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INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5) FUNCTION

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Abstract. A necessary and sufficient condition in terms of lower cut sets is given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that whose $F_\sigma-$kernel of sets are $F_\sigma-$sets.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [17]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [17].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [22] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 10].

In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 21].

Results of Katětov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are

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used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that whose $F_\sigma-$kernel of sets are $F_\sigma-$sets.

A real-valued function f defined on a topological space X is called *contra-Baire-1 (Baire-.5)* if the preimage of every open subset of $\mathbb{R}$ is a $G_\delta-$set in X [23].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [15].

A property P defined relative to a real-valued function on a topological space is a $B-.5-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are $B-.5-$properties, the following terminology is used: (i) A space X has the *weak* $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$. (ii) A space X has the $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g < h < f$.

In this paper, for a topological space that whose $F_\sigma-$kernel of sets are $F_\sigma-$sets, is given a sufficient condition for the weak $B-.5-$insertion property. Also for a space with the weak $B-.5-$insertion property, we give a necessary and sufficient condition for the space to have the $B-.5-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result and Applications

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [6, 16, 20], A^Λ is called the *kernel* of A.

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:
$G_\delta(A) = \cup\{O : O \subseteq A, O is G_\delta - set\}$ and
$F_\sigma(A) = \cap\{F : F \supseteq A, F is F_\sigma - set\}$.

$F_\sigma(A)$ is called the $F_\sigma - kernel$ of A.

The following first two definitions are modifications of conditions considered in [13, 14].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ \nu$ implies $x \ \rho \ \nu$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \ \bar{\rho} \ B$.

3) If $A \ \rho \ B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, that $F_\sigma-$kernel sets in X are $F_\sigma-$ sets , with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.*

Proof. [19, Theorem 2.1]. □

Definition 2.5. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_\delta-$set for any real number t.

The abbreviations $usc, lsc, cusB_.5$ and $clsB_.5$ are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1 ([13, 14])**.** A space X has the weak $c-$insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences, we suppose that X is a topological space that whose $F_\sigma-$kernel of sets are $F_\sigma-$sets.

Corollary 2.1. *For each pair of disjoint $F_\sigma-$sets F_1, F_2, there are two $G_\delta-$sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$.*

Proof. [19, Corollary 2.1]. □

Remark 2 ([24])**.** A space X has the weak $c-$insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 2.2. *For every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set if and only if X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.*

Proof. [19, Corollary 2.2]. □

Theorem 2.2. *Let P_1 and P_2 be $B-.5-$property and X be a space that satisfies the weak $B-.5-$insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the $B-.5-$insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by Baire-.5 functions.*

Proof. [18, Theorem 2.1]. □

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 2.1. *The following conditions on the space X are equivalent:*

(i) For every G of $G_\delta-$set we have $F_\sigma(G)$ is a $G_\delta-$set.

(ii) For each pair of disjoint $G_\delta-$sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.

The proof of Lemma 2.1 is a direct consequence of the definition $F_\sigma-$kernel of sets.

Lemma 2.2. *The following conditions on the space X are equivalent:*

(i) Every two disjoint $F_\sigma-$sets of X can be separated by $G_\delta-$sets of X.

(ii) If F is a $F_\sigma-$set of X which is contained in a $G_\delta-$set G, then there exists a $G_\delta-$set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

Proof. [19, Lemma 2.2]. □

Lemma 2.3. *Suppose that X is a topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If F_1 and F_2 be two disjoint $F_\sigma-$sets of X, then there exists a Baire-.5 function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. [19, Lemma 2.3]. □

Lemma 2.4. *Suppose that X is the topological space such that every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets. The following conditions are equivalent:*

(i) Every countable convering of $G_\delta-$sets of X has a refinement consisting of $G_\delta-$sets such that, for every $x \in X$, there exists a $G_\delta-$set containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection there exists a decreasing sequence $\{G_n\}$ of $G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

Proof. (i) $\Rightarrow$ (ii). suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets. By hypothesis (i) and Lemma 2.2, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a $G_\delta-$set and $F_\sigma(V_n) \subseteq F_n^c$. By setting $F_n = (F_\sigma(V_n))^c$, we obtain a decreasing sequence of $G_\delta-$sets with the required properties.

(ii) $\Rightarrow$ (i). Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets, we set for $n \in \mathbb{N}, F_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of $G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets W_n of X in the following manner:

W_1 is a $G_\delta-$set of X such that $G_1^c \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$.

W_2 is a $G_\delta-$set of X such that $F_\sigma(W_1) \cup G_2^c \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 2.2, W_n exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of $G_\delta-$sets. Moreover, we have

(i) $F_\sigma(W_n) \subseteq W_{n+1}$

(ii) $G_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$.

Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $G_\delta-$sets and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$S_1 \cap H_1, \quad S_1 \cap H_2$

$S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3$

$S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4$

and continue ad infinitum. These sets are $G_\delta-$sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $G_\delta-$set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $G_\delta-$sets, and for every point in X we can find a $G_\delta-$set containing the point that intersects only finitely many elements of that refinement. □

Remark 3 ([12, 13])**.** A space X has the $c-$insertion property for (usc, lsc) if and only if X is normal and countably paracompact.

Corollary 2.3. *X has the $B-.5-$insertion property for $(cusB-.5, clsB-.5)$ if and only if every two disjoint $F_\sigma-$sets of X can be separated by $G_\delta-$sets, and in addition, every countable covering of $G_\delta-$sets has a refinement that consists of $G_\delta-$sets such that, for every point of X we can find a $G_\delta-$set containing that point such that, it intersects only a finite number of refining members.*

Proof. Suppose that F_1 and F_2 are disjoint $F_\sigma-$sets . Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_2}$.

Since F_2 is a $F_\sigma-$set, and F_1^c is a $G_\delta-$set, g is $cusB-.5$,f is $clsB-.5$ and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function h such that,

$g < h < f$. Now by setting $G_1 = \{x \in X : h(x) < 1\}$ and $G_2 = \{x \in X : h(x) > 1\}$. We can say that G_1 and G_2 are disjoint $G_\delta-$sets that contain F_1 and F_2, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$, $f(x) = \frac{1}{n+1}$. Since $\bigcap_{n=0}^{\infty} F_n = \emptyset$ and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}, f$ is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then $\{x \in X : f(x) > r\} = X$ is a $G_\delta-$set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, we can find $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Now suppose that k is the least natural number such that $\frac{1}{k+1} \leq r$. Hence $\frac{1}{k} > r$ and consequently, $\{x \in X : f(x) > r\} = X \setminus F_k$ is a $G_\delta-$set. Therefore, f is $clsB - .5$. By setting $g = 0$, we have g is $cusB - .5$ and $g < f$. Hence by hypothesis there exists a Baire-.5 function h on X such that, $g < h < f$.

By setting $G_n = \{x \in X : h(x) < \frac{1}{n+1}\}$, we have G_n is a $G_\delta-$set. But for every $x \in F_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$, $0 < h(x) < \frac{1}{n+1}$, i.e., $x \in G_n$ therefore $F_n \subseteq G_n$ and since $h > 0$ it follows that $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Hence by Lemma 2.4, the conditions hold.

On the other hand, since every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets, by Corollary 2.1, X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that, g is $cusB - .5$ and f is $clsB - .5$. For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since g is $cusB - .5$, and f is $clsB - .5$, $f - g$ is $clsB - .5$. Hence $A(f - g, 3^{-n+1})$ is a $F_\sigma-$set of X. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of $F_\sigma-$sets and furthermore since $0 < f - g$, it follows that

$$\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset.$$

Now by Lemma 2.4, there exists a decreasing sequence $\{D_n\}$ of $G_\delta-$sets such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 2.3, $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of $F_\sigma-$sets can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, there exists a Baire-.5 function h defined on X such that, $g < h < f$, i.e., X has the $B - .5-$insertion property for $(cusB - .5, clsB - .5)$. ☐

Remark 4 ([14]). A space X has the $c-$insertion property for (lsc, usc) iff X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of

X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection such that $G_n \subseteq F_n$ for each n.

Corollary 2.4. *For every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set and in addition for every decreasing sequence $\{G_n\}$ of $G_\delta-$sets with empty intersection, there exists a decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection such that for every $n \in \mathbb{N}, G_n \subseteq F_n$ if and only if X has the $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.*

Proof. Since for every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set, by Corollary 2.2, X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$. Now suppose that f and g are real-valued functions defined on X with $g < f, g$ is $clsB-.5$, and f is $cusB-.5$. Set $A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) < 3^{-n+1}\}$. Then since $f-g$ is $cusB-.5$, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection. By hypothesis, there exists a decreasing sequence $\{D_n\}$ of $F_\sigma-$sets with empty intersection such that, for every $n \in \mathbb{N}, A(f-g, 3^{-n+1}) \subseteq D_n$. Hence $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are two disjoint $G_\delta-$sets and therefore by Lemma 2.1, we have

$$F_\sigma(A(f-g, 3^{-n+1})) \cap F_\sigma((X \setminus D_n)) = \emptyset$$

and therefore by Lemma 2.3, $X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separable by Baire-.5 functions. Therefore by Theorem 2.2, there exists a Baire-.5 function h on X such that, $g < h < f$, i.e., X has the $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.

On the other hand, suppose that G_1 and G_2 are two disjoint $G_\delta-$sets. Since $G_1 \cap G_2 = \emptyset$, we have $G_2 \subseteq G_1^c$. We set $f(x) = 2$ for $x \in G_1^c$, $f(x) = \frac{1}{2}$ for $x \notin G_1^c$ and $g = \chi_{G_2}$.

Then since G_2 is a $G_\delta-$set and G_1^c is an $F_\sigma-$set, we conclude that g is $clsB-.5$ and f is $cusB-.5$ and furthermore $g < f$. By hypothesis, there exists a Baire-.5 function h on X such that, $g < h < f$. Now we set $F_1 = \{x \in X : h(x) \leq \frac{3}{4}\}$ and $F_2 = \{x \in X : h(x) \geq 1\}$. Then F_1 and F_2 are two disjoint $F_\sigma-$sets containing G_1 and G_2, respectively. Hence $F_\sigma(G_1) \subseteq F_1$ and $F_\sigma(G_2) \subseteq F_2$ and consequently $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$. By Lemma 2.1, for every G of $G_\delta-$set, the set $F_\sigma(G)$ is a $G_\delta-$set.

Now suppose that $\{G_n\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection.

We set $G_0 = X$ and $f(x) = \frac{1}{n+1}$ for $x \in G_n \setminus G_{n+1}$. Since $\bigcap_{n=0}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}$ there exists $x \in G_n \setminus G_{n+1}, f$ is well-defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then $\{x \in X : f(x) < r\} = \emptyset$ is a $G_\delta-$set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, there exists $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Suppose that k is the least natural number with this property. Hence $\frac{1}{k} > r$. Now if $\frac{1}{k+1} < r$ then $\{x \in X : f(x) < r\} = G_k$ is a $G_\delta-$set and if $\frac{1}{k+1} = r$ then $\{x \in X : f(x) < r\} = G_{k+1}$ is a $G_\delta-$set. Hence f is a $cusB - .5$ on X. By setting $g = 0$, we have conclude, g is $clsB - .5$ on X and in addition $g < f$. By hypothesis there exists a Baire-.5 function h on X suvh that, $g < h < f$.

Set $F_n = \{x \in X : h(x) \leq \frac{1}{n+1}\}$. This set is an $F_\sigma-$set. But for every $x \in G_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$, $h(x) < \frac{1}{n+1}$, which means that $x \in F_n$ and consequently $G_n \subseteq F_n$.

By definition of $F_n, \{F_n\}$ is a decreasing sequence of $F_\sigma-$sets and since $h > 0, \bigcap_{n=1}^{\infty} F_n = \emptyset$. Thus the conditions hold. □

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References

1. A. Al-Omari & M.S. Md Noorani: Some properties of contra-b-continuous and almost contra-b-continuous functions. European J. Pure. Appl. Math. **2** (2009), no. 2, 213-230.
2. F. Brooks: Indefinite cut sets for real functions. Amer. Math. Monthly **78** (1971), 1007-1010.
3. M. Caldas & S. Jafari: Some properties of contra-$\beta-$continuous functions. Mem. Fac. Sci. Kochi. Univ. **22** (2001), 19-28.
4. J. Dontchev: The characterization of some peculiar topological space via $\alpha-$ and $\beta-$sets. Acta Math. Hungar. **69** (1995), no. 1-2, 67-71.
5. J. Dontchev: Contra-continuous functions and strongly S-closed space. Intrnat. J. Math. Math. Sci. **19** (1996), no. 2, 303-310.
6. J. Dontchev & H. Maki: On sg-closed sets and semi$-\lambda-$closed sets. Questions Answers Gen. Topology **15** (1997), no. 2, 259-266.
7. E. Ekici: On contra-continuity. Annales Univ. Sci. Bodapest **47** (2004), 127-137.
8. E. Ekici: New forms of contra-continuity. Carpathian J. Math. **24** (2008), no. 1, 37-45.
9. A.I. El-Magbrabi: Some properties of contra-continuous mappings. Int. J. General Topol. **3** (2010), no. 1-2, 55-64.

10. M. Ganster & I. Reilly: A decomposition of continuity. Acta Math. Hungar. **56** (1990), no. 3-4, 299-301.
11. S. Jafari & T. Noiri: Contra-continuous function between topological spaces. Iranian Int. J. Sci. **2** (2001), 153-167.
12. S. Jafari & T. Noiri: On contra-precontinuous functions. Bull. Malaysian Math. Sc. Soc. **25** (2002), 115-128.
13. M. Katětov: On real-valued functions in topological spaces. Fund. Math. **38** (1951), 85-91.
14. M. Katětov: Correction to 'On real-valued functions in topological spaces'. Fund. Math. **40** (1953), 203-205.
15. E. Lane: Insertion of a continuous function. Pacific J. Math. **66** (1976), 181-190.
16. S.N. Maheshwari & R. Prasad: On $R_{Os}-$spaces, Portugal. Math. **34** (1975), 213-217.
17. H. Maki: Generalized $\Lambda-$sets and the associated closue operator. The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement (1986), 139-146.
18. M. Mirmiran: Insertion of a function belonging to a certain subclass of $\mathbb{R}^X$. Bull. Iran. Math. Soc. **28** (2002), no. 2, 19-27.
19. M. Mirmiran & B. Naderi: Strong insertion of a contra-Baire-1 (Baire-.5) function between two comparable real-valued functions. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **26** (2019), no, 1, 1-12.
20. M. Mrsevic: On pairwise R and pairwise R_1 bitopological spaces. Bull. Math. Soc. Sci. Math. R. S. Roumanie **30** (1986), 141-145.
21. A.A. Nasef: Some properties of contra-continuous functions. Chaos Solitons Fractals **24** (2005), 471-477.
22. M. Przemski: A decomposition of continuity and $\alpha-$continuity. Acta Math. Hungar. **61** (1993), no. 1-2, 93-98.
23. H. Rosen: Darboux Baire-.5 functions. Proc. Amer. Math. Soc. **110** (1990), no. 1, 285-286.
24. M.H. Stone: Boundedness properties in function-lattices. Canad. J. Math. **1** (1949), 176-189.

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