$ and $a \leq \top \odot a$, for all $a \in L$,

(CL2) if $a \leq b$ and $c \leq d$, then $a \odot c \leq b \odot d$,

(CL3) $a \odot \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a \odot b_i)$ and $\bigvee_{j\in\Gamma} a_j \odot b = \bigvee_{j\in\Gamma}(a_j \odot b)$.

We can define an implication operator:

$$a \to b = \bigvee\{c \mid a \odot c \leq b\}.$$

**Definition 2.4** ([1, 4, 14])**.** A complete lattice $(L, \leq, \perp, \top)$ is called an *ecl-premonoid* $(L, \leq, \odot, *)$ with a GL-monoid $(L, \leq, *)$ and a cl-premonoid $(L, \leq, \odot)$ which satisfy the following condition:

(D) $(a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)$, for all $a, b, c, d \in L$.

An ecl-premonoid $(L, \leq, \odot, *)$ is called an M-*ecl-premonoid* if it satisfies the following condition:

(M) $a \leq a \odot a$ for all $a \in L$.

In this paper, we always assume that $(L, \leq, \odot, *)$ is an ecl-premonoid unless otherwise specified.

**Lemma 2.5** ([1, 4, 13])**.** *Let $(L, \leq, \odot, *)$ be an ecl-premonoid. For each $a, b, c, d, a_i, b_i \in L$ and for $\uparrow \in \{\to, \Rightarrow\}$, we have the following properties.*

(1) *If $b \leq c$, then $a \odot b \leq a \odot c$ and $a * b \leq a * c$.*

(2) $a \odot b \leq c$ *iff* $a \leq b \to c$. *Moreover,* $a * b \leq c$ *iff* $a \leq b \Rightarrow c$.
(3) *If* $b \leq c$, *then* $a \uparrow b \leq a \uparrow c$ *and* $c \uparrow a \leq b \uparrow a$.
(4) $a \leq b$ *iff* $a \Rightarrow b = \top$.
(5) $a * b \leq a \odot b$, $a \to b \leq a \Rightarrow b$ *and* $a * (b \odot c) \leq (a * b) \odot c$.
(6) $(a \uparrow b) \odot (c \uparrow d) \leq (a \odot c) \uparrow (b \odot d)$.
(7) $(b \uparrow c) \leq (a \odot b) \uparrow (a \odot c)$.
(8) $(b \uparrow c) \leq (a \uparrow b) \uparrow (a \uparrow c)$ *and* $(b \uparrow a) \leq (a \uparrow c) \uparrow (b \uparrow c)$.
(9) $(b \to c) \leq (a \uparrow b) \to (a \uparrow c)$ *and* $(b \uparrow a) \leq (a \to c) \to (b \uparrow c)$
(10) $a_i \uparrow b_i \leq (\bigwedge_{i\in\Gamma} a_i) \uparrow (\bigwedge_{i\in\Gamma} b_i)$.
(11) $a_i \uparrow b_i \leq (\bigvee_{i\in\Gamma} a_i) \uparrow (\bigvee_{i\in\Gamma} b_i)$.
(12) $(c \uparrow a) * (b \to d) \leq (a \to b) \to (c \uparrow d)$.

**Definition 2.6** ([4, 13])**.** For $L^X = \{f \mid f : X \to L \text{ is a function}\}$, a mapping $\mathcal{F} : L^X \to L$ is called an $(L,*)$-*filter* on $X$ if it satisfies the following conditions:
(F1) $\mathcal{F}(\perp_X) = \perp$ and $\mathcal{F}(\top_X) = \top$, where $\perp_X(x) = \perp, \top_X(x) = \top$ for $x \in X$.
(F2) $\mathcal{F}(f * g) \geq \mathcal{F}(f) * \mathcal{F}(g)$, for each $f, g \in L^X$,
(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.
The pair $(X, \mathcal{F})$ is called an $(L,*)$-*filter space.* We denote by $F_*(X)$ the set of all $(L,*)$-filters on $X$.

**Theorem 2.7** ([13])**.** *Let* $\mathcal{U}, \mathcal{V} \in F_*(X \times X)$. *We define* $\mathcal{U} \circ_\odot \mathcal{V} : L^{X\times X} \to L$ *as follows:*

$$(\mathcal{U} \circ_\odot \mathcal{V})(w) = \bigvee\{\mathcal{U}(u) \odot \mathcal{V}(v) \mid u \circ v \leq w\}$$

*where* $u \circ v(x,z) = \bigvee_{y\in X}(u(x,y) * v(y,z))$.
(1) $u \circ v = \perp_{X\times X}$ *implies* $\mathcal{U}(u) \odot \mathcal{V}(v) = \perp$ *iff* $(\mathcal{U} \circ_\odot \mathcal{V}) \in F_*(X \times X)$.
(2) *If* $\mathcal{U}(1_\triangle) = \top$ *where* $1_\triangle(x,x) = \top$ *and* $1_\triangle(x,y) = \perp$ *for* $x \neq y \in X$, $\mathcal{U} \circ \mathcal{U} \geq \mathcal{U}$.
(3) $[(x,x)] \circ_* [(x,x)] = [(x,x)]$.
(4) $\bigwedge_{x\in X}[(x,x)] \circ_* \bigwedge_{x\in X}[(x,x)] = \bigwedge_{x\in X}[(x,x)]$.

**Definition 2.8** ([13])**.** A map $\Lambda : F_*(X\times X) \to L$ is called an $(L,*,\odot)$-*quasiuniform convergence structure* on $X$ if it satisfies the following conditions:
(QC1) $\Lambda([(x,x)]) = \top$, for each $x \in X$.
(QC2) If $\mathcal{U} \leq \mathcal{V}$, then $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{V})$.
(QC3) $\Lambda(\mathcal{U}) \odot \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \odot \mathcal{V})$.
(QC4) $\Lambda(\mathcal{U}) \odot \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \circ_\odot \mathcal{V})$ where $\mathcal{U} \circ_\odot \mathcal{V} \in F_*(X \times X)$.
The pair $(X, \Lambda)$ is called an $(L,*,\odot)$-*quasiuniform convergence space.*

An $(L,*,\odot)$-quasiuniform convergence space is called an $(L,*,\odot)$-*uniform convergence space* if it satisfies the following condition;

(U) $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{U}^{-1})$ where $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $u^{-1}(x,y) = u(y,x)$ for $x, y \in X$.

We say $\Lambda_1$ is *finer* than $\Lambda_2$ (or $\Lambda_2$ is *coarser* than $\Lambda_1$) iff $\Lambda_1 \leq \Lambda_2$.

We define $\Lambda_\top, \Lambda_\bot : F_*(X \times X) \to [0,1]$ as follows:

$$\Lambda_\top(\mathcal{W}) = \begin{cases} \top, & \text{if } \mathcal{W} \geq [(x,x)], \forall x \in X \\ \bot, & \text{otherwise.} \end{cases} \qquad \Lambda_\bot(\mathcal{W}) = \top,\ \forall \mathcal{W} \in F_*(X \times X)$$

Then $\Lambda_\top$ (resp. $\Lambda_\bot$) is the finest (resp. coarsest) $(L,*,\odot)$-quasiuniform convergence structure.

Let $(X, \Lambda_X)$ and $(Y, \Lambda_Y)$ be $(L,*,\odot)$-quasiuniform convergence spaces. A map $\psi : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is called *quasiuniformly continuous* if for all $\mathcal{U} \in F_*(X \times X)$, $\Lambda_X(\mathcal{U}) \leq \Lambda_Y((\psi \times \psi)^{\Rightarrow}(\mathcal{U}))$.

## 3. $(L,*,\odot)$-quasiuniform Convergence Spaces induced by Operators

**Theorem 3.1.** *Let $M : F_*(X \times X) \to L^{L^{X\times X}}$ be maps satisfying the following conditions:*

(M1) *$M([(x,x)]) \uparrow [(x,x)] = \top$, for each $\uparrow \in \{\to, \Rightarrow\}$ and $x \in X$.*

(M2) *If $\mathcal{U} \leq \mathcal{V}$, then $M(\mathcal{U}) \geq M(\mathcal{V})$.*

(M3) *$M(\mathcal{U} \odot \mathcal{V}) \leq M(\mathcal{U}) \odot M(\mathcal{V})$.*

(M4) *$M(\mathcal{U} \circ_\odot \mathcal{V}) \leq M(\mathcal{U}) \circ_\odot M(\mathcal{V})$.*

*For each $\uparrow \in \{\to, \Rightarrow\}$, we define a map $\Lambda^{M\uparrow} : F_*(X \times X) \to L$ as follows:*

$$\Lambda^{M\uparrow}(\mathcal{U}) = \bigwedge_{u \in L^{X\times X}} (M(\mathcal{U})(u) \uparrow \mathcal{U}(u)).$$

*Then the following properties hold.*

(1) *$\Lambda^{M\uparrow}$ is an $(L,*,\odot)$ quasi-uniform convergence structure.*

(2) *If $\psi : (X, M_X) \to (Y, M_Y)$ is a map such that $M_Y((\psi \times \psi)^{\Rightarrow}(\mathcal{U}))(v) \leq M_X(\mathcal{U})((\psi \times \psi)^{\leftarrow}(v))$ for each $\mathcal{U} \in F_*(X \times X)$, then $\psi : (X, \Lambda_X^{M\uparrow}) \to (Y, \Lambda_Y^{M\uparrow})$ is quasi-uniformly continuous.*

*Proof.* (1) (QC1) Since $M([(x,x)]) \uparrow [(x,x)] = \top$,

$$\Lambda^{M\uparrow}([(x,x)]) = \bigwedge_{u \in L^{X\times X}} (M([(x,x)])(u) \uparrow [(x,x)](u)) = \top.$$

(QC3) For each $\mathcal{U},\mathcal{V}\in F_*(X\times X)$, by Lemma 2.5(6),

$$\begin{aligned}
&\Lambda^{M\uparrow}(\mathcal{U})\odot\Lambda^{M\uparrow}(\mathcal{V})\\
&=\Big(\bigwedge_{u\in L^{X\times X}}(M(\mathcal{U})(u)\uparrow\mathcal{U}(u))\Big)\odot\Big(\bigwedge_{v\in L^{X\times X}}(M(\mathcal{V})(v)\uparrow\mathcal{V}(v))\Big)\\
&\le\bigwedge_{u\in L^{X\times X}}\bigwedge_{v\in L^{X\times X}}\Big((M(\mathcal{U})(u)\uparrow\mathcal{U}(u))\odot(M(\mathcal{V})(v)\uparrow\mathcal{V}(v))\Big)\\
&\le\bigwedge_{u\in L^{X\times X}}\bigwedge_{v\in L^{X\times X}}\Big(M(\mathcal{U})(u)\odot M(\mathcal{V})(v)\uparrow\mathcal{U}(u)\odot\mathcal{V}(v)\Big)\\
&\le\bigwedge_{u\in L^{X\times X}}\Big(M(\mathcal{U})(u)\odot M(\mathcal{V})(u)\uparrow\mathcal{U}(u)\odot\mathcal{V}(u)\Big)\\
&\le\bigwedge_{u\in L^{X\times X}}\Big(M(\mathcal{U}\odot\mathcal{V})(u)\uparrow(\mathcal{U}\odot\mathcal{V})(u)\Big)\\
&=\Lambda^{M\uparrow}(\mathcal{U}\odot\mathcal{V}).
\end{aligned}$$

(QC4) For each $\mathcal{U},\mathcal{V}\in F_*(X\times X)$, by Lemma 2.5(6),

$$\begin{aligned}
&\Lambda^{M\uparrow}(\mathcal{U}\circ_\odot\mathcal{V})\\
&=\bigwedge_{u\in L^{X\times X}}\Big(M(\mathcal{U}\circ_\odot\mathcal{V})(u)\uparrow(\mathcal{U}\circ_\odot\mathcal{V})(u)\Big)\\
&\ge\bigwedge_{u\in L^{X\times X}}\Big((M(\mathcal{U})\circ_\odot M(\mathcal{V}))(u)\uparrow(\mathcal{U}\circ_\odot\mathcal{V})(u)\Big)\\
&\ge\bigwedge_{u\in L^{X\times X}}\Big(\bigvee_{u_1\circ u_2\le u}(M(\mathcal{U})(u_1)\odot M(\mathcal{V})(u_2))\uparrow(\mathcal{U}\circ_\odot\mathcal{V})(u))\Big)\\
&=\bigwedge_{u\in L^{X\times X}}\bigwedge_{u_1\circ u_2\le u}\Big(M(\mathcal{U})(u_1)\odot M(\mathcal{V})(u_2)\uparrow(\mathcal{U}\circ_\odot\mathcal{V})(u)\Big)\\
&\ge\bigwedge_{u\in L^{X\times X}}\bigwedge_{u_1\circ u_2\le u}\Big(M(\mathcal{U})(u_1)\odot M(\mathcal{V})(u_2)\uparrow\mathcal{U}(u_1)\odot\mathcal{V}(u_2)\Big)\\
&\ge\bigwedge_{u_1\in L^{X\times X}}\bigwedge_{u_2\in L^{X\times X}}\Big((M(\mathcal{U})(u_1)\uparrow\mathcal{U}(u_1))\odot(M(\mathcal{V})(u_2)\uparrow\mathcal{V}(u_2))\Big)\\
&\ge\Big(\bigwedge_{u_1\in L^{X\times X}}(M(\mathcal{U})(u_1)\uparrow\mathcal{U}(u_1))\Big)\odot\Big(\bigwedge_{u_2\in L^{X\times X}}(M(\mathcal{V})(u_2)\uparrow\mathcal{V}(u_2))\Big)\\
&=\Lambda^{M\uparrow}(\mathcal{U})\odot\Lambda^{M\uparrow}(\mathcal{V}).
\end{aligned}$$

(2) For each $\mathcal{U}\in F_*(X\times X)$, by Lemma 2.5(8),

$$\begin{aligned}
&\Lambda_X^{M\uparrow}(\mathcal{U})\uparrow\Lambda_Y^{M\uparrow}((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))\\
&\ge\Big(\bigwedge_{u\in L^{X\times X}}(M_X(\mathcal{U})(u)\uparrow\mathcal{U}(u))\Big)\\
&\uparrow\Big(\bigwedge_{v\in L^{Y\times Y}}(M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v)\uparrow(\psi\times\psi)^{\Rightarrow}(\mathcal{U})(v))\Big)\\
&\ge\Big(\bigwedge_{v\in L^{Y\times Y}}(M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))\uparrow\mathcal{U}((\psi\times\psi)^{\leftarrow}(v)))\Big)\uparrow\\
&\Big(\bigwedge_{v\in L^{Y\times Y}}(M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v)\uparrow(\psi\times\psi)^{\Rightarrow}(\mathcal{U})(v))\Big)\\
&\ge\bigwedge_{v\in L^{Y\times Y}}\Big((M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))\uparrow\mathcal{U}((\psi\times\psi)^{\leftarrow}(v)))\uparrow\\
&(M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v)\uparrow\mathcal{U}((\psi\times\psi)^{\leftarrow}(v)))\Big)\\
&\ge\bigwedge_{v\in L^{Y\times Y}}\Big(M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v)\uparrow M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))\Big).
\end{aligned}$$

Since $M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v)\le M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))$ for each $v\in L^{Y\times Y}$,$\mathcal{U}\in F_*(X\times X)$, by Lemma 2.5(4),

$$\bigwedge_{v\in L^{Y\times Y}} \Big(M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v) \Rightarrow M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))\Big) = \top.$$

Hence $\Lambda^{M\uparrow}_X(\mathcal{U}) \Rightarrow \Lambda^{M\uparrow}_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U})) = \top$. Thus $\psi : (X, \Lambda^{M\uparrow}_X) \to (Y, \Lambda^{M\uparrow}_Y)$ is quasi-uniformly continuous. □

**Example 3.2.** Let $(L = [0,1], \leq, \odot, *, 0, 1)$ be an M-ecl-premonoid. Let a map $M_X : F_*(X\times X) \to [0,1]^{[0,1]^X}$ defined as $M_X(\mathcal{U}) = \bigwedge_{x\in X}[(x,x)]$.

(1) Let $(L = [0,1], \leq, \wedge, *, 0, 1)$ be an M-ecl-premonoid. Since

$$M_X(\mathcal{U}) = \bigwedge_{x\in X}[(x,x)] \leq [(x,x)],$$

$M_X(\mathcal{U}\odot\mathcal{V}) = \bigwedge_{x\in X}[(x,x)] \leq \bigwedge_{x\in X}[(x,x)] \odot \bigwedge_{x\in X}[(x,x)] = M_X(\mathcal{U}) \odot M_X(\mathcal{V})$ and

$$\begin{aligned}(M_X(\mathcal{U}) \circ_\wedge M_X(\mathcal{V}))(u) &\geq M_X(\mathcal{U})(u) \odot M_X(\mathcal{U})(1_\triangle)\\ &= \bigwedge_{x\in X}[(x,x)](u) \odot \bigwedge_{x\in X}[(x,x)](1_\triangle) \geq \bigwedge_{x\in X}[(x,x)](u),\end{aligned}$$

it satisfies the following conditions (M1), (M2) and (M3). For each $\uparrow\in\{\to,\Rightarrow\}$,

$$\Lambda^{M_X\uparrow}(\mathcal{U}) = \bigwedge_{u\in L^{X\times X}}(\bigwedge_{x\in X}[(x,x)](u) \uparrow \mathcal{U}(u)) = \bigwedge_{u\in L^{X\times X}}(\bigwedge_{x\in X} u(x,x) \uparrow \mathcal{U}(u)).$$

Then $\Lambda^{M_X\uparrow}$ is an $(L, *, \odot)$-quasi-uniform convergence structure.

Let $\psi : (X, M_X) \to (Y, M_Y)$ be a map with $M_Y(\mathcal{V}) = \bigwedge_{y\in Y}[(y,y)]$ for all $\mathcal{V} \in F(Y\times Y)$. Since $M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v) = \bigwedge_{y\in Y} v(y,y) \leq \bigwedge_{x\in X} v(\psi(x),\psi(x)) = M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))$ for each $v \in L^{Y\times Y}$, then $\psi : (X, \Lambda^{M_X\uparrow}) \to (Y, \Lambda^{M_Y\uparrow})$ is uniformly continuous.

**Example 3.3.** Let $X = \{a,b,c\}$ be a set and $(L = [0,1], \leq, \wedge, *, 0, 1)$ an M-ecl-premonoid with $a * b = (a+b-1)\vee 0$. Put $u \in [0,1]^{X\times X}$ as follows:

$$u(a,a) = u(b,b) = 1, u(c,c) = 0.4, \;\; u(a,b) = u(b,a) = 0.6,$$

$$u(a,c) = u(c,a) = 0.5, u(b,c) = u(c,b) = 0.4.$$

Define $[0,1]$-filter as $\mathcal{U} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w \geq 1_\triangle,\\ 0.2, & \text{if } u \leq w \not\geq 1_\triangle,\\ 0, & \text{otherwise.}\end{cases}$$

Since $v \circ 1_\triangle = v$, we obtain $\mathcal{U} \circ_\wedge \mathcal{U} = \mathcal{U} = \mathcal{U}^{-1}$ and $0.2 = \mathcal{U}(u) \leq [(c,c)](u) = 0.4$. Put $M_X(\mathcal{W}) = \mathcal{U}$ for all $\mathcal{W} \in F_*(X\times X)$. Then $M_X$ satisfies the conditions (M1)-(M4). For each $\uparrow\in\{\to,\Rightarrow\}$, we obtain an $(L, *, \wedge)$ uniform convergence structure

$\Lambda^{M_X\uparrow} : F_*(X\times X) \to [0,1]$ as follows:

$$\begin{array}{ll}\Lambda^{M_X\uparrow}(\mathcal{W}) & = \bigwedge_{v\in L^{X\times X}}(M_X(\mathcal{W})(v)\uparrow \mathcal{W}(v)) = \bigwedge_{v\in L^{X\times X}}(\mathcal{V}(v)\uparrow \mathcal{W}(v))\\ \Lambda^{M_X\uparrow}(\mathcal{W}^{-1}) & = \bigwedge_{v\in L^{X\times X}}(M_X(\mathcal{W}^{-1})(v)\uparrow \mathcal{W}^{-1}(v))\\ & = \bigwedge_{v\in L^{X\times X}}(\mathcal{V}(v)\uparrow \mathcal{W}^{-1}(v)) = \bigwedge_{v\in L^{X\times X}}(\mathcal{V}^{-1}(v)\uparrow \mathcal{W}(v^{-1}))\\ & = \bigwedge_{v\in L^{X\times X}}(\mathcal{V}(v^{-1})\uparrow \mathcal{W}(v^{-1})) = \Lambda^{M_X\uparrow}(\mathcal{W})\end{array}$$

where $a\Rightarrow b = (1-a+b)\wedge 1$ and

$$a\to b = \begin{cases} 1, & \text{if } a\le b,\\ b, & \text{if } a\not\le b.\end{cases}$$

**Example 3.4.** Let $X=\{a,b,c\}$ be a set, $(L=[0,1],\le,\odot,*,0,1)$ an ecl-premonoid with $a*b=a\cdot b$, $a\odot b = a^{\frac{1}{3}}\cdot b^{\frac{1}{3}}$ and $u\in[0,1]^{X\times X}$ defined as follows:

$$u(a,a)=u(b,b)=u(c,c)=1,\ \ u(a,b)=0.5, u(b,a)=0.6,$$

$$u(a,c)=u(c,a)=0.5, u(b,c)=0.6, u(c,b)=0.4.$$

Define $[0,1]$-filter as $\mathcal{U}:[0,1]^{X\times X}\to[0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X},\\ 0.6^n, & \text{if } u^n\le w\not\ge u^{n-1}, n\in N,\\ 0, & \text{otherwise.}\end{cases}$$

where $u^{n+1}=u^n * u$ and $u^0 = 1_{X\times X}$.

Since $u^n\circ u^n = u^n$, we obtain

$$(\mathcal{U}\circ_\odot \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X},\\ 0.6^n\odot 0.6^n, & \text{if } u^n\le w\not\ge u^{n-1}, n\in N,,\\ 0, & \text{otherwise.}\end{cases}$$

$$(\mathcal{U}\odot \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X},\\ 0.6^n\odot 0.6^n, & \text{if } u^n\le w\not\ge u^{n-1}, n\in N,\\ 0, & \text{otherwise.}\end{cases}$$

Put $M_X(\mathcal{W})=\mathcal{U}$ for all $\mathcal{W}\in F_*(X\times X)$.

(1) Let $(L=[0,1],\le,\wedge,*,0,1)$ be an M-ecl-premonoid with $a*b=a\cdot b$ with

$$a\Rightarrow b = \begin{cases} 1, & \text{if } a\le b,\\ \frac{b}{a}, & \text{if } a\not\le b,\end{cases}\quad a\to b = \begin{cases} 1, & \text{if } a\le b,\\ b, & \text{if } a\not\le b.\end{cases}$$

Since $\mathcal{U}\circ_\wedge\mathcal{U} = \mathcal{U}\wedge\mathcal{U} = \mathcal{U}$, $M$ satisfies the conditions (M1)-(M4). For each $\uparrow\in\{\to,\Rightarrow\}$, we obtain an $(L,*,\wedge)$ quasi-uniform convergence structures $\Lambda^{M_X\uparrow}: F_*(X\times X)\to[0,1]$ as follows:

$$\begin{array}{ll}\Lambda^{M_X\uparrow}(\mathcal{W}) & = \bigwedge_{v\in L^{X\times X}}(M_X(\mathcal{W})(v)\uparrow \mathcal{W}(v))\\ & = \bigwedge_{v\in L^{X\times X}}(\mathcal{U}(v)\uparrow \mathcal{W}(v))\\ & = \bigwedge_{n\in N}(0.6^n\uparrow \mathcal{W}(u^n)).\end{array}$$

So, we have

$$\Lambda^{M_X\Rightarrow}(\mathcal{W}) = \begin{cases} 1, & \text{if } 0.6^n \le \mathcal{W}(u^n), \forall n \in N \\ \frac{\mathcal{W}(u^n)}{0.6^n}, & \text{if } 0.6^n \not\le \mathcal{W}(u^n), \end{cases}$$

$$\Lambda^{M_X\rightarrow}(\mathcal{W}) = \begin{cases} 1, & \text{if } 0.6^n \le \mathcal{W}(u^n), \forall n \in N \\ \mathcal{W}(u^n), & \text{if } 0.6^n \not\le \mathcal{W}(u^n). \end{cases}$$

Since $1 = \Lambda^{M_X\rightarrow}(\mathcal{U}) = 0.6 \rightarrow \mathcal{U}(u) \not\le \Lambda^{M_X\rightarrow}(\mathcal{U}^{-1}) = 0.6 \rightarrow \mathcal{U}(u^{-1}) = 0.6 \rightarrow 0.36 = 0.36$, $\Lambda^{M_X\rightarrow}$ is not an $(L,*,\wedge)$ uniform convergence structure on $X$. Since $1 = \Lambda^{M_X\Rightarrow}(\mathcal{U}) = 0.6 \Rightarrow \mathcal{U}(u) \not\le \Lambda^{M_X\Rightarrow}(\mathcal{U}^{-1}) = (0.6 \Rightarrow \mathcal{U}(u^{-1})) = \frac{1}{6}$, $\Lambda^{M_X\Rightarrow}$ is not an $(L,*,\wedge)$ uniform convergence structure on $X$.

Let $\psi : (X, M_X^x) \rightarrow (Y, M_Y^{\psi(x)})$ be a map with $M_Y(\mathcal{V}) = (\psi\times\psi)^{\Rightarrow}(\mathcal{U})$ for all $\mathcal{V} \in F_*(Y\times Y)$. Then $M_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))(v) = (\psi\times\psi)^{\Rightarrow}(\mathcal{U})(v) = \mathcal{U}((\psi\times\psi)^{\leftarrow}(v)) = M_X(\mathcal{U})((\psi\times\psi)^{\leftarrow}(v))$ for each $\mathcal{U} \in F_*(X\times X)$. Thus $\psi : (X, \Lambda^{M_X\uparrow}) \rightarrow (Y, \Lambda^{M_X\uparrow})$ is uniformly continuous.

(2) Let $(L=[0,1], \le, \odot, *, 0, 1)$ be an M-ecl-premonoid with $a*b = a\cdot b$, $a\odot b = a^{\frac{1}{3}}\cdot b^{\frac{1}{3}}$ with

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \le b, \\ \frac{b}{a}, & \text{if } a \not\le b, \end{cases} \quad a \rightarrow b = \begin{cases} 1, & \text{if } a \le b, \\ (\frac{b}{a^3})^{\frac{1}{3}}, & \text{if } a \not\le b. \end{cases}$$

We obtain an $(L,*,\odot)$ quasi-uniform convergence structures $\Lambda^{M_X\Rightarrow}, \Lambda^{M_X\rightarrow} : F_*(X\times X) \rightarrow [0,1]$ as follows:

$$\Lambda^{M_X\Rightarrow}(\mathcal{W}) = \begin{cases} 1, & \text{if } 0.6^n \le \mathcal{W}(u^n), \forall n \in N \\ \frac{\mathcal{W}(u^n)}{0.6^n}, & \text{if } 0.6^n \not\le \mathcal{W}(u^n), \end{cases}$$

$$\Lambda^{M_X\rightarrow}(\mathcal{W}) = \begin{cases} 1, & \text{if } 0.6^n \le \mathcal{W}(u^n), \forall n \in N \\ (\frac{\mathcal{W}(u^n)}{0.6^{3n}})^{\frac{1}{3}}, & \text{if } 0.6^n \not\le \mathcal{W}(u^n). \end{cases}$$

**Theorem 3.5.** *Let $\Lambda_1$ and $\Lambda_2$ be $(L,*,\odot)$-quasi-uniform convergence spaces on $X$. We define a map $\Lambda_1 \odot_* \Lambda_2 : F_*(X\times X) \rightarrow L$ as follows:*

$$(\Lambda_1 \odot_* \Lambda_2)(\mathcal{U}) = \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \le \mathcal{U}\}.$$

*Then $\Lambda_1 \odot_* \Lambda_2$ is an $(L,*,\odot)$-quasi-uniform convergence space on $X$ which is coarser than $\Lambda_1$ and $\Lambda_2$. Moreover, $\Lambda_1 *_* \Lambda_2$ is the finest $(L,*,*)$ -quasi-uniform convergence spaces on $X$ which is coarser than $\Lambda_1$ and $\Lambda_2$.*

*Proof.* (QUC1) Since $[(x,x)] * [(x,x)] \le [(x,x)]$,

$$(\Lambda_1 \odot_* \Lambda_2)([(x,x)]) \ge \Lambda_1([(x,x)]) \odot \Lambda_1([(x,x)]) = \top$$

Since $(\mathcal{U}_1 \odot \mathcal{V}_1) * (\mathcal{U}_2 \odot \mathcal{V}_2) \leq (\mathcal{U}_1 * \mathcal{U}_2) \odot (\mathcal{V}_1 * \mathcal{V}_2)$,

$$\begin{aligned}
&(\Lambda_1 \odot_* \Lambda_2)(\mathcal{U}) \odot (\Lambda_1 \odot_* \Lambda_2)(\mathcal{V}) \\
&= \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}\} \odot \bigvee\{\Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&= \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \odot \Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}, \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&\leq \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \odot \Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}, \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&\leq \bigvee\{\Lambda_1(\mathcal{U}_1 \odot \mathcal{V}_1) \odot \Lambda_2(\mathcal{U}_2 \odot \mathcal{V}_2) \mid (\mathcal{U}_1 \odot \mathcal{V}_1) * (\mathcal{U}_2 \odot \mathcal{V}_2) \leq \mathcal{U} \odot \mathcal{V}\} \\
&\leq (\Lambda_1 \odot_* \Lambda_2)(\mathcal{U} \odot \mathcal{V}).
\end{aligned}$$

Since $(\mathcal{U}_1 \circ_\odot \mathcal{V}_1) * (\mathcal{U}_2 \circ_\odot \mathcal{V}_2) \leq (\mathcal{U}_1 * \mathcal{U}_2) \circ_\odot (\mathcal{V}_1 * \mathcal{V}_2)$,

$$\begin{aligned}
&(\Lambda_1 \odot_* \Lambda_2)(\mathcal{U}) \odot (\Lambda_1 \odot_* \Lambda_2)(\mathcal{V}) \\
&= \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}\} \odot \bigvee\{\Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&= \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \odot \Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}, \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&\leq \bigvee\{\Lambda_1(\mathcal{U}_1) \odot \Lambda_2(\mathcal{U}_2) \odot \Lambda_1(\mathcal{V}_1) \odot \Lambda_2(\mathcal{V}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}, \mathcal{V}_1 * \mathcal{V}_2 \leq \mathcal{V}\} \\
&\leq \bigvee\{\Lambda_1(\mathcal{U}_1 \circ_\odot \mathcal{V}_1) \odot \Lambda_2(\mathcal{U}_2 \circ_\odot \mathcal{V}_2) \mid (\mathcal{U}_1 \circ_\odot \mathcal{V}_1) * (\mathcal{U}_2 \circ_\odot \mathcal{V}_2) \leq \mathcal{U} \circ_\odot \mathcal{V}\} \\
&\leq (\Lambda_1 \odot_* \Lambda_2)(\mathcal{U} \circ_\odot \mathcal{V}).
\end{aligned}$$

Since $\mathcal{U} * [(x,x)] \leq \mathcal{U}$, then $(\Lambda_1 \odot_* \Lambda_2)(\mathcal{U}) \geq \Lambda_1(\mathcal{U}) \odot \Lambda_2([(x,x)]) = \Lambda_1(\mathcal{U})$. Similarly, $\Lambda_1 \odot_* \Lambda_2 \geq \Lambda_2$.

If $\odot = *$ and $\Lambda_i \leq \Lambda$ for $i \in \{1,2\}$, we have $\Lambda_1 *_* \Lambda_2 \leq \Lambda$ from:

$$\begin{aligned}
(\Lambda_1 *_* \Lambda_2)(\mathcal{U}) &= \bigvee\{\Lambda_1(\mathcal{U}_1) * \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}\} \\
&\leq \bigvee\{\Lambda(\mathcal{U}_1) * \Lambda(\mathcal{U}_2) \mid \mathcal{U}_1 * \mathcal{U}_2 \leq \mathcal{U}\} \leq \Lambda(\mathcal{U}).
\end{aligned}$$

□

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[a]Department of Mathematics, Gangneung-Wonju National Gangneung 25457, Korea
*Email address*: `jmko@gwnu.ac.kr`

[b]Department of Mathematics, Gangneung-Wonju National Gangneung 25457, Korea
*Email address*: `yck@gwnu.ac.kr`