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SOME INEQUALITIES FOR THE WEIGHTED CHAOTICALLY GEOMETRIC MEAN

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Abstract. In this paper we obtain some new inequalities for the weighted chaotically geometric mean of two positive operators on a complex Hilbert space.

1. Introduction

For positive operators A and B consider the *weighted arithmetic* and *chaotically geometric* means

$$A\nabla_{\alpha}B := (1-\alpha)A + \alpha B$$

and

$$A\lozenge_{\alpha}B := \exp\left[(1-\alpha)\ln A + \alpha \ln B\right]$$

with $\alpha \in [0,1]$.

We recall that *Specht's ratio* is defined by [6]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1,\infty), \\ \\ 1 & \text{if } h = 1. \end{cases} \tag{1.1}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

It has been shown in [4] that, if $0 < mI \le A,\ B \le MI$, for some scalars $m < M$ and $h := \frac{M}{m}$, then

$$S^{-1}(h)\, A\lozenge_{\alpha}B \le A\nabla_{\alpha}B \le S(h)\, A\lozenge_{\alpha}B \tag{1.2}$$

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for any $\alpha \in [0,1]$, where I is the identity operator.

With the same assumptions for A and B we also have the additive version obtained in [5]

$$-L(m,M)\ln S(h) I \le A\nabla_\alpha B - A\lozenge_\alpha B \le L(m,M)\ln S(h) I, \tag{1.3}$$

where

$$L(m,M) := \begin{cases} \frac{M-m}{\ln M-\ln m}, & \text{if } M \neq m, \\ M, & \text{if } M \neq m \end{cases}$$

is the *logarithmic mean.*

Motivated by these results, we establish in this paper other inequalities involving the chaotically geometric mean.

2. The Results

We have:

Theorem 2.1. *If $0 < mI \le A,\ B \le MI$ for some scalars $m < M$, then*

$$\begin{aligned} & -\frac{2}{\ln M-\ln m}\left|\ln A\nabla_\alpha \ln B - \ln G(mM) I\right| \\ & \le L(m,M)\left(\ln A\nabla_\alpha \ln B + U(m,M) I\right) - \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2 I - A\lozenge_\alpha B \\ & \le \frac{2}{\ln M-\ln m}\left|\ln A\nabla_\alpha \ln B - \ln G(mM) I\right|, \end{aligned} \tag{2.1}$$

where

$$U(m,M) := \frac{m\ln M - M\ln m}{M-m},\ G(m,M) := \sqrt{mM}$$

and

$$\ln A\nabla_\alpha \ln B = (1-\alpha)\ln A + \alpha \ln B.$$

Proof. Recall the following result obtained by Dragomir in 2006 [1] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ & \le \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_j x_j\right) \end{aligned} \tag{2.2}$$

$$\leq n \max_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space E, $x_1, ..., x_n$ are vectors in C and $p_1, ..., p_n$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (2.2) that

$$\begin{aligned}(2.3)\qquad & 2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y] \\ & \leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$.

If we take $\Phi(x) = \exp(x)$, then we get from (2.3)

$$\begin{aligned}(2.4)\qquad & 2\min\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\exp(x) + (1-\nu)\exp(y) - \exp[\nu x + (1-\nu)y] \\ & \leq 2\max\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$.

Let $0 < m < M$ and take $x = \ln M$ and $y = \ln m$ to get

$$\begin{aligned}(2.5)\qquad & 2\min\{\nu, 1-\nu\}\left(\frac{m+M}{2} - \sqrt{mM}\right) \\ & \leq \nu M + (1-\nu)m - \exp[\nu\ln M + (1-\nu)\ln m] \\ & \leq 2\max\{\nu, 1-\nu\}\left(\frac{m+M}{2} - \sqrt{mM}\right)\end{aligned}$$

for any $\nu \in [0, 1]$.

Let $z \in [\ln m, \ln M]$ and take $\nu \in [0, 1]$ such that $\nu\ln M + (1-\nu)\ln m = z$, namely $\nu = \frac{z-\ln m}{\ln M - \ln m}$.

Since

$$\begin{aligned}\min\{\nu, 1-\nu\} &= \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{z-\ln m}{\ln M - \ln m} - \frac{1}{2}\right| \\ &= \frac{1}{2} - \frac{1}{\ln M - \ln m}\left|z - \ln G(mM)\right|\end{aligned}$$

and

$$\max\{\nu, 1-\nu\} = \frac{1}{2} + \frac{1}{\ln M - \ln m}\left|z - \ln G(mM)\right|,$$

then by (2.5) we have

$$\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(1-\frac{2}{\ln M-\ln m}\left|z-\ln G\left(mM\right)\right|\right) \tag{2.6}$$
$$\leq \frac{z-\ln m}{\ln M-\ln m}M+\frac{\ln M-z}{\ln M-\ln m}m-\exp z$$
$$\leq \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(1+\frac{2}{\ln M-\ln m}\left|z-\ln G\left(mM\right)\right|\right)$$

for any $z\in[\ln m,\ln M]$.

If X is a selfadjoint operator with $\mathrm{Sp}\left(X\right)\subset[\ln m,\ln M]$, then by (2.6) we have

$$\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I-\frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right|\right) \tag{2.7}$$
$$\leq \frac{X-\ln mI}{\ln M-\ln m}M+\frac{\ln MI-X}{\ln M-\ln m}m-\exp X$$
$$\leq \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I+\frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right|\right).$$

Since

$$\frac{X-\ln mI}{\ln M-\ln m}M+\frac{\ln MI-X}{\ln M-\ln m}m \tag{2.8}$$
$$=\frac{M-m}{\ln M-\ln m}X+\frac{m\ln M-M\ln m}{\ln M-\ln m}I$$
$$=L\left(m,M\right)\left(X+\frac{m\ln M-M\ln m}{M-m}I\right)$$
$$=L\left(m,M\right)\left(X+U\left(m,M\right)I\right),$$

then by (2.7) and (2.8) we get

$$\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I-\frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right|\right) \tag{2.9}$$
$$\leq L\left(m,M\right)\left(X+U\left(m,M\right)I\right)-\exp X$$
$$\leq \frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2\left(I+\frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right|\right),$$

that is equivalent to

$$-\frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right| \tag{2.10}$$
$$\leq L\left(m,M\right)\left(X+U\left(m,M\right)I\right)-\frac{1}{2}\left(\sqrt{M}-\sqrt{m}\right)^2 I-\exp X$$
$$\leq \frac{2}{\ln M-\ln m}\left|X-\ln G\left(mM\right)I\right|.$$

This inequality is of interest in itself.

If we take $X = \ln A\nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\operatorname{Sp}(X) \subset [\ln m, \ln M]$ and by (2.10) we get the desired result (2.1). $\square$

We also have:

Theorem 2.2. *With the assumptions of Theorem 2.1 we have*

$$\begin{aligned} 0 &\le L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\lozenge_\alpha B \\ &\le L(m,M)(\ln MI - \ln A\nabla_\alpha \ln B)(\ln A\nabla_\alpha \ln B - \ln mI) \\ &\le \frac{1}{4}(\ln M - \ln m)(M-m)I, \end{aligned} \tag{2.11}$$

for any $\alpha \in [0,1]$.

Proof. We use the following inequality for convex functions, see for instance [2]: If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$ we have

$$\begin{aligned} 0 &\le (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b) \\ &\le \nu(1-\nu)(b-a)\left[f'(b) - f'(a)\right]. \end{aligned} \tag{2.12}$$

If we write this inequality for the convex function $f(x) = \exp(x)$, then we have

$$\begin{aligned} 0 &\le (1-\nu)\exp a + \nu \exp b - \exp((1-\nu)a + \nu b) \\ &\le \nu(1-\nu)(b-a)[\exp b - \exp a], \end{aligned} \tag{2.13}$$

for any $a, b \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $0 < m < M$ and take $b = \ln M$ and $a = \ln m$ to get

$$\begin{aligned} 0 &\le \nu M + (1-\nu)m - \exp[\nu \ln M + (1-\nu)\ln m] \\ &\le \nu(1-\nu)(\ln M - \ln m)(M-m). \end{aligned} \tag{2.14}$$

Let $z \in [\ln m, \ln M]$ and take $\nu \in [0,1]$ such that $\nu \ln M + (1-\nu)\ln m = z$, namely $\nu = \frac{z - \ln m}{\ln M - \ln m}$. Then by (2.14) and upon some simple calculations we get

$$0 \le L(m,M)(z + U(m,M)) - \exp z \le L(m,M)(\ln M - z)(z - \ln m)$$

for any $z \in [\ln m, \ln M]$.

Since

$$(\ln M - z)(z - \ln m) \le \frac{1}{4}(\ln M - \ln m)^2,$$

then we have

$$0 \le L(m,M)(z+U(m,M)) - \exp z \le L(m,M)(\ln M - z)(z - \ln m) \le \frac{1}{4}(\ln M - \ln m)(M-m) \tag{2.15}$$

for any $z \in [\ln m, \ln M]$.

If X is a selfadjoint operator with $\mathrm{Sp}(X) \subset [\ln m, \ln M]$, then by (2.15) we have

$$\begin{aligned} 0 &\le L(m,M)(X+U(m,M)I) - \exp X \\ &\le L(m,M)(\ln MI - X)(X - \ln mI) \\ &\le \frac{1}{4}(\ln M - \ln m)(M-m)I, \end{aligned} \tag{2.16}$$

which is an inequality of interest in itself as well.

If we take $X = \ln A \nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\mathrm{Sp}(X) \subset [\ln m, \ln M]$ and by (2.16) we get the desired result (2.11). □

We also have:

Theorem 2.3. *With the assumptions of Theorem 2.1 we have*

$$\begin{aligned} &\frac{1}{2}m(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B) \\ &\le L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\Diamond_\alpha B \\ &\le \frac{1}{2}M(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B), \end{aligned} \tag{2.17}$$

for any $\alpha \in [0,1]$.

Proof. We use the following result for twice differentiable functions, see for instance [3]:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that

$$d \le f''(t) \le D \text{ for any } t \in \mathring{I}, \tag{2.18}$$

then

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)d(b-a)^2 &\le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a + \nu b) \\ &\le \frac{1}{2}\nu(1-\nu)D(b-a)^2 \end{aligned} \tag{2.19}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.

If we write this inequality for the convex function $f(x) = \exp(x)$, then we have

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)(b-a)^2 \exp(\min\{a,b\}) \\ &\leq (1-\nu)\exp a + \nu \exp b - \exp((1-\nu)a + \nu b) \\ &\leq \frac{1}{2}\nu(1-\nu)(b-a)^2 \exp(\max\{a,b\}), \end{aligned} \tag{2.20}$$

for any $a, b \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $0 < m < M$ and take $b = \ln M$ and $a = \ln m$ to get

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)(\ln M - \ln m)^2 m \\ &\leq (1-\nu)m + \nu M - \exp((1-\nu)\ln m + \nu \ln M) \\ &\leq \frac{1}{2}\nu(1-\nu)(\ln M - \ln m)^2 M, \end{aligned} \tag{2.21}$$

for any $\nu \in [0,1]$.

Let $z \in [\ln m, \ln M]$. If we take $\nu = \frac{z - \ln m}{\ln M - \ln m} \in [0,1]$, then we get

$$\begin{aligned} \frac{1}{2}(z-\ln m)(\ln M - z)m &\leq L(m,M)(z + U(m,M)) - \exp z \\ &\leq \frac{1}{2}(z - \ln m)(\ln M - z)M, \end{aligned} \tag{2.22}$$

for any $z \in [\ln m, \ln M]$.

If X is a selfadjoint operator with $\operatorname{Sp}(X) \subset [\ln m, \ln M]$, then by (2.22) we have

$$\begin{aligned} \frac{1}{2}m(X - \ln mI)(\ln MI - X) &\leq L(m,M)(X + U(m,M)I) - \exp X \\ &\leq \frac{1}{2}M(X - \ln mI)(\ln MI - X), \end{aligned} \tag{2.23}$$

which is an inequality of interest in itself as well.

If we take $X = \ln A \nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\operatorname{Sp}(X) \subset [\ln m, \ln M]$ and by (2.23) we get the desired result (2.17). □

Remark 2.4. Since

$$(\ln A\nabla_\alpha \ln B - \ln mI)(\ln MI - \ln A\nabla_\alpha \ln B) \leq \frac{1}{4}(\ln M - \ln m)^2 I,$$

then from (2.17) we get the following simpler upper bound

$$L(m,M)(\ln A\nabla_\alpha \ln B + U(m,M)I) - A\lozenge_\alpha B \leq \frac{1}{8}M(\ln M - \ln m)^2 I. \tag{2.24}$$

In [7], M. Tominaga obtained the following reverses of *Young's inequality*

$$\left(a^{1-\nu}b^\nu \leq\right)(1-\nu)a + \nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \tag{2.25}$$

and

$$(0 \leq) \, (1-\nu) \, a + \nu b - a^{1-\nu} b^{\nu} \leq L\,(a,b) \ln S\left(\frac{a}{b}\right) \tag{2.26}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Theorem 2.5. *With the assumptions of Theorem 2.1 we have*

$$(A \lozenge_\alpha B \leq) \, L\,(m,M) \, (\ln A \nabla_\alpha \ln B + U\,(m,M)\, I) \leq S\left(\frac{M}{m}\right) A \lozenge_\alpha B \tag{2.27}$$

and

$$\begin{aligned}(0 \leq) \, & L\,(m,M) \, (\ln A \nabla_\alpha \ln B + U\,(m,M)\, I) - A \lozenge_\alpha B \\ & \leq L\,(m,M) \ln S\left(\frac{M}{m}\right) I\end{aligned} \tag{2.28}$$

for any $\alpha \in [0,1]$.

Proof. For $0 < m < M$ we have by (2.25) that

$$(1-\nu)\, m + \nu M \leq S\left(\frac{m}{M}\right) m^{1-\nu} M^{\nu} = S\left(\frac{M}{m}\right) \exp\left((1-\nu) \ln m + \nu \ln M\right) \tag{2.29}$$

for any $\nu \in [0,1]$.

Similarly,

$$(1-\nu)\, m + \nu M - \exp\left((1-\nu) \ln m + \nu \ln M\right) \leq L\,(m,M) \ln S\left(\frac{M}{m}\right) \tag{2.30}$$

for any $\nu \in [0,1]$.

Let $z \in [\ln m, \ln M]$. If we take $\nu = \frac{z - \ln m}{\ln M - \ln m} \in [0,1]$ in (2.29) and (2.29) then we get

$$L\,(m,M)\,(z + U\,(m,M)) \leq S\left(\frac{M}{m}\right) \exp z \tag{2.31}$$

and

$$L\,(m,M)\,(z + U\,(m,M)) - \exp z \leq L\,(m,M) \ln S\left(\frac{M}{m}\right) \tag{2.32}$$

for any $z \in [\ln m, \ln M]$.

If X is a selfadjoint operator with $\mathrm{Sp}\,(X) \subset [\ln m, \ln M]$, then by (2.31) and (2.32) we have

$$L\,(m,M)\,(X + U\,(m,M)\, I) \leq S\left(\frac{M}{m}\right) \exp X \tag{2.33}$$

and

$$L\,(m,M)\,(X + U\,(m,M)\, I) - \exp X \leq L\,(m,M) \ln S\left(\frac{M}{m}\right) I, \tag{2.34}$$

which are inequalities of interest in themselves.

If we take $X = \ln A \nabla_\alpha \ln B$, $\alpha \in [0,1]$, then $\mathrm{Sp}(X) \subset [\ln m, \ln M]$ and by (2.33) and (2.34) we deduce the desired results (2.27) and (2.28). $\square$

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