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COMMON COUPLED FIXED POINT RESULTS FOR HYBRID PAIR OF MAPPING UNDER GENERALIZED $(\psi, \theta, \varphi)-$CONTRACTION WITH APPLICATION

Amrish Handa

Abstract. We introduce (CLRg) property for hybrid pair $F : X \times X \to 2^X$ and $g : X \to X$. We also introduce joint common limit range (JCLR) property for two hybrid pairs $F,\ G : X \times X \to 2^X$ and $f,\ g : X \to X$. We also establish some common coupled fixed point theorems for hybrid pair of mappings under generalized $(\psi, \theta, \varphi)-$contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. As an application, we study the existence and uniqueness of the solution to an integral equation. We also give an example to demonstrate the degree of validity of our hypothesis. The results we obtain generalize, extend and improve several recent results in the existing literature.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} \max\left\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A,\ B \in CL(X)$, where $D(x,\ A) = \inf_{a\in A} d(x,\ a)$ denotes the distance from x to $A \subset X$. For simplicity, if $x \in X$, we denotes $g(x)$ by gx.

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Nadler [23] extended the famous Banach Contraction Principle from single-valued mapping to multivalued mapping. Markin [22] initiated to study the existence of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric which was further studied by many authors under different contractive conditions. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics.

The Banach contraction principle is one of very popular tools in solving the existence in many problems of mathematical analysis. Due to its simplicity and usefulness, there are a lot of generalizations of this principle in the literature. Ran and Reurings [25] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and López [24] extended the result of Ran and Reurings [25] and applied their main theorems to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

Guo and Lakshmikantham [15] introduced the notion of coupled fixed point and initiated the investigation of multidimensional fixed point theory. Later on, Gnana-Bhaskar and Lakshmikantham [5] obtained some coupled fixed point theorems for mapping $F : X \times X \to X$ (where X is a partially ordered metric space) by defining the notion of mixed monotone mapping. After that, Lakshmikantham and Ciric [19] proved coupled fixed/coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ by introducing the concept of the mixed g-monotone property. They also illustrated these results by proving the existence and uniqueness of the solution for periodic boundary value problems. Samet et al. [28] claimed that most of the coupled fixed point theorems for single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems. Many authors focused on coupled fixed point theory including ([3], [4], [7], [8], [18], [21], [26], [29], [32]).

The concepts related to coupled fixed point theory for multivalued mappings were extended by Abbas et al. [1] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few researcher gave attention to coupled fixed point problems for hybrid pair of mappings including ([9], [10], [11], [12], [13], [14], [20], [30]).

In [1], Abbas et al. introduced the following for multivalued mappings:

Definition 1. Let X be a nonempty set, $F : X \times X \to 2^X$ and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a *coupled fixed point* of F if $x \in F(x, y)$ and $y \in F(y, x)$.

(2) a *coupled coincidence point* of hybrid pair (F, g) if $gx \in F(x,y)$ and $gy \in F(y,x)$.

(3) a *common coupled fixed point* of hybrid pair (F, g) if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C(F,g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 2. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair (F, g) is called *$w-$compatible* if $gF(x,y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

Definition 3. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called *$F-$weakly commuting* at some point $(x,y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Lemma 4 ([27])**.** *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b\in B} d(a, b)$.*

Sintunavarat and Kumam [31] defined the notion of common limit in the range property in fuzzy metric space. Chauhan et al. [6] introduce the notion of the joint common limit in the range of mappings property called (JCLR) property and proved a common fixed point theorem for a pair of weakly compatible mappings using (JCLR) property in fuzzy metric space.

Definition 5 ([31])**.** Suppose that (X, d) is a metric space and $f,\ g : X \to X$ are two mappings. Then f and g are said to satisfy the *common limit in the range of g property* (CLRg-property) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \text{ for some } x \in X.$$

Definition 6 ([6])**.** For mappings $F,\ G,\ f,\ g : X \to X$, the pairs $(F,\ f)$ and $(G,\ g)$ are said to have (JCLR)*property* if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and x, $y \in X$ such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gy_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = fx = gy.$$

Khan and Sumitra [18] established the concept of (CLRg) property for mappings $F : X \times X \to X$ and $g : X \to X$.

Definition 7 ([18])**.** Let $(X,\ d)$ be a metric space and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then F and g are said to satisfy the *common limit in the range of g property* (CLRg-property) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some x, y in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = gx, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = gy. \end{aligned}$$

In [2], Ahmed and Nafadi introduced the notion of common limit range property (CLR property) for two hybrid pairs of mappings in fuzzy metric spaces and proved common fixed point theorems using (CLR) property for these mappings with implicit relation.

Definition 8 ([2])**.** Mappings $F : X \to CB(X)$ and $g : X \to X$ are said to satisfy the *common limit in the range of g property* (CLRg-property) if there exist sequences $\{x_n\}$ in X, some x in X and A in $CB(X)$ such that

$$\lim_{n\to\infty} gx_n = gx \in A = \lim_{n\to\infty} Fx_n.$$

Definition 9 ([2])**.** Mappings $F,\ G : X \to CB(X)$ and $f,\ g : X \to X$ are said to satisfy *joint common limit in the range (JCLR) property* if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some $x,\ y$ in X and $A,\ B$ in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} fx_n &= fx \in A = \lim_{n\to\infty} Fx_n, \\ \lim_{n\to\infty} gy_n &= gy \in B = \lim_{n\to\infty} Fy_n. \end{aligned}$$

Definition 10 ([29])**.** An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

In this paper, we introduce (CLRg) property for hybrid pair $F : X \times X \to 2^X$ and $g : X \to X$. We also introduce joint common limit range (JCLR) property for two hybrid pairs $F, G : X \times X \to 2^X$ and $f, g : X \to X$. We prove a common coupled fixed point theorems for hybrid pair of mappings under generalized $(\psi, \theta, \varphi)-$contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. As an application, we study the existence and uniqueness of the solution to an integral equation. We modify, improve, sharpen, enrich and generalize the results of Alotaibi and Alsulami [3], Alsulami [4], Gnana-Bhaskar and Lakshmikantham [5], Harjani et al. [16], Harjani and Sadarangani [17], Lakshmikantham and Ciric [19], Luong and Thuan [21], Nieto and Rodriguez-Lopez [24], Ran and Reurings [25], Razani and Parvaneh [26] and many other famous results in the literature. The effectiveness of our generalization is demonstrated with the help of an example.

2. Main Results

Definition 11. Let $(X,\ d)$ be a metric space. Mappings $F : X \times X \to 2^X$ and $g : X \to X$ are said to satisfy the *common limit in the range of g property* (CLRg-property) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some x, y in X and A, B in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= gx \in A = \lim_{n\to\infty} F(x_n,\ y_n), \\ \lim_{n\to\infty} gy_n &= gy \in B = \lim_{n\to\infty} F(y_n,\ x_n). \end{aligned}$$

Definition 12. Let $(X,\ d)$ be a metric space. For mappings $f,\ g : X \to X$ and $F,\ G : X \times X \to 2^X$, the pairs $(F,\ f)$ and $(G,\ g)$ are said to have *joint common limit range (JCLR) property* if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some x, y, u, v in X and A, B, C, D in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= A,\ \lim_{n\to\infty} G(u_n,\ v_n) = B, \\ \text{then } \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = fx = gu \in A \cap B, \\ \text{and } \lim_{n\to\infty} F(y_n,\ x_n) &= C,\ \lim_{n\to\infty} G(v_n,\ u_n) = D, \\ \text{then } \lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = fy = gv \in C \cap D. \end{aligned}$$

Theorem 13. *Let $(X,\ d)$ be a metric space. Suppose $F : X \times X \to K(X)$ and $g : X \to X$ are two mappings for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} \psi(H(F(x,\ y),\ F(u,\ v))) &\leq \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ &\quad -\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}), \end{aligned} \tag{2.1}$$

for all $x,\ y,\ u,\ v \in X$*, where* $\theta(0) = \varphi(0) = 0$ *and* $\psi(t) - \theta(t) + \varphi(t) > 0$ *for all* $t > 0$*. Furthermore assume that* $F(X \times X) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X*. Then* F *and* g *have a coupled coincidence point. Moreover,* F *and* g *have a common coupled fixed point, if one of the following conditions holds:*

(a) F *and* g *are* $w-$*compatible.* $\lim_{n\to\infty} g^n x = u$ *and* $\lim_{n\to\infty} g^n y = v$ *for some* $(x,\ y) \in C(F,\ g)$ *and for some* $u,\ v \in X$ *and* g *is continuous at* u *and* v*.*

(b) g *is* $F-$*weakly commuting for some* $(x,\ y) \in C(F,\ g)$ *and* gx *and* gy *are fixed points of* g*, that is,* $g^2x = gx$ *and* $g^2y = gy$*.*

(c) g *is continuous at* x *and* y*.* $\lim_{n\to\infty} g^n u = x$ *and* $\lim_{n\to\infty} g^n v = y$ *for some* $(x,\ y) \in C(F,\ g)$ *and for some* $u,\ v \in X$*.*

(d) $g(C(F,\ g))$ *is a singleton subset of* $C(F,\ g)$*.*

Proof. Let $x_0,\ y_0 \in X$ be arbitrary. Then $F(x_0,\ y_0)$ and $F(y_0,\ x_0)$ are well defined. Choose $gx_1 \in F(x_0,\ y_0)$ and $gy_1 \in F(y_0,\ x_0)$, as $F(X \times X) \subseteq g(X)$. Since $F : X \times X \to K(X)$, therefore by Lemma 4, there exist $z_1 \in F(x_1,\ y_1)$ and $z_2 \in F(y_1,\ x_1)$ such that

$$\begin{aligned} d(gx_1,\ z_1) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)),\\ d(gy_1,\ z_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist $x_2,\ y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1,\ gx_2) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)),\\ d(gy_1,\ gy_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$ such that

$$\begin{aligned} d(gx_{n+1},\ gx_{n+2}) &\leq H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1})),\\ d(gy_{n+1},\ gy_{n+2}) &\leq H(F(y_n,\ x_n),\ F(y_{n+1},\ x_{n+1})), \end{aligned}$$

which, by the monotonicity of ψ and (2.1), implies

$$\begin{aligned} &\psi(d(gx_{n+1},\ gx_{n+2}))\\ \leq\ &\psi(H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1})))\\ \leq\ &\theta(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\})\\ &-\varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}). \end{aligned}$$

Similarly

$$\begin{aligned} & \psi(d(gy_{n+1},\ gy_{n+2})) \\ \leq\ & \theta(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}) \\ & -\varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\{\psi(d(gx_{n+1},\ gx_{n+2})),\ \psi(d(gy_{n+1},\ gy_{n+2}))\} \\ \leq\ & \theta(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}) \\ & -\varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}). \end{aligned}$$

Since ψ is non-decreasing, it follows that

$$\begin{aligned} & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\}) \\ \leq\ & \theta(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}) \\ & -\varphi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}). \end{aligned} \tag{2.2}$$

But we have $\psi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) - \theta(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) + \varphi(\max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}) > 0$.Then

$$\begin{aligned} & \frac{\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})}{\psi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\})} \\ \leq\ & \frac{\theta\left(\max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}), \\ d(gy_n,\ gy_{n+1}) \end{array}\right\}\right) - \varphi\left(\max\left\{\begin{array}{c} d(gx_n,\ gx_{n+1}), \\ d(gy_n,\ gy_{n+1}) \end{array}\right\}\right)}{\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right)} < 1. \end{aligned}$$

Thus

$$\begin{aligned} & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\}) \\ <\ & \psi(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned} & \max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} \\ <\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}. \end{aligned}$$

This shows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ defined by

$$\delta_n = \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\},$$

is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} = \delta. \tag{2.3}$$

We shall prove that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Taking $n \to \infty$ in (2.2), by using the property of ψ, θ, φ and (2.3), we obtain

$$\psi(\delta) \le \theta(\delta) - \varphi(\delta),$$

so

$$\psi(\delta) - \theta(\delta) + \varphi(\delta) \le 0,$$

which is a contradiction. Thus, by (2.3), we get

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} = 0. \tag{2.4}$$

We now claim that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are Cauchy sequences in X. Suppose, to the contrary, that at least one of the sequences $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}_{n=0}^{\infty}$and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}_{n=0}^{\infty}$ such that

$$\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\} \ge \varepsilon \text{ for } n(k) > m(k) > k.$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$\max\{d(gx_{n(k)-1},\ gx_{m(k)}),\ d(gy_{n(k)-1},\ gy_{m(k)})\}) < \varepsilon.$$

Now, by triangle inequality, we have

$$\begin{aligned}
\varepsilon &\le r_k = \max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\} \\
&\le \max\{d(gx_{n(k)},\ gx_{n(k)-1}),\ d(gy_{n(k)},\ gy_{n(k)-1})\} \\
&\quad + \max\{d(gx_{n(k)-1},\ gx_{m(k)}),\ d(gy_{n(k)-1},\ gy_{m(k)})\} \\
&< \max\{d(gx_{n(k)},\ gx_{n(k)-1}),\ d(gy_{n(k)},\ gy_{n(k)-1})\} + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.4), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\} = \varepsilon. \tag{2.5}$$

By the triangle inequality, we have

$$\begin{aligned}
&\max\{d(gx_{n(k)+1},\ gx_{m(k)+1}),\ d(gy_{n(k)+1},\ gy_{m(k)+1})\} \\
\le\ &\max\{d(gx_{n(k)+1},\ gx_{n(k)}),\ d(gy_{n(k)+1},\ gy_{n(k)})\} \\
&+ \max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\} \\
&+ \max\{d(gx_{m(k)},\ gx_{m(k)+1}),\ d(gy_{m(k)},\ gy_{m(k)+1})\}.
\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (2.4) and (2.5), we have

$$\lim_{k\to\infty} \max\{d(gx_{n(k)+1},\ gx_{m(k)+1}),\ d(gy_{n(k)+1},\ gy_{m(k)+1})\} = \varepsilon. \tag{2.6}$$

Now, by the monotonicity of ψ and (2.1), implies

$$\begin{aligned}
& \psi(d(gx_{n(k)+1},\ gx_{m(k)+1})) \\
\leq\ & \psi(H(F(x_{n(k)},\ y_{n(k)}),\ F(x_{m(k)},\ y_{m(k)}))) \\
\leq\ & \theta(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}) \\
& -\varphi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \psi(d(gy_{n(k)+1},\ gy_{m(k)+1})) \\
\leq\ & \theta(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}) \\
& -\varphi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
& \max\{\psi(d(gx_{n(k)+1},\ gx_{m(k)+1})),\ \psi(d(gy_{n(k)+1},\ gy_{m(k)+1}))\} \\
\leq\ & \theta(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}) \\
& -\varphi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}).
\end{aligned}$$

Since ψ is non-decreasing, it follows that

$$\begin{aligned}
& \psi(\max\{d(gx_{n(k)+1},\ gx_{m(k)+1}),\ d(gy_{n(k)+1},\ gy_{m(k)+1})\}) \\
\leq\ & \theta(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}) \\
& -\varphi(\max\{d(gx_{n(k)},\ gx_{m(k)}),\ d(gy_{n(k)},\ gy_{m(k)})\}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (2.5), (2.6), we have

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction due to $\varepsilon > 0$. This shows that $\{gx_n\}_{n=0}^{\infty}$ and $\{gx_n\}_{n=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $x,\, y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{2.7}$$

Now, since $gx_{n+1} \in F(x_n,\, y_n)$ and $gy_{n+1} \in F(y_n,\, x_n)$, by using condition (2.1) and by the monotonicity of ψ, we get

$$\begin{aligned}
& \psi(D(gx_{n+1},\ F(x,\ y))) \\
\leq\ & \psi(H(F(x_n,\ y_n),\ F(x,\ y))) \\
\leq\ & \theta(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}) \\
& -\varphi(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}).
\end{aligned}$$

On taking $n \to \infty$ in the above inequality and by using the property of ψ, θ, φ and (2.7), we get

$$D(gx,\ F(x,\ y)) = 0. \text{ Similarly } D(gy,\ F(y,\ x)) = 0,$$

which implies that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, (x, y) is a coupled coincidence point of F and g. Hence $C(F, g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x, y) \in C(F, g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{2.8}$$

where $u, v \in X$. Since g is continuous at u and v, we have, by (2.8), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{2.9}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g), \text{ for all } n \geq 1,$$

that is,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x), \text{ for all } n \geq 1. \tag{2.10}$$

Now, by using (2.1), (2.10) and by the monotonicity of ψ, we obtain

$$\begin{aligned} & \psi(D(g^n x,\ F(u,\ v))) \\ \leq\ & \psi(H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v))) \\ \leq\ & \theta(\max\{d(g^n x,\ gu),\ d(g^n y,\ gv)\}) \\ & -\varphi(\max\{d(g^n x,\ gu),\ d(g^n y,\ gv)\}). \end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, by using (2.8), (2.9) and by the continuity of ψ, we get

$$D(gu,\ F(u,\ v)) = 0. \text{ Similarly } D(gv,\ F(v,\ u)) = 0,$$

which implies that

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u). \tag{2.11}$$

Now, from (2.9) and (2.11), we have

$$u = gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, (u, v) is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\, y) \in C(F,\, g)$, g is $F-$weakly commuting, that is $g^2x \in F(gx,\, gy)$ and $g^2y \in F(gy,\, gx)$ and $g^2x = gx$ and $g^2y = gy$. Thus $gx = g^2x \in F(gx,\, gy)$ and $gy = g^2y \in F(gy,\, gx)$, that is, $(gx,\, gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,\, y) \in C(F,\, g)$ and for some $u,\, v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$. Since g is continuous at x and y, then x and y are fixed points of g, that is, $gx = x$ and $gy = y$. Since $(x,\, y) \in C(F,\, g)$, so we obtain $x = gx \in F(x,\, y)$ and $y = gy \in F(y,\, x)$, that is, $(x,\, y)$ is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F,\, g)) = \{(x,\, x)\}$. Then $\{x\} = \{gx\} = F(x,\, x)$. Hence $(x,\, x)$ is a common coupled fixed point of F and g. □

If we put $g = I$ (the identity mapping) in the Theorem 13, we get the following result:

Corollary 14. *Let $(X,\, d)$ be a complete metric space, $F : X \times X \to K(X)$ be a mapping for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\, +\infty) \to [0,\, +\infty)$ and a lower semi-continuous function $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ such that*

$$\psi(H(F(x,\ y),\ F(u,\ v))) \le \theta(\max\{d(x,\ u),\ d(y,\ v)\}) - \varphi(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all $x,\, y,\, u,\, v \in X$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Then F has a coupled fixed point.

If we take $\psi(t) = \theta(t)$ in Theorem 13, we obtain the following corollary.

Corollary 15. *Let $(X,\, d)$ be a metric space. Suppose $F : X \times X \to K(X)$ and $g : X \to X$ are two mappings for which there exist an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ such that*

$$\begin{aligned} \psi(H(F(x,\ y),\ F(u,\ v))) \ &\le\ \psi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ &\quad -\varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}), \end{aligned} \tag{2.12}$$

for all $x,\, y,\, u,\, v \in X$, where $\varphi(0) = 0$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 13 holds.

If we put $g = I$ (the identity mapping) in Corollary 15, we get the following result:

Corollary 16. *Let $(X,\ d)$ be a complete metric space, $F : X \times X \to K(X)$ be a mapping for which there exist an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\psi(H(F(x,\ y),\ F(u,\ v))) \leq \psi(\max\{d(x,\ u),\ d(y,\ v)\}) - \varphi(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all $x,\ y,\ u,\ v \in X$, where $\varphi(0) = 0$. Then F has a coupled fixed point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Theorem 13, we get the following corollary.

Corollary 17. *Let $(X,\ d)$ be a metric space. Suppose $F : X \times X \to K(X)$ and $g : X \to X$ are two mappings satisfying*

$$H(F(x,\ y),\ F(u,\ v)) \leq k \max\{d(gx,\ gu),\ d(gy,\ gv)\}), \tag{2.13}$$

for all $x,\ y,\ u,\ v \in X$, where $k < 1$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 13 holds.

If we put $g = I$ (the identity mapping) in Corollary 17, we get the following result:

Corollary 18. *Let $(X,\ d)$ be a complete metric space and $F : X \times X \to K(X)$ be a mapping satisfying*

$$H(F(x,\ y),\ F(u,\ v)) \leq k \max\{d(x,\ u),\ d(y,\ v)\},$$

for all $x,\ y,\ u,\ v \in X$, where $k < 1$. Then F has a coupled fixed point.

If we take F to be a singleton set in Theorem 13, then we get the following result:

Corollary 19. *Let $(X,\ d)$ be a metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are two mappings for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned}\psi(d(F(x,\ y),\ F(u,\ v))) &\leq \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ &\quad - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}),\end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Furthermore $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point.

Put $g = I$ (the identity mapping) in Corollary 19, we get the following result:

Corollary 20. *Let $(X,\ d)$ be a complete metric space. Assume $F : X \times X \to X$ is a mapping for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} & \psi(d(F(x,\ y),\ F(u,\ v))) \\ \leq\ & \theta(\max\{d(x,\ u),\ d(y,\ v)\}) - \varphi(\max\{d(x,\ u),\ d(y,\ v)\}), \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Then F has a coupled fixed point.

Theorem 21. *Let $(X,\ d)$ be a metric space. Suppose $F : X \times X \to CB(X)$ and $g : X \to X$ are two mappings for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying (2.1), for all $x,\ y,\ u,\ v \in X$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$ and $(F,\ g)$ satisfies $(CLRg)$ property. Then F and g have a coupled coincidence point. Moreover, if one of the conditions $(a) - (d)$ of Theorem 13 holds, then F and g have a common coupled fixed point.*

Proof. Since $(F,\ g)$ satisfies $(CLRg)$ property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some $x,\ y$ in X and $A,\ B$ in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= gx \in A = \lim_{n\to\infty} F(x_n,\ y_n), \\ \lim_{n\to\infty} gy_n &= gy \in B = \lim_{n\to\infty} F(y_n,\ x_n). \end{aligned} \tag{2.14}$$

Now, by contractive condition (2.1), we have

$$\begin{aligned} \psi(H(F(x_n,\ y_n),\ F(x,\ y))) \ \leq\ & \theta(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}) \\ & -\varphi(\max\{d(gx_n,\ gx),\ d(gy_n,\ gy)\}). \end{aligned}$$

On taking $n \to \infty$ in the above inequality and by using the property of ψ, θ, φ and (2.14), we get

$$\psi(H(A,\ F(x,\ y))) \leq \theta(0) - \varphi(0) = 0 - 0 = 0,$$

which, by (ii_ψ), implies

$$H(A,\ F(x,\ y)) = 0,\ \text{similarly}\ H(B,\ F(y,\ x)) = 0.$$

Since $gx \in A$ and $gy \in B$,

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, (x, y) is a coupled coincidence point of F and g. Hence $C(F, g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x, y) \in C(F, g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{2.15}$$

where $u, v \in X$. Since g is continuous at u and v, we have, by (2.15), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{2.16}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g),\ \text{ for all } n \geq 1,$$

that is, for all $n \geq 1$,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x). \tag{2.17}$$

Now, by using contractive condition (2.1), (2.17) and by the monotonicity of ψ, we obtain

$$\begin{aligned}
& \psi(D(g^n x,\ F(u,\ v))) \\
\leq\ & \psi(H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v))) \\
\leq\ & \theta(\max\{d(g^n x,\ gu),\ d(g^n y,\ gv)\}) \\
& -\varphi(\max\{d(g^n x,\ gu),\ d(g^n y,\ gv)\}).
\end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (2.15), (2.16), we get

$$\psi(D(gu,\ F(u,\ v))) \leq \theta(0) - \varphi(0) = 0 - 0 = 0,$$

which, by (ii_ψ), implies

$$D(gu,\ F(u,\ v)) = 0,\ \text{similarly } D(gv,\ F(v,\ u)) = 0,$$

which implies that

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u), \tag{2.18}$$

Now, from (2.16) and (2.18), we have

$$u =\ gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, (u, v) is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\, y) \in C(F,\, g)$, g is $F-$weakly commuting, that is $g^2x \in F(gx,\, gy)$ and $g^2y \in F(gy,\, gx)$ and $g^2x = gx$ and $g^2y = gy$. Thus $gx = g^2x \in F(gx,\, gy)$ and $gy = g^2y \in F(gy,\, gx)$, that is, $(gx,\, gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,\, y) \in C(F,\, g)$ and for some $u,\, v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$. Since g is continuous at x and y, then x and y are fixed points of g, that is, $gx = x$ and $gy = y$. Since $(x,\, y) \in C(F,\, g)$, so we obtain $x = gx \in F(x,\, y)$ and $y = gy \in F(y,\, x)$, that is, $(x,\, y)$ is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F,\, g)) = \{(x,\, x)\}$. Then $\{x\} = \{gx\} = F(x,\, x)$. Hence $(x,\, x)$ is a common coupled fixed point of F and g. □

If we take $\psi(t) = \theta(t)$ in Theorem 21, we obtain the following corollary.

Corollary 22. *Let $(X,\, d)$ be a metric space. Suppose $F : X \times X \to CB(X)$ and $g : X \to X$ are two mappings for which there exist an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ satisfying (2.12), for all $x,\, y,\, u,\, v \in X$, where $\varphi(0) = 0$ and $(F,\, g)$ satisfies $(CLRg)$ property. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 13 holds.*

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1 - k)t$ with $k < 1$ in Theorem 21, we get the following corollary.

Corollary 23. *Let $(X,\, d)$ be a metric space. Suppose $F : X \times X \to CB(X)$ and $g : X \to X$ are two mappings satisfying (2.13), for all $x,\, y,\, u,\, v \in X$, where $k < 1$ and $(F,\, g)$ satisfies $(CLRg)$ property. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 13 holds.*

Example 24. Suppose that $X = [0,\, 1]$, equipped with the metric $d : X \times X \to [0,\, +\infty)$ defined as $d(x,\, y) = \max\{x,\, y\}$ and $d(x,\, x) = 0$ for all $x,\, y \in X$. Let $F : X \times X \to K(X)$ be defined as

$$F(x,\ y) = \begin{cases} \{0\},\ \text{for } x,\ y = 1, \\ \left[0,\ \frac{x^2+y^2}{6}\right],\ \text{for } x,\ y \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$gx = x^2,\ \text{for all } x \in X.$$

and $\psi(t) = \theta(t) = t$ and $\varphi(t) = \frac{2t}{3}$ for $t \geq 0$. Now, for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$, we have

Case (a). If $x^2 + y^2 = u^2 + v^2$, then

$$\begin{aligned} & \psi(H(F(x,\ y),\ F(u,\ v))) \\ = \;& H(F(x,\ y),\ F(u,\ v)) \\ = \;& \frac{u^2+v^2}{6} \\ \leq \;& \frac{1}{6}\max\{x^2,\ u^2\} + \frac{1}{6}\max\{y^2,\ v^2\} \\ \leq \;& \frac{1}{6}d(gx,\ gu) + \frac{1}{6}d(gy,\ gv) \\ \leq \;& \frac{1}{3}(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ \leq \;& \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}). \end{aligned}$$

Case (b). If $x^2 + y^2 \neq u^2 + v^2$ with $x^2 + y^2 < u^2 + v^2$, then

$$\begin{aligned} & \psi(H(F(x,\ y),\ F(u,\ v))) \\ = \;& H(F(x,\ y),\ F(u,\ v)) \\ = \;& \frac{u^2+v^2}{6} \\ \leq \;& \frac{1}{6}\max\{x^2,\ u^2\} + \frac{1}{6}\max\{y^2,\ v^2\} \\ \leq \;& \frac{1}{6}d(gx,\ gu) + \frac{1}{6}d(gy,\ gv) \\ \leq \;& \frac{1}{3}(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ \leq \;& \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}). \end{aligned}$$

Similarly, we obtain the same result for $u^2 + v^2 < x^2 + y^2$. Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$. Again, for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$, we have

$$\begin{aligned} & \psi(H(F(x,\ y),\ F(u,\ v))) \\ = \;& H(F(x,\ y),\ F(u,\ v)) \\ = \;& \frac{x^2+y^2}{6} \\ \leq \;& \frac{1}{6}\max\{x^2,\ u^2\} + \frac{1}{6}\max\{y^2,\ v^2\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{6}d(gx,\ gu) + \frac{1}{6}d(gy,\ gv) \\
&\leq \frac{1}{3}(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\
&\leq \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) - \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).
\end{aligned}
$$

Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $(F,\ g)$ satisfies the contractive condition (2.1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 13 and Theorem 21 are satisfied and $z = (0,\ 0)$ is a common coupled fixed point of hybrid pair $(F,\ g)$. The function $F : X \times X \to K(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.

3. Applications

In this section, based on the results in [17], we propose an application to our results. Consider the integral equation

$$
x(t) = \int_0^T K(t,\ s,\ x(s))ds + h(t),\ t \in [0,\ T], \tag{3.1}
$$

where $T > 0$. We introduce the following space:

$$
C[0,\ T] = \{u : [0,\ T] \to \mathbb{R} : u \text{ is continuous on } [0,\ T]\},
$$

equipped with the metric

$$
d(x,\ y) = \sup_{t\in[0,\ T]} |x(t) - y(t)|\,,\ \text{for each } x,\ y \in C[0,\ T].
$$

It is clear that $(C[0,\ T],\ d)$ is a complete metric space.

Now, we state the main result of this section.

Theorem 25. *We assume that the following hypotheses hold:*

(i) $K_1,\ K_2 : [0,\ T] \times [0,\ T] \times \mathbb{R} \to \mathbb{R}$ *and* $h : [0,\ T] \to \mathbb{R}$ *are continuous,*

(ii) there exists a continuous function $G : [0,\ T] \times [0,\ T] \to [0,\ +\infty)$ *such that*

$$
|K(t,\ s,\ x(s)) - K(t,\ s,\ y(s))| \leq G(t,\ s) \cdot \frac{|x(s) - y(s)|}{6},
$$

for all $s,\ t \in C[0,\ T]$ *and* $x,\ y \in \mathbb{R}$,

(iii) $\sup_{t\in[0,\ T]} \int_0^T G(t,\ s)^2 ds \leq \frac{1}{T}$.

Then the integral equation (3.1) has a solution $(u^*, v^*) \in C[0,\ T]$.

Proof. We first define $\psi,\ \theta,\ \varphi : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$\psi(t) = \theta(t) = t \text{ and } \varphi(t) = \frac{2t}{3} \text{ for } t \geq 0,$$

and define $F : C[0,\ T] \times C[0,\ T] \to C[0,\ T]$ by

$$F(x,\ y)(t) = \int_0^T [K(t,\ s,\ x(s)) + K(t,\ s,\ y(s))]ds + h(t),$$

for all $t \in [0,\ T]$ and $x,\ y \in C[0,\ T]$. Now, for all $x,\ y,\ u,\ v \in C[0,\ T]$, due to (ii) and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|F(x,\ y)(t) - F(u,\ v)(t)| \\
\leq\ & \int_0^T |K(t,\ s,\ x(s)) - K(t,\ s,\ u(s))|\, ds \\
&+ \int_0^T |K(t,\ s,\ y(s)) - K(t,\ s,\ v(s))|\, ds \\
\leq\ & \int_0^T G(t,\ s) \cdot \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{6} \right) ds \\
\leq\ & \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{6} \right)^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&|F(x,\ y)(t) - F(u,\ v)(t)| \\
\leq\ & \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{6} \right)^2 ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.2}$$

Taking (iii) into account, we estimate the first integral in (3.2) as follows:

$$\left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}. \tag{3.3}$$

For the second integral in (3.2) we proceed in the following way:

$$\left(\int_0^T \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{6} \right)^2 ds \right)^{\frac{1}{2}} \leq \sqrt{T} \cdot \frac{d(x,\ u) + d(y,\ v)}{6}. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we conclude that

$$\begin{aligned}
|F(x,\ y)(t) - F(u,\ v)(t)| &\leq \frac{1}{6} d(x,\ u) + \frac{1}{6} d(y,\ v) \\
&\leq \frac{1}{3} \max\{d(x,\ u),\ d(y,\ v)\}.
\end{aligned}$$

It yields

$$\psi(d(F(x,\ y),\ F(u,\ v))) \leq \theta(\max\{d(x,\ u),\ d(y,\ v)\}) - \varphi(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all x, y, u, $v \in C[0,\ T]$. Hence, all hypotheses of Corollary 20 are satisfied. Thus, F has a coupled fixed point $(u^*,\ v^*) \in C[0,\ T] \times C[0,\ T]$ which is a solution of (3.1). □

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Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M. P.), India
Email address: amrishhanda83@gmail.com