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GENERALIZED SECOND-ORDER DIFFERENTIAL EQUATIONS WITH TWO-POINT BOUNDARY CONDITIONS

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Abstract. In this paper we define higher-order Stieltjes derivatives, and using Schaefer's fixed point theorem we investigate the existence of solutions for a class of differential equations involving second-order Stieltjes derivatives with two-point boundary conditions. The equations include ordinary and impulsive differential equations, and difference equations.

1. Introduction

Ordinary and impulsive differential equations, and difference equations with various boundary conditions arise in diverse real world phenomena in mathematical physics, mechanics, engineering, biology and so on. For comprehensive study on the subject, see, e.g., [2].

In this paper we define higher-order Stieltjes derivatives, and using Schaefer's fixed point theorem we investigate the existence of solutions for a class of second-order differential equations involving Stieltjes derivatives with two-point boundary conditions. The equations include ordinary and impulsive differential equations, and difference equations.

2. Preliminaries

In this section we state some materials that are needed in this paper.

Let $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ be the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integers, respectively.

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Let

$$\mathbf{G}([a,b]) = \{f : [a,b] \to \mathbf{R} | \forall t \in [a,b], \text{ both } f(t+) \text{ and } f(t-) \text{ exist}\},$$

where $f(t+)$ and $f(t-)$ represent the right- and the left-hand limits respectively, and $f(a-) = f(a), f(b+) = f(b)$, and we define

$$\mathbf{G}_{\mathrm{L}}([a,b]) = \{f \in \mathbf{G}([a,b]) : \forall t \in [a,b], f(t-) = f(t)\}.$$

For $f \in \mathbf{G}([a,b])$ we define

$$|f|_\infty = \sup_{s \in [a,b]} |f(s)|.$$

Then $(\mathbf{G}([a,b]), |\cdot|_\infty)$ is a Banach space. For details of $\mathbf{G}([a,b])$, see, e.g., [3].

For $\mathbf{G}([a,b])$ we have the following fundamental result.

Theorem 2.1 ([3, p.17, Corollary 3.2])**.** *Let $f \in \mathbf{G}([a,b])$. Then the set of discontinuities of f is countable.*

A *neighborhood of t in* $[a,b]$ is an open interval in $[a,b]$ that contains t. Let a function $g : [a,b] \longrightarrow \mathbf{R}$ be nondecreasing. Then we say that g is *locally constant at* t if there exists a neighborhood of t in $[a,b]$, where g is constant. Otherwise, we say that the function g is *not locally constant at* t.

Notation. For functions $g : [a,b] \longrightarrow \mathbf{R}$ and $f : E \longrightarrow \mathbf{R}(E \supset g([a,b]))$, $f \circ g$ represents the composite of f and g, i.e.,

$$f \circ g(t) = f(g(t)),$$

and for $f \in \mathbf{G}([a,b])$ and for a function g that is nondecreasing on $[a,b]$, we define

$$\begin{aligned}
&\mathbf{C}(f) = \{t \in [a,b] : f \text{ is continuous at } t\}, \\
&\mathbf{D}(f) = [a,b] - \mathbf{C}(f), \\
&\mathbf{J}(g) = \{t \in [a,b] : g \text{ is not locally constant at } t\}, \\
&\mathbf{K}(g) = [a,b] - \mathbf{J}(g).
\end{aligned}$$

Throughout this paper

$$\alpha_k : [a,b] \longrightarrow \mathbf{R}(k \in \mathbf{N})$$

is a nondecreasing left-continuous function.

Let $t \in [a,b]$. If $\{s \in \mathbf{J}(\alpha_k) : t \le s\} \ne \emptyset$, then we define

$$\sigma_k(t) = \inf\{s \in \mathbf{J}(\alpha_k) : t \le s\}.$$

We define

$$\alpha = (\,\alpha_1,\, \alpha_2,\, \alpha_3, \cdots, \alpha_n),\ n \in \mathbf{N},$$

and for $(a, n) \in \mathbf{R} \times \mathbf{N}$

$$\overline{a, a+n} = a, a+1, a+2, \cdots, a+n.$$

For example, $\overline{2,7} = 2, 3, \cdots, 7$.

Let g be a nondecreasing function defined on $[a, b]$. Then we define

$$\mathbf{G}_{\mathrm{L}}^{g}([a, b]) = \{f \in \mathbf{G}_{\mathrm{L}}([a, b]) : \mathbf{C}(g) \subset \mathbf{C}(f)\}.$$

Then we have the following result.

Theorem 2.2. *The space*

$$\big(\mathbf{G}_{\mathrm{L}}^{g}([a, b]), |\cdot|_\infty\big)$$

is a Banach space.

Proof. It is obvious that $\mathbf{G}_{\mathrm{L}}^{g}([a, b])$ is a linear space. Let $X = \mathbf{G}_{\mathrm{L}}^{g}([a, b])$ and let $f \in \overline{X}$. Then there exists a sequence $\{f_n\} \subset X$ such that $f_n \to f$ as $n \to \infty$. Since $\mathbf{G}([a, b])$ is a Banach space and $\{f_n\} \subset \mathbf{G}([a, b])$, $f \in \mathbf{G}([a, b])$. Now we have

$$|f(t) - f(t-\eta)| \le |f(t) - f_n(t)| + |f_n(t) - f_n(t-\eta)| + |f_n(t-\eta) - f(t-\eta)|.$$

So for every $\varepsilon > 0$ there exist $n_0 \in \mathbf{N}$ and $\eta_0 > 0$ such that $\forall \eta \in (0, \eta_0)$, we have

$$\begin{aligned} |f(t) - f(t-\eta)| &\le |f(t) - f_{n_0}(t)| + |f_{n_0}(t) - f_{n_0}(t-\eta)| + |f_{n_0}(t-\eta) - f(t-\eta)| \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned} \tag{2.1}$$

since $\{f_n\} \subset \mathbf{G}_{\mathrm{L}}([a, b])$. This implies that $f \in \mathbf{G}_{\mathrm{L}}([a, b])$. Let $t \in \mathbf{C}(g)$. Then, since $\forall n \in \mathbf{N},\ \mathbf{C}(g) \subset \mathbf{C}(f_n)$, by similar process to (2.1) we can verify that f is continuous at t, i.e., $\mathbf{C}(g) \subset \mathbf{C}(f)$. Thus $f \in X$. This implies that $X = \mathbf{G}_{\mathrm{L}}^{g}([a, b])$ is a closed subspace of a Banach space $\mathbf{G}([a, b])$. So $\mathbf{G}_{\mathrm{L}}^{g}([a, b])$ is also a Banach space. This completes the proof. □

Definition 2.3 ([4])**.** Let $f, g : [a, b] \longrightarrow \mathbf{R}$, where g is nondecreasing. Assume that g is not locally constant at $t \in [a, b]$. If $t \in (a, b)$, then we define

$$\frac{\mathrm{d}f(t)}{\mathrm{d}g(t)} = \lim_{\eta,\delta \to 0+} \frac{f(t+\eta) - f(t-\delta)}{g(t+\eta) - g(t-\delta)},$$

provided that the limit exists. And for $t = a$ or $t = b$ we define

$$\frac{\mathrm{d}f(a)}{\mathrm{d}g(a)} = \lim_{\eta\to 0+} \frac{f(a+\eta)-f(a)}{g(a+\eta)-g(a)}, \quad \frac{\mathrm{d}f(b)}{\mathrm{d}g(b)} = \lim_{\delta\to 0+} \frac{f(b)-f(b-\delta)}{g(b)-g(b-\delta)},$$

respectively, provided that the limits exist.

If both f and g are constant on some neighborhood of t in $[a,b]$, then we define $\frac{\mathrm{d}f(t)}{\mathrm{d}g(t)} = 0$. Frequently we use $f'_g(t)$ instead of $\frac{\mathrm{d}f(t)}{\mathrm{d}g(t)}$.

Remark 2.4. In the above definition, $f'_g(t)$ is called the *Stieltjes derivative* of f at t with respect to g.

Definition 2.5. (Higher-order Stieltjes Derivatives)

Let $n \in \mathbf{N} \cap [2,\infty)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. Assume that

$$\mathbf{C}(\alpha_1) = \mathbf{C}(\alpha_2) = \cdots = \mathbf{C}(\alpha_n),\ \mathbf{J}(\alpha_1) = \mathbf{J}(\alpha_2) = \cdots = \mathbf{J}(\alpha_n),$$

and

$$b \in \mathbf{J}(\alpha_k),\ \forall k = \overline{1,n}.$$

Here we define

$$\forall k = \overline{1,n},\ \mathbf{C}(\alpha_k) = \mathbf{C}(\alpha),\ \mathbf{J}(\alpha_k) = \mathbf{J}(\alpha).$$

Let $\mathbf{D}(\alpha) = [a,b] - \mathbf{C}(\alpha)$ and $\mathbf{K}(\alpha) = [a,b] - \mathbf{J}(\alpha)$. Since $\forall k = \overline{1,n},\ \mathbf{J}(\alpha_k) = \mathbf{J}(\alpha)$, it is obvious that

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n.$$

We define $\sigma = \sigma_k,\ \forall k = \overline{1,n}$. Suppose that

$$f \in \mathbf{G}_{\mathrm{L}}([a,b]) \text{ and } \forall s \in [a,b],\ f \circ \sigma(s) = f(s).$$

If $f'_{\alpha_1}(t)$ exists, then we define $f^{(1)}_{\alpha}(t) = f'_{\alpha_1}(t)$. Let

$$\mathbf{T}_1 = \{\alpha : \mathbf{J}(\alpha) = [a,b] \text{ and } \mathbf{D}(\alpha)(\subset (a,b)) \text{ is a finite set}\}, \text{ and } \mathbf{T}_2 = \{\alpha : \alpha \notin \mathbf{T}_1\}.$$

(Case I) Assume that $\alpha \in \mathbf{T}_1$ and that $\mathbf{C}(\alpha) \subset \mathbf{C}(f)$. For every $k = \overline{1,n-1}$ we define $f^{(k+1)}_{\alpha}(t)$ inductively as follows:

If $\forall s \in [a,b],\ f^{(k)}_{\alpha}(s)$ exists, and $\mathbf{C}(\alpha) \subset \mathbf{C}(f^{(k)}_{\alpha}),\ f^{(k)}_{\alpha} \in \mathbf{G}([a,b])$, and $(f^{(k)}_{\alpha})'_{\alpha_{k+1}}(t)$ exists, then we define

$$f^{(k+1)}_{\alpha}(t) = (f^{(k)}_{\alpha})'_{\alpha_{k+1}}(t).$$

(Case II) Assume that $\alpha \in \mathbf{T}_2$. For every $k = \overline{1,n-1}$ we define $f^{(k+1)}_{\alpha}(t)$ inductively as follows:

If $\forall s \in [a,b]$, $f_\alpha^{(k)}(s)$ exists, and $f_\alpha^{(k)} \circ \sigma \in \mathbf{G}_{\mathrm{L}}([a,b])$, and $(f_\alpha^{(k)} \circ \sigma)'_{\alpha_{k+1}}(t)$ exists, then we define

$$f_\alpha^{(k+1)}(t) = (f_\alpha^{(k)} \circ \sigma)'_{\alpha_{k+1}}(t).$$

Remark 2.6. We also use notations $f'_\alpha, f''_\alpha, f'''_\alpha$ instead of $f_\alpha^{(1)}, f_\alpha^{(2)}, f_\alpha^{(3)}$, respectively. And for some important examples of Definition 2.5, see Corollary 3.7, 3.8 and 3.9.

Throughout this paper we use the Kurzweil-Stieltjes integral(sometimes the integral is called as the Perron-Stieltjes integral, see, e.g., [11, 13]), and the Stieltjes derivative. For the integral and Stieltjes derivative, and various notations and results that are needed here, see, e.g., [4, 5, 6, 7, 10, 11, 13] and the references cited there.

We use the following results frequently.

Theorem 2.7 ([13, Theorem 2.15])**.** *Assume that $f \in \mathbf{G}([a,b])$ and $g \in \mathbf{BV}([a,b])$. Then both $\int_a^b f(s)\,\mathrm{d}g(s)$ and $\int_a^b g(s)\,\mathrm{d}f(s)$ exist.*

Theorem 2.8 ([4, 5])**.** *Assume that a function $g : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and is not locally constant at $t \in [a,b]$. If f is continuous at t or g is not continuous at t, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}g(t)} \int_a^t f(s)\,\mathrm{d}g(s) = f(t).$$

Theorem 2.9 ([4, 5])**.** *Assume that a function $g : [a,b] \longrightarrow \mathbf{R}$ is nondecreasing, and that if g is constant on some neighborhood of t in $[a,b]$, then there exists a neighborhood of t in $[a,b]$ such that both f and g are constant there. Suppose that $f'_g(t)$ exists for every $t \in [a,b] - \{c_1, c_2, ...\}$, where f is continuous at every $t \in \{c_1, c_2, ...\}$. Then we have*

$$(K^*) \int_a^b f'_g(s)\,\mathrm{d}g(s) = f(b) - f(a).$$

Remark 2.10. In Theorem 2.9 $(K^*) \int_a^b$ represents the Kurzweil*-Stieltjes integral. Kurzweil-Stieltjes integrability means Kurzweil*-Stieltjes integrability, and their values are equal each other, see, e.g., [8].

Theorem 2.11 ([11, p.34, Corollary 4.13]). *Assume that* $g \in \mathbf{G}([a,b])$ *and* $\int_a^b f(s)\,\mathrm{d}g(s)$ *exists. Then for every* $t \in [a,b]$ *we have*

$$\lim_{\eta \to 0+} \int_a^{t \pm \eta} f(s)\,\mathrm{d}g(s) = \int_a^t f(s)\,\mathrm{d}g(s) \pm f(t)\Delta^{\pm} g(t),$$

where $\Delta^+ g(t) = g(t+) - g(t),\ \Delta^- g(t) = g(t) - g(t-)$.

Remark 2.12. Let $F(t) = \int_a^t f(s)\,\mathrm{d}g(s)$. Then by Theorem 2.11, $\mathbf{C}(g) \subset \mathbf{C}(F)$ and, if g is left-continuous or right-continuous at t, then F is also left-continuous or right-continuous there, respectively. And, by the definition of the Kurzweil-Stieltjes integral, if g is constant on $[c,d] \subset [a,b]$, then F is also constant there.

Let $E \subset \mathbf{R}$ and let λ be a nondecreasing function defined on $[a,b]$, and let $\lambda^*_{[a,b]}(E)$ be the $\lambda_{[a,b]}-$measure of E (for details, see [9, Section 3.7]). Here $\lambda^*_{[a,b]}$ is called the Lebesgue-Stieltjes measure induced by $\lambda_{[a,b]}$.

Lemma 2.13. *Let* λ *be a nondecreasing function defined on* $[a,b]$. *Then*

$$\lambda^*_{[a,b]}[\mathbf{K}(\lambda)] = 0.$$

Proof. Since λ is nondecreasing, we have

$$\mathbf{K}(\lambda) \cap (a,b) = \bigcup_{i=1}^{\infty} (a_i, b_i),$$

where λ is constant on (a_i, b_i). Then by [9, Exercise 3.2.3], we have

$$\lambda^*_{[a,b]}[(a_i,b_i)] = \lambda(b_i-) - \lambda(a_i+) = 0,\ \forall i \in \mathbf{N}.$$

So we have

$$\lambda^*_{[a,b]}[\mathbf{K}(\lambda) \cap (a,b)] \le \sum_{i=1}^{\infty} \lambda^*_{[a,b]}[(a_i,b_i)] = 0. \tag{2.2}$$

If $a \in \mathbf{K}(\lambda)$ or $b \in \mathbf{K}(\lambda)$, then by [9, Proposition 3.2.2] we have

$$\lambda^*_{[a,b]}(\{a\}) = \lambda(a+) - \lambda(a-) = \lambda(a) - \lambda(a) = 0, \text{ or similarly } \lambda^*_{[a,b]}(\{b\}) = 0. \tag{2.3}$$

Thus by (2.2) and (2.3) the proof is complete. □

Also we have the following result.

Theorem 2.14 ([9, Proposition 6.3.1])**.** *Let f, g be functions defined on $[a,b]$ and let λ be a nondecreasing function defined on $[a,b]$. Assume that there is a set $E \subset [a,b]$ with $\lambda^*_{[a,b]}(E) = 0$ such that $f(t) = g(t),\ \forall t \in [a,b] - E$. Then $\int\limits_a^b f(s)\,\mathrm{d}\lambda(s)$ exists if and only if $\int\limits_a^b g(s)\,\mathrm{d}\lambda(s)$ does, in which case $\int\limits_a^b f(s)\,\mathrm{d}\lambda(s) = \int\limits_a^b g(s)\,\mathrm{d}\lambda(s)$.*

Definition 2.15. Let $f : [a,b] \times \mathbf{R} \longrightarrow \mathbf{R}$ and let $F_x(s) = f(s, x(s))\ \forall s \in [a,b]$, where $x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$. Then we say that f satisfies *condition* ($\mathbf{H}$) if
(1) for every fixed $s \in [a,b]$, $f(s,\cdot) : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous,
(2) $\forall x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b]),\ F_x \in \mathbf{G}([a,b])$ and $\mathbf{C}(\alpha_1) \subset \mathbf{C}(F_x)$.

Example 2.16. Let $a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = b$. Assume that $\mathbf{D}(\alpha_1) = \{t_1, t_2, \cdots, t_p\},\ x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$.

Let

$$f_{i+1} : [t_i, t_{i+1}] \times \mathbf{R} \longrightarrow \mathbf{R},\ (i = \overline{0,p}\ )$$

be a continuous function on the domain, also let

$$g_i : \mathbf{R} \longrightarrow \mathbf{R},\ (i = \overline{1,p}\ )$$

be continuous. If we define

$$f(s,x) = \begin{cases} f_1(s,x), & (s,x) \in [a,t_1) \times \mathbf{R}, \\ f_{p+1}(s,x), & (s,x) \in (t_p, b] \times \mathbf{R}, \\ f_{i+1}(s,x), & (s,x) \in (t_i, t_{i+1}) \times \mathbf{R},\ (i = \overline{1,p-1}\ ) \\ g_i(x), & (s,x) \in \{t_i\} \times \mathbf{R},\ (i = \overline{1,p}\ ), \end{cases}$$

then the function f satisfies condition (H).

In the above definition, by Theorem 2.7, $\int\limits_a^b f(s, x(s))\,\mathrm{d}\alpha_1(s)$ exists.

3. Existence of Solutions

In this section we study the existence of solutions for the following second-order generalized differential equation with two-point boundary conditions:

$$\begin{cases} x''_\alpha(t) = f(t, x(t)),\ \forall t \in \mathbf{J}(\alpha), \\ x(0) = a,\ x(T^*) = b,\ T^* \in (0,T], \\ \forall t \in \mathbf{D}(\alpha),\ x'_\alpha(t) = 0 \text{ if } \alpha \in \mathbf{T}_1, \end{cases} \tag{3.1}$$

where $x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T]),\ x''_\alpha(t) = 0, \forall t \in \mathbf{K}(\alpha)$, and $f : [0,T] \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfies condition ($\mathbf{H}$).

Throughout this paper, for every $t \in [0,T]$, we define

$$A(t) = \begin{cases} 1, & t \in \mathbf{C}(\alpha) \\ 0, & t \notin \mathbf{C}(\alpha) \end{cases} \quad \text{if } \alpha \in \mathbf{T}_1,\ A(t) = 1 \text{ if } \alpha \in \mathbf{T}_2, \text{ and } \eta(t) = \int_0^t A\, \mathrm{d}\alpha_1.$$

Lemma 3.1. *Let $y \in \mathbf{G}([0,T])$ with $\mathbf{C}(\alpha) \subset \mathbf{C}(y)$. Then*

$$x(t) = a + \frac{\eta(t)}{\eta(T^*)}\left[b - a - \int_0^{T^*}\int_0^s A(s)y(v)\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s)\right] + \int_0^t\int_0^s A(s)y(v)\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s) \tag{3.2}$$

is the unique solution of the boundary value problem

$$\begin{cases} x''_\alpha(t) = y(t),\ \forall t \in \mathbf{J}(\alpha), \\ x(0) = a,\ x(T^*) = b,\ T^* \in (0,T], \\ \forall t \in \mathbf{D}(\alpha),\ x'_\alpha(t) = 0 \text{ if } \alpha \in \mathbf{T}_1, \end{cases} \tag{3.3}$$

where $x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T])$, $x''_\alpha(t) = 0, \forall t \in \mathbf{K}(\alpha)$.

Proof. (I) First, suppose that x is a solution of the equation (3.3). Let $t \in [0,T]$. Then by Lemma 2.13, $\alpha^*_{2[0,T]}(\mathbf{K}(\alpha)) = 0$, and

$$x''_\alpha(t) = y(t), \forall t \in \mathbf{J}(\alpha) = [0,T] - \mathbf{K}(\alpha).$$

So by Theorem 2.14 we get

$$\int_0^t x''_\alpha(v)\, \mathrm{d}\alpha_2(v) = \int_0^t y(v)\, \mathrm{d}\alpha_2(v). \tag{3.4}$$

(In case that $\alpha \in \mathbf{T}_1$) Since $\mathbf{C}(\alpha) \subset \mathbf{C}(x'_\alpha)$, by Theorem 2.9 for every $t \in \mathbf{C}(\alpha)$, we have

$$\int_0^t x''_\alpha(v)\, \mathrm{d}\alpha_2(v) = \int_0^t (x'_\alpha)'_{\alpha_2}(v)\, \mathrm{d}\alpha_2(v)$$
$$= x'_\alpha(t) - x'_\alpha(0) = x'_\alpha \circ \sigma(t) - x'_\alpha \circ \sigma(0),$$

because $\forall t \in [0,T]$, $\sigma(t) = t$. So by (3.4) we get

$$x'_\alpha \circ \sigma(t) = x'_\alpha \circ \sigma(0) + \int_0^t y(v)\, \mathrm{d}\alpha_2(v), \forall t \in \mathbf{C}(\alpha), \tag{3.5}$$

and by the hypotheses in this theorem

$$x'_{\alpha} \circ \sigma(t) = x'_{\alpha}(t) = 0, \ \forall t \in \mathbf{D}(\alpha). \tag{3.6}$$

Hence by (3.5) and (3.6) we obtain

$$x'_{\alpha} \circ \sigma(t) = A(t)\left[x'_{\alpha} \circ \sigma(0) + \int_0^t y(v)\,\mathrm{d}\alpha_2(v)\right]. \tag{3.7}$$

(In case that $\alpha \in \mathbf{T}_2$) Since $x'_{\alpha} \circ \sigma \in \mathbf{G}_{\mathrm{L}}([0,T])$, by Theorem 2.9 we have

$$\begin{aligned}\int_0^t x''_{\alpha}(v)\,\mathrm{d}\alpha_2(v) &= \int_0^t (x'_{\alpha} \circ \sigma)'_{\alpha_2}(v)\,\mathrm{d}\alpha_2(v) \\ &= x'_{\alpha} \circ \sigma(t) - x'_{\alpha} \circ \sigma(0), \forall t \in [0,T],\end{aligned} \tag{3.8}$$

and so by (3.4) and (3.8), and since $A(t) \equiv 1$, we get

$$\begin{aligned}x'_{\alpha} \circ \sigma(t) &= x'_{\alpha} \circ \sigma(0) + \int_0^t y(v)\,\mathrm{d}\alpha_2(v) \\ &= A(t)\left[x'_{\alpha} \circ \sigma(0) + \int_0^t y(v)\,\mathrm{d}\alpha_2(v)\right].\end{aligned} \tag{3.9}$$

Thus, by (3.7) and (3.9), in any case of Definition 2.5, we have

$$x'_{\alpha} \circ \sigma(t) = A(t)\left[x'_{\alpha} \circ \sigma(0) + \int_0^t y(v)\,\mathrm{d}\alpha_2(v)\right]. \tag{3.10}$$

So, since by Lemma 2.13 $\alpha^*_{1[0,T]}(\mathbf{K}(\alpha)) = 0$ and $\forall s \in \mathbf{J}(\alpha),\ x'_{\alpha}(s) = x'_{\alpha} \circ \sigma(s)$, and $x \in \mathbf{G}_{\mathrm{L}}([0,T])$, by Theorem 2.9 and 2.14 we get

$$\begin{aligned}\int_0^t x'_{\alpha} \circ \sigma(s)\,\mathrm{d}\alpha_1(s) &= \int_0^t x'_{\alpha}(s)\,\mathrm{d}\alpha_1(s) = \int_0^t x'_{\alpha_1}(s)\,\mathrm{d}\alpha_1(s) \\ &= x(t) - x(0) = x(t) - a.\end{aligned} \tag{3.11}$$

Thus by (3.10) and (3.11) we have

$$\begin{aligned}x(t) &= a + \int_0^t A(s)\left[x'_{\alpha} \circ \sigma(0) + \int_0^s y(v)\,\mathrm{d}\alpha_2(v)\right]\mathrm{d}\alpha_1(s) \\ &= a + x'_{\alpha} \circ \sigma(0)\eta(t) + \int_0^t\int_0^s A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).\end{aligned} \tag{3.12}$$

This yields

$$b = x(T^*) = a + x'_\alpha \circ \sigma(0)\eta(T^*) + \int_0^{T^*}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).$$

So we have

$$x'_\alpha \circ \sigma(0) = \frac{1}{\eta(T^*)}\left[b - a - \int_0^{T^*}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right]. \tag{3.13}$$

Thus by (3.12) and (3.13) we get

$$x(t) = a + \frac{\eta(t)}{\eta(T^*)}\left[b - a - \int_0^{T^*}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right] + \int_0^{t}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s). \tag{3.14}$$

(II) Now suppose that x is defined by (3.2). Then it is obvious that

$$x(0) = a,\ x(T^*) = b. \tag{3.15}$$

And by Remark 2.12 $x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T])$. Since α_1 is constant on $[t, \sigma(t)]$, by Remark 2.12 we have

$$\eta \circ \sigma(t) = \int_0^{\sigma(t)} A(s)\,\mathrm{d}\alpha_1(s) = \int_0^{t} A(s)\,\mathrm{d}\alpha_1(s),$$

and

$$\int_0^{\sigma(t)}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s) = \int_0^{t}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).$$

This implies that $x \circ \sigma(t) = x(t),\ \forall t \in [0,T]$. Now let

$$c = \frac{1}{\eta(T^*)}\left[b - a - \int_0^{T^*}\int_0^{s} A(s)y(v)\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right].$$

(In case that $\alpha \in \mathbf{T}_1$) Since, by Remark 2.12, for every $t \in \mathbf{C}(\alpha)$ both A and $\int_0^{(\cdot)} y(v)\,\mathrm{d}\alpha_2(v)$ are continuous at t, by Theorem 2.8 we have

$$x'_\alpha(t) = c + \int_0^{t} y(v)\,\mathrm{d}\alpha_2(v),\ \forall t \in \mathbf{C}(\alpha), \tag{3.16}$$

and, since for $t \in \mathbf{D}(\alpha)$, $A(t) = 0$, again by Theorem 2.8 we get

$$x'_\alpha(t) = 0, \ \forall t \in \mathbf{D}(\alpha). \tag{3.17}$$

Hence (3.16) and (3.17) yield

$$x'_\alpha(t) = A(t)\left[c + \int_0^t y(v)\,\mathrm{d}\alpha_2(v)\right], \ \forall t \in [0,T].$$

So, since $\mathbf{D}(\alpha)$ is a finite set and $\forall t \notin \mathbf{D}(\alpha), A(t) = 1$, by the definition of the Stieltjes derivative and by Theorem 2.8, and since $\mathbf{C}(\alpha) \subset \mathbf{C}(y)$, we have

(3.18)

$$x''_\alpha(t) = (x'_\alpha)'_{\alpha_2}(t) = \frac{\mathrm{d}}{\mathrm{d}\alpha_2(t)}\left[c + \int_0^t y(v)\,\mathrm{d}\alpha_2(v)\right] = y(t), \ \forall t \in \mathbf{J}(\alpha) = [0,T].$$

(In case that $\alpha \in \mathbf{T}_2$) Since $\forall s \in [0,T]$, $A(s) \equiv 1$ and $\mathbf{C}(\alpha) = \mathbf{C}(\alpha_1) = \mathbf{C}(\alpha_2)$, by Theorem 2.8 and Remark 2.12, differentiating both sides of (3.2), we have

$$x'_\alpha(t) = c + \int_0^t y(v)\,\mathrm{d}\alpha_2(v), \ \forall t \in \mathbf{J}(\alpha). \tag{3.19}$$

Since α_2 is constant on $[t, \sigma(t)]$, by (3.19) and Remark 2.12 we have

$$x'_\alpha \circ \sigma(t) = c + \int_0^{\sigma(t)} y(v)\,\mathrm{d}\alpha_2(v) = c + \int_0^t y(v)\,\mathrm{d}\alpha_2(v), \ \forall t \in [0,T],$$

and also, since α_2 is left-continuous, by Remark 2.12, $x'_\alpha \circ \sigma \in \mathbf{G}_{\mathrm{L}}([0,T])$. So, since by hypotheses $\mathbf{C}(\alpha) \subset \mathbf{C}(y)$, by Theorem 2.8 we get

$$x''_\alpha(t) = (x'_\alpha \circ \sigma)'_{\alpha_2}(t) = y(t), \ \forall t \in \mathbf{J}(\alpha). \tag{3.20}$$

Thus by (3.15), (3.18) and (3.20) the proof is complete. □

Now we state some materials to prove the existence of solutions for Eq.(3.1).

We say that a set $\mathcal{A} \subset \mathbf{G}([a,b])$ has *uniform one-sided limits at* $t_0 \in [a,b]$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathcal{A}$ we have: if $t_0 < t < t_0 + \delta$ then $|x(t) - x(t_0+)| < \varepsilon$; if $t_0 - \delta < t < t_0$ then $|x(t) - x(t_0-)| < \varepsilon$.

A set $\mathcal{A} \subset \mathbf{G}([a,b])$ is called *equi-regulated on* $[a,b]$ if it has uniform one-sided limits at every point $t_0 \in [a,b]$.

For relative compactness of a set $\mathcal{A} \subset \mathbf{G}([a,b])$, we have the following result.

Theorem 3.2 ([1, Corollary 2.4])**.** *A set $\mathcal{A} \subset \mathbf{G}([a,b])$ is relatively compact if and only if it is equi-regulated on $[a,b]$ and for every $t \in [a,b]$ the set $\{x(t) : x \in \mathcal{A}\}$ is bounded in $\mathbf{R}$.*

Corollary 3.3. *Let $\mathcal{A} \subset \mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$. Assume that $\mathcal{A}$ is equi-regulated on $[a,b]$ and for every $t \in [a,b]$ the set $\{x(t) : x \in \mathcal{A}\}$ is bounded in $\mathbf{R}$. Then $\mathcal{A}$ is relatively compact in the space $\mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$.*

Proof. Since $\mathcal{A} \subset \mathbf{G}([a,b])$, by Theorem 3.2 $\mathcal{A}$ is relatively compact in the space $\mathbf{G}([a,b])$. This means that $\mathcal{A} \subset \mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$ is totally bounded for the norm $|\cdot|_\infty$. So, since by Theorem 2.2 $\mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$ is complete, $\mathcal{A}$ is relatively compact in the space $\mathbf{G}_{\mathrm{L}}^{\alpha_1}([a,b])$. The proof is complete. □

We have the following fixed point theorem.

Theorem 3.4 ([12, p.29])**.** (SCHAEFER'S FIXED POINT THEOREM)
Let $(\mathbf{X}, |\cdot|)$ be a Banach space and let $N : \mathbf{X} \longrightarrow \mathbf{X}$ is a completely continuous operator. If the set

$$\{x \in \mathbf{X} : x = kNx \text{ for some } k \in (0,1)\}$$

is bounded, then N has a fixed point.

Using the above theorem we can obtain the following result.

Theorem 3.5. *Assume that hypotheses*
(H1) *The function $f : [0,T] \times \mathbf{R} \longrightarrow \mathbf{R}$ satisfies condition* (**H**).
(H2) *There exist non-negative functions $a, b \in \mathbf{G}([0,T])$ and a number $\lambda \in [0,1)$ such that*

$$|f(t,x)| \le a(t)|x|^\lambda + b(t), \ \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{3.21}$$

are satisfied. Then the equation (3.1) *has at least one solution.*

Proof. Transform the problem (3.1) into a fixed-point problem. Consider the operator $N : \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T]) \longrightarrow \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T])$ defined by

$$Nx(t) = N_1 x(t) + N_2 x(t),$$

where

$$N_1x(t) = a + \frac{\eta(t)}{\eta(T^*)}\left[b - a - \int_0^{T^*}\int_0^s A(s)f(v, x(v))\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s)\right],$$

$$N_2x(t) = \int_0^t\int_0^s A(s)f(v, x(v))\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s).$$

By Remark 2.12 we see that $Nx \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0, T])$, and by Lemma 3.1 x satisfies the problem (3.1) if and only if $x = Nx$.

(I)(N is continuous.) Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathbf{G}_{\mathrm{L}}^{\alpha_1}([0, T])$. Then there is a number $M \geq 0$ with $|x_n|_\infty + |x|_\infty \leq M$. This yields

$$\begin{aligned}
&|f(v, x_n(v)) - f(v, x(v))| \leq |f(v, x_n(v))| + |f(v, x(v))| \\
&\leq a(v)|x_n(v)|^\lambda + b(v) + a(v)|x(v)|^\lambda + b(v) \\
&\leq a(v)(|x_n|_\infty^\lambda + |x|_\infty^\lambda) + 2b(v) \\
&\leq 2[a(v)M^\lambda + b(v)].
\end{aligned}$$

Note that by Theorem 2.7 $\int_0^T [a(v)M^\lambda + b(v)]\, \mathrm{d}\alpha_2(v)$ exists. And by (H1), for every $v \in [0, T]$, we have

$$|f(v, x_n(v)) - f(v, x(v))| \longrightarrow 0, \text{ as } n \to \infty.$$

Let $L_1 = \frac{\eta(T)}{\eta(T^*)}$. Then, since $|A(s)| \leq 1$, by the Dominated Convergence Theorem[10, Corollary 1.32], we get, as $n \longrightarrow \infty$,

$$\begin{aligned}
&|Nx_n(t) - Nx(t)| \\
&\leq \frac{\eta(t)}{\eta(T^*)} \int_0^{T^*}\int_0^s |A(s)| \cdot |f(v, x_n(v)) - f(v, x(v))|\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s) \\
&\quad + \int_0^t\int_0^s |A(s)| \cdot |f(v, x_n(v)) - f(v, x(v))|\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s) \\
&\leq L_1 \int_0^T\int_0^T |f(v, x_n(v)) - f(v, x(v))|\, \mathrm{d}\alpha_2(v)\, \mathrm{d}\alpha_1(s)
\end{aligned}$$

$$+\int_0^T\int_0^T |f(v,x_n(v))-f(v,x(v))|\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\longrightarrow 0.$$

This implies that N is a continuous operator.

(II)(N is a compact operator.)

Let $\mathcal{A}$ be a bounded subset of $\mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T])$ with a bound L. Let $x\in\mathcal{A}$, and let $t_m,t_n\longrightarrow t+$ or $t-$, $(t_m\le t_n)$ as $m,n\longrightarrow\infty$. Here, since $|\eta(t_n)-\eta(t_m)| = |\int_{t_m}^{t_n} A(s)\,\mathrm{d}\alpha_1(s)|\le|\,\alpha_1(t_n)-\alpha_1(t_m)|\to 0$ as $n,m\to\infty$, and since $\forall v\in[0,T]$, $|A(v)|\le 1$, $|x(v)|\le|x|_\infty\le L$, we get

$$\begin{aligned}
&|N_1x(t_n)-N_1x(t_m)| \\
&\le\frac{|\eta(t_n)-\eta(t_m)|}{\eta(T^*)}\left[|b-a|+\int_0^{T^*}\int_0^s |A(s)|\cdot|f(v,x(v))|\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right]\\
&\le\frac{|\eta(t_n)-\eta(t_m)|}{\eta(T^*)}\left[|b-a|+\int_0^{T^*}\int_0^s [a(v)|x|_\infty^\lambda+b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right]\\
&\le\frac{|\eta(t_n)-\eta(t_m)|}{\eta(T^*)}\left[|b-a|+\int_0^{T^*}\int_0^s [a(v)L^\lambda+b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right]\\
&\longrightarrow 0+\ \text{ as } n,m\to\infty \text{ uniformly for all } x\in\mathcal{A},
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
&|N_2x(t_n)-N_2x(t_m)|\\
&\le\int_{t_m}^{t_n}\int_0^s |A(s)|\cdot|f(v,x(v))|\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\\
&\le\int_{t_m}^{t_n}\int_0^T [a(v)|x|_\infty^\lambda+b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\\
&\le|\,\alpha_1(t_n)-\ \alpha_1(t_m)|\int_0^T [a(v)L^\lambda+b(v)]\,\mathrm{d}\alpha_2(v)\\
&\longrightarrow 0+\ \text{ as } n,m\to\infty \text{ uniformly for all } x\in\mathcal{A}.
\end{aligned}\tag{3.23}$$

Thus by (3.22) and (3.23)

$$|Nx(t_n)-Nx(t_m)|\longrightarrow 0+\ \text{ as } n,m\to\infty \text{ uniformly for all } x\in\mathcal{A}.$$

This implies that $N(\mathcal{A})$ is equi-regulated on $[0,T]$.

Now for $x \in \mathcal{A}$ we have

$$\begin{aligned}(3.24)\quad |N_1x(t)| &\le |a| + \frac{\eta(t)}{\eta(T^*)}\left[|b-a| + \int_0^{T^*}\int_0^s |A(s)|\cdot|f(v,x(v))|\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\right]\\ &\le |a| + L_1|b-a| + L_1\int_0^T\int_0^s [a(v)|x|_\infty^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s),\end{aligned}$$

and

$$\begin{aligned}(3.25)\qquad |N_2x(t)| &\le \int_0^t\int_0^s |A(s)|\cdot|f(v,x(v))|\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\\ &\le \int_0^T\int_0^s [a(v)|x|_\infty^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).\end{aligned}$$

So (3.24) and (3.25) imply

$$\begin{aligned}(3.26)\quad |Nx(t)| &\le |N_1x(t)| + |N_2x(t)|\\ &\le |a| + L_1|b-a| + (L_1+1)\int_0^T\int_0^s [a(v)|x|_\infty^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s)\\ &\le |a| + L_1|b-a| + (L_1+1)\int_0^T\int_0^s [a(v)L^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).\end{aligned}$$

This implies that the set $\{Nx : x \in \mathcal{A}\}$ is bounded. Hence, by Corollary 3.3, $N(\mathcal{A})$ is relatively compact, i.e., the operator N is compact.

(III)($\mathbf{K} = \{x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T]) : x = kNx$ for some $k \in (0,1)\}$ is bounded.)

Let $x \in \mathbf{K}$. Then, considering (3.26), for every $t \in [0,T]$ we have

$$\begin{aligned}|x(t)| &= |kNx(t)| \le |Nx(t)|\\ &\le |a| + L_1|b-a| + (L_1+1)\int_0^T\int_0^s [a(v)|x|_\infty^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).\end{aligned}$$

This implies that

$$|x|_\infty \le |a| + L_1|b-a| + (L_1+1)\int_0^T\int_0^s [a(v)|x|_\infty^\lambda + b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s).$$

Thus, if we assume that $|x|_\infty > 1$, then $1/|x|_\infty^\lambda \le 1$. So we have

$$|x|_\infty^{1-\lambda} \le |a| + L_1|b-a| + (L_1+1)\int_0^T\int_0^s [a(v)+b(v)]\,\mathrm{d}\alpha_2(v)\,\mathrm{d}\alpha_1(s) \equiv \gamma.$$

That is,

$$|x|_\infty \le \gamma^{1/(1-\lambda)} \equiv M_1.$$

Hence

$$|x|_\infty \le \max\{1, M_1\}.$$

This implies that $\mathbf{K}$ is bounded.

So by (I), (II) and (III), and by Theorem 3.4 there exists $x \in \mathbf{G}_{\mathrm{L}}^{\alpha_1}([0,T])$ such that $x = Nx$. The proof is complete. □

Now let

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < T = t_{n+1}.$$

Then for $k = \overline{1,n}$ we define

$$\phi(t) = \begin{cases} t, & \text{if } t \in [0,t_1], \\ t+k, & \text{if } t \in (t_k, t_{k+1}], \end{cases}$$

and for $k = \overline{1, T-2}$ ($T \in [5,\infty) \cap \mathbf{N}$) we define

$$\psi(t) = \begin{cases} 0, & \text{if } t \in [0,1], \\ k, & \text{if } t \in (k,k+1], \\ t, & \text{if } t \in (T-1,T]. \end{cases}$$

From now on, a function $g : (c,d) \longrightarrow \mathbf{R}$ is defined as $g(t) = t + \lambda, \forall t \in (c,d)$, where $\lambda \in \mathbf{R}$. We need the following result.

Lemma 3.6. *Let $f : (c,d) \longrightarrow \mathbf{R}$. If f is continuous at $t \in (c,d)$ and $f_g'(t)$ exists, then $f'(t)$ exists also and*

$$f_g'(t) = f'(t).$$

Proof. Since $f_g'(t)$ exists, for every $\varepsilon > 0$ there is $\rho > 0$ such that for every $\eta, \delta \in (0,\rho)$ we have

$$\left|\frac{f(t+\eta)-f(t-\delta)}{\eta+\delta} - f_g'(t)\right| = \left|\frac{f(t+\eta)-f(t-\delta)}{g(t+\eta)-g(t-\delta)} - f_g'(t)\right| < \varepsilon.$$

Since f is continuous at t, we have

$$\left|\frac{f(t+\eta)-f(t)}{\eta} - f_g'(t)\right| = \lim_{\delta\to 0+}\left|\frac{f(t+\eta)-f(t-\delta)}{\eta+\delta} - f_g'(t)\right| \le \varepsilon.$$

This implies that $f'_+(t)$ exists and $f'_+(t) = f'_g(t)$. Similarly we can show that $f'_-(t)$ exists and $f'_-(t) = f'_g(t)$. This completes the proof. □

Corollary 3.7. *Assume the same conditions as in Theorem 3.5, where* $\alpha_1(t) = \alpha_2(t) = t,\ \forall t \in [0,T]$. *Then the following second-order differential equation with two-point boundary conditions*

$$x''(t) = f(t, x(t)),\ \forall t \in [0,T], \tag{3.27}$$

$$x(0) = a,\ x(T) = b,$$

has al least one solution.

Proof. Obviously $\alpha = (\alpha_1, \alpha_2) \in \mathbf{T}_1$. Since by Definition 2.5 $\mathbf{C}(x) = \mathbf{C}(x'_\alpha) = [0,T]$, by Lemma 3.6 $x'_\alpha(t) = x'_{\alpha_1}(t) = x'(t)$ and this implies $\mathbf{C}(x) = \mathbf{C}(x') = [0,T]$, and so again by Lemma 3.6 we get

$$x''_\alpha(t) = (x'_\alpha)'_{\alpha_2}(t) = (x')'_{\alpha_2}(t) = (x')'(t) = x''(t).$$

Thus by Theorem 3.5 the proof is complete. □

For Corollary 3.8 for $f \in \mathbf{G}([0,T])$ we define $\Delta f(t_k) = f(t_k+) - f(t_k-),\ \forall k = \overline{1,n}$.

Corollary 3.8. *Assume the same conditions as in Theorem 3.5, where* $\alpha_1 = \alpha_2 = \phi$. *Then the following second-order differential equation with impulses*

$$\begin{cases} x''(t) = f(t, x(t)), t \neq t_k,\ k = \overline{1,n}, \\ \Delta x'(t_k) = f(t_k, x(t_k)),\ \Delta x(t_k) = 0, \\ x(0) = a,\ x(T) = b \end{cases} \tag{3.28}$$

has al least one solution.

Proof. Note that $\alpha = (\alpha_1, \alpha_2) = (\phi, \phi) \in \mathbf{T}_1, \mathbf{D}(\alpha) = \{t_1, t_2, \cdots, t_n\}$.

Assume that $t \in \mathbf{C}(\alpha)$. Then by Definition 2.5 both x and x'_α are also continuous at t. So by Lemma 3.6 and by the similar process to the proof of Corollary 3.7 we have

$$x'_\alpha(t) = x'(t) \text{ and } x''_\alpha(t) = x''(t). \tag{3.29}$$

And so, since by Definition 2.5 $x'_\alpha \in \mathbf{G}([0,T]), \forall k = \overline{1,n},\ x'_\alpha(t_k\pm) = x'(t_k\pm)$ exists. Note that

$$\lim_{\eta,\delta \to 0+} [\phi(t_k + \eta) - \phi(t_k - \delta)] = \phi(t_k+) - \phi(t_k-) = 1.$$

Considering the above facts, by the definition of Stieltjes derivatives, we have

$$x'_{\alpha}(t_k) = \lim_{\eta,\delta\to 0+} \frac{x(t_k+\eta) - x(t_k-\delta)}{\phi(t_k+\eta) - \phi(t_k-\delta)} = \frac{x(t_k+) - x(t_k-)}{1} = \Delta x(t_k), \tag{3.30}$$

and by (3.29)

$$\begin{aligned} x''_{\alpha}(t_k) &= (x'_{\alpha})'_{\alpha_2}(t_k) = (x')'_{\phi}(t_k) = \lim_{\eta,\delta\to 0+} \frac{x'(t_k+\eta) - x'(t_k-\delta)}{\phi(t_k+\eta) - \phi(t_k-\delta)} \\ &= x'(t_k+) - x'(t_k-) = \Delta x'(t_k). \end{aligned} \tag{3.31}$$

Thus (3.29), (3.30) and (3.31), and Theorem 3.5 complete the proof. □

For Corollary 3.9 we define

$$\Delta x(n) = x(n+1) - x(n),\ \Delta^2 x(n) = \Delta x(n+1) - \Delta x(n),\ \forall n \in \mathbf{N}.$$

Here Δ and Δ^2 are called as the first- and the second-order forward difference operators, respectively.

Corollary 3.9. *Assume the same conditions as in Theorem 3.5, where $\alpha_1 = \alpha_2 = \psi$. Then the following second-order difference equation with two-point boundary conditions*

$$\begin{aligned} \Delta^2 x(n) &= f(n, x(n)),\ n = \overline{1, T-3}, T \in \mathbf{N}, \\ x(0) &= a,\ x(T-1) = b, \end{aligned} \tag{3.32}$$

has al least one solution.

Proof. Note that $\alpha = (\alpha_1, \alpha_2) = (\psi, \psi) \in \mathbf{T}_2$, $\mathbf{J}(\alpha) = \{1, 2, \cdots, T-1\} \cup (T-1, T]$. Now, for sufficiently small $\eta, \delta > 0$ we have

$$\forall n = \overline{1, T-2},\ \psi(n+\eta) - \psi(n-\delta) = 1,\ \sigma(n+\eta) = n+1,\ \sigma(n-\delta) = n.$$

This implies that since by Definition 2.5 $x = x \circ \sigma$ on $[0, T]$ we have, $\forall n = \overline{1, T-2}$,

$$\begin{aligned} x'_{\alpha}(n) &= \lim_{\eta,\delta\to 0+} \frac{x(n+\eta) - x(n-\delta)}{\alpha_1(n+\eta) - \alpha_1(n-\delta)} = \lim_{\eta,\delta\to 0+} \frac{x \circ \sigma(n+\eta) - x \circ \sigma(n-\delta)}{\psi(n+\eta) - \psi(n-\delta)} \\ &= x(n+1) - x(n) = \Delta x(n). \end{aligned} \tag{3.33}$$

Then, for $n = \overline{1, T-3}$, by (3.33) we get

$$\begin{aligned} x''_{\alpha}(n) &= (x'_{\alpha} \circ \sigma)'_{\alpha_2}(n) = \lim_{\eta,\delta\to 0+} \frac{x'_{\alpha} \circ \sigma(n+\eta) - x'_{\alpha} \circ \sigma(n-\delta)}{\alpha_2(n+\eta) - \alpha_2(n-\delta)} \\ &= \lim_{\eta,\delta\to 0+} \frac{x'_{\alpha}(n+1) - x'_{\alpha}(n)}{\psi(n+\eta) - \psi(n-\delta)} = x'_{\alpha}(n+1) - x'_{\alpha}(n) \\ &= \Delta x(n+1) - \Delta x(n) = \Delta^2 x(n). \end{aligned} \tag{3.34}$$

Thus, (3.34) and Theorem 3.5, where $T^* = T - 1$, complete the proof. □

Remark 3.10. Considering the above corollaries, we conclude that the higher-order Stieltjes derivatives can be used to unify various equations.

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