J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2019.26.3.209 ISSN(Online) 2287-6081
Volume 26, Number 3 (August 2019), Pages 209–214

THE ARTINIAN QUOTIENT OF CODIMENSION $n+1$

Yong-Su Shin

Abstract. We investigate all kinds of the Hilbert function of the Artinian quotient of the coordinate ring of a linear star configuration in $\mathbb{P}^n$ of type $(n+1)$ (or $(n+1)$-general points in $\mathbb{P}^n$), which generalizes the result [7, Theorem 3.1].

1. Introduction

Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over a field $\Bbbk$ of characteristic 0 and I be a homogeneous ideal of R. A standard graded $\Bbbk$-algebra $A = R/I = \oplus_{i\ge 0} A_i$ has the *weak Lefschetz property* (WLP) if there is a linear form ℓ such that the multiplication by $\times\ell : A_i \to A_{i+1}$ has maximal rank for every $i \ge 0$, and A has the *strong Lefschetz property* (SLP) if $\times\ell^d : A_i \to A_{i+d}$ has maximal rank for every $i \ge 0$ and $d \ge 1$. The Hilbert function of $A = R/I$, $\mathbf{H}_A : \mathbb{N} \to \mathbb{N}$, is defined by $\mathbf{H}_A(t) = \dim_{\Bbbk} R_t - \dim_{\Bbbk} I_t$. If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by $\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}(R/I_{\mathbb{X}}, t)$.

In [1], the authors found the graded minimal free resolution of a star configuration in $\mathbb{P}^n$ of codimention 2 before the general case (see Definition 2.1 in Section 2). In 2014 [5], Park and Shin gave a general definition of a star configuration in $\mathbb{P}^n$ of codimension r, and found the minimal graded free resolution of a general star configuration in $\mathbb{P}^n$.

In [7], the author found the Hilbert function of the Artinian quotient of 3-general points in $\mathbb{P}^2$ (or a linear star configuration in $\mathbb{P}^2$ of type 3) and proved that the Artinian quotient has the SLP. In this paper we focus on the following question.

Received by the editors June 27, 2019. Accepted August 13, 2019
2010 *Mathematics Subject Classification.* Primary 13A02, Secondary 16W50.
Key words and phrases. Hilbert function, star configuration, generic Hilbert function, weak Lefschetz property, strong Lefschetz property.
This paper was supported by a grant from Sungshin Women's University.

Question 1.1. Let $\mathbb{X}$ be a set of $(n+1)$-general points in $\mathbb{P}^n$ (or a linear star configuration in $\mathbb{P}^n$) and $\mathbb{Y}$ be a star configuration in $\mathbb{P}^n$ of type $t \ge n+1$.

(a) What is the Hilbert function of the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$?

(b) Does the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

In this paper, we find a complete answer to Question 1.1. In other words, we show that the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has a specific type of Hilbert function and the SLP (see Theorem 3.2), which generalizes the result [7, Theorem 3.1] (see Corollary 3.3).

2. A Star Configuration in $\mathbb{P}^n$

We first recall the definition of a star configuration in $\mathbb{P}^n$ in [5], and then introduce some related results.

Definition 2.1. Let $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{k}$. For positive integers r and s with $1 \le r \le \min\{n, s\}$, suppose $F_1, \dots, F_s$ are general forms in R of degrees $d_1, \dots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \dots, F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type (r, s). In particular, if $F_1, \dots, F_s$ are general linear forms in R, then we call $\mathbb{X}$ *a linear star-configuration in* $\mathbb{P}^n$ *of type* (r, s).

If $n = r$, then we call $\mathbb{X}$ *a star configuration in* $\mathbb{P}^n$ *of type* s instead of *type* (n, s).

The following corollary is the results of Carlini, Guardo, and Van Tuyl [2, Theorem 2.5], Geramita, Harbourne, and Migliore [3, Proposition 2.9], and Park and Shin [5, Corollary 2.4].

Corollary 2.2. *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type s with $s \ge n \ge 2$. Then $\mathbb{X}$ has generic Hilbert function i.e.,*

$$\mathbf{H}_{\mathbb{X}}(i) = \min\left\{ \deg(\mathbb{X}), \binom{i+n}{n} \right\}$$

for every $i \ge 0$.

Proposition 2.3 ([6, Proposition 2.6])**.** *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type s with $s \ge n \ge 2$. Then*

$$\sigma_{\mathbb{X}} = \left[\textstyle\sum_{i=1}^{s} d_i \right] - (n-1),$$

where

$$\sigma_{\mathbb{X}} = \min\{\, i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

We recall the result in [4].

Proposition 2.4 ([4, Proposition 5.3])**.** *Let $\mathbb{X}$ be a set of $(n+1)$-general points in $\mathbb{P}^n$, and let A be the Artinian quotient of a coordinate ring of $\mathbb{X}$ having Hilbert function of the form*

$$\mathbf{H}_A \;:\; 1 \quad n+1 \quad \cdots \quad n+1 \quad h_s \quad \cdots \quad h_t,$$

where $2 \le s \le t$. Then A has the SLP.

3. The Artinian Quotient of a Linear Star Configuration in $\mathbb{P}^n$ of Type $(n+1)$

In this section, we find the Hilbert function of the Artinian quotient of coordinate rings of a linear star configuration in $\mathbb{P}^n$ of type $(n+1)$ and a general star configuration in $\mathbb{P}^n$ of type t with $t \ge (n+1)$. We can prove the main theorem (Theorem 3.2) using [5, Theorem 3.4], but we introduce an easier proof here without the theorem.

Lemma 3.1. *Let $\mathbb{X}$ be a set of $(n+1)$-general points in $\mathbb{P}^n$ (or a linear star configuration in $\mathbb{P}^n$ of type $(n+1)$) and $\mathbb{Y}$ be a star configuration in $\mathbb{P}^n$ of type t with $t \ge n+1$ defined by forms of degree $d_1 \ge d_2 \ge \cdots \ge d_t$. Define $d = \sum_{i=n}^{t} d_i$ and $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$. Then*

$$\mathbf{H}_A(d+1) = 0.$$

Proof. Recall that $I_{\mathbb{Y}}$ has a minimal generator in degree d. Hence

$$\mathbf{H}_{\mathbb{Y}}(d) \le \binom{n+d}{d} - 1, \quad \text{and thus,} \quad \mathbf{H}_{\mathbb{Y}}(d+1) \le \binom{n+d}{d} - (n+1).$$

Since $\mathbb{X}$ is a set of $(n+1)$-general points in $\mathbb{P}^n$, we get that

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d+1) = (n+1) + \mathbf{H}_{\mathbb{Y}}(d+1) = \mathbf{H}_{\mathbb{X}}(d+1) + \mathbf{H}_{\mathbb{Y}}(d+1).$$

By equation (3.1), $\mathbf{H}_A(d+1) = 0$, as we wished. □

Theorem 3.2. *Let $\mathbb{X}$ be a set of $(n+1)$-general points in $\mathbb{P}^n$ (or a linear star configuration in $\mathbb{P}^n$ of type $(n+1)$) and $\mathbb{Y}$ be a star configuration in $\mathbb{P}^n$ of type t with $t \ge n+1$ defined by forms of degree $d_1 \ge d_2 \ge \cdots \ge d_t$ with $d_1 > 1$. Define*

$d = \sum_{i=n}^{t} d_i$. *Then the Artinian quotient* $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ *has the SLP having Hilbert function*

$$\mathbf{H}_A \ : \ 1 \quad n+1 \quad \cdots \quad n+1 \quad \overset{d\text{-th}}{h_d} \quad 0,$$

where

(i) $h_d = 0$ *if either* $d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t$ *with* $s \geq n+1$ *or* $d_1 = \cdots = d_u > d_{u+1} = \cdots = d_s \geq d_{s+1} \geq \cdots \geq d_t$ *with* $1 \leq u \leq (n-1) < s \leq t$ *and* $\binom{s-u}{(n-1)-u} \geq n+1$,

(ii) $h_d = 1$ *if* $d_1 = \cdots = d_n > d_{n+1} \geq \cdots \geq d_t$, *and*

(iii) $h_d = 2n - s - 1$ *if* $d_1 = \cdots = d_u > d_{u+1} = \cdots = d_s \geq d_{s+1} \geq \cdots \geq d_t$ *with* $1 \leq u \leq (n-1) < s \leq t$ *and* $\binom{s-u}{(n-1)-u} \leq n+1$.

Proof. We first find the Hilbert function of A in degrees $d-1$ and d. Note that by [5, Theorem 3.4] $I_{\mathbb{Y}}$ has no minimal generators in degree $d-1$, and thus, $I_{\mathbb{X}\cup\mathbb{Y}}$ has no minimal generators in degree $d-1$, as well. Hence

$$\mathbf{H}_{\mathbb{Y}}(d-1) = \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d-1) = \binom{n+(d-1)}{n}.$$

Using the exact sequence

$$0 \to R/I_{\mathbb{X}\cup\mathbb{Y}} \to R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0, \tag{3.1}$$

we have that $\mathbf{H}_A(d-1) = n+1$. We now find $\mathbf{H}_A(d)$.

(a) Let $d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t$ with $s \geq n+1$. First, since $d_1 \geq \cdots \geq d_t$, we see that, by [5, Theorem 3.4], the initial degree of $I_{\mathbb{Y}}$ is d. Recall that $\mathbb{X}$ is a set of $(n+1)$-general points in $\mathbb{P}^n$ and $\binom{s}{n-1} \geq n+1$. Hence

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{n+d}{n} - \binom{s}{n-1}, \quad \text{and so,} \quad \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \binom{n+d}{n} - \binom{s}{n-1} + (n+1).$$

By equation (3.1), $\mathbf{H}_A(d) = 0$.

(b) Let $d_1 = \cdots = d_n > d_{n+1} \cdots \geq d_t$. Recall that $I_{\mathbb{Y}}$ has $\binom{n}{n-1} = n$-minimal generators in degree d. Since $\mathbb{X}$ is a set of $(n+1)$-general points in $\mathbb{P}^n$,

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{n+d}{n} - n, \quad \text{and thus,} \quad \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \binom{n+d}{n}.$$

By equation (3.1), $\mathbf{H}_A(d) = 1$.

(c) Let $d_1 = \cdots = d_u > d_{u+1} = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t$ with $1 \leq u \leq (n-1) < s \leq t$. Then $I_{\mathbb{Y}}$ has $\binom{s-u}{(n-1)-u}$-minimal generators in degree d. So

$$\mathbf{H}_{\mathbb{Y}}(d) = \binom{n+d}{n} - \binom{s-u}{(n-1)-u}.$$

$$\mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(d) = \begin{cases} \mathbf{H}_{\mathbb{Y}}(d) + (n+1), & \text{if } \binom{s-u}{(n-1)-u} > n+1, \\ \binom{n+d}{n}, & \text{if } \binom{s-u}{(n-1)-u} \leq n+1. \end{cases}$$

By equation (3.1),

$$\mathbf{H}_A(d) = \begin{cases} 0, & \text{if } \binom{s-u}{(n-1)-u} > n+1, \\ 2n-s-1, & \text{if } \binom{s-u}{(n-1)-u} \leq n+1. \end{cases}$$

By Lemma 3.1, the Hilbert function of A is as follows.

(i) If $d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t$ with $s \geq n+1$ or $d_1 = \cdots = d_u > d_{u+1} = \cdots = d_s \geq d_{s+1} \geq \cdots \geq d_t$ with $1 \leq u \leq (n-1) < s \leq t$ and $\binom{s-u}{(n-1)-u} > n+1$, then

$$\mathbf{H}_A \ : \ 1 \quad n+1 \quad \cdots \quad n+1 \quad \overset{d\text{-th}}{0}.$$

(ii) If $d_1 = \cdots = d_n > d_{n+1} \geq \cdots \geq d_t$, then

$$\mathbf{H}_A \ : \ 1 \quad n+1 \quad \cdots \quad n+1 \quad \overset{d\text{-th}}{1} \quad 0.$$

(iii) $d_1 = \cdots = d_u > d_{u+1} = \cdots = d_s \geq d_{s+1} \geq \cdots \geq d_t$ with $1 \leq u \leq (n-1) < s \leq t$ and $\binom{s-u}{(n-1)-u} \leq n+1$, then

$$\mathbf{H}_A \ : \ 1 \quad n+1 \quad \cdots \quad n+1 \quad \overset{d\text{-th}}{2n-s-1} \quad 0.$$

Therefore, by Proposition 2.4, A has the SLP. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3 ([7, Theorem 3.1]). *Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^2$ of type 3 and $\mathbb{Y}$ be a star configuration in $\mathbb{P}^2$ of type t with $t \geq 3$ defined by forms of degree $d_1 \geq d_2 \geq \cdots \geq d_t$ with $d_1 > 1$. Define $d = \sum_{i=2}^{t} d_i$. Then the Artinian star configuration quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP with Hilbert function*

$$\mathbf{H}_A \ : \ 1 \quad 3 \quad \cdots \quad 3 \quad \overset{d\text{-th}}{h_d} \quad 0,$$

where

$$h_d = \begin{cases} 0, & \text{for } d_1 = \cdots = d_s > d_{s+1} \geq \cdots \geq d_t \text{ with } s \geq 3, \\ 1, & \text{for } d_1 = d_2 > d_3 \geq \cdots \geq d_t, \quad \text{and} \\ 2, & \text{for } d_1 > d_2 \geq \cdots \geq d_t. \end{cases}$$

REFERENCES

1. J. Ahn & Y.S. Shin: The Minimal Free Resolution of a Star-Configuration in $\mathbb{P}^n$ and the Weak-Lefschetz Property. J. Korean Math. Soc. **49** (2012), no. 2, 405-417.
2. E. Carlini, E. Guardo & A. Van Tuyl: Star configurations on generic hypersurfaces. J. Algebra **407** (2014) 1-20.
3. A.V. Geramita, B. Harbourne & J.C. Migliore: Star Configurations in $\mathbb{P}^n$. J. Algebra **376** (2013), 279-299.
4. Y.R. Kim & Y.S. Shin: The Artinian Point Star Configuration Quotient and the Strong Lefschetz Property. In prepartation.
5. J.P. Park & Y.S. Shin: The Minimal Free Resolution of a Star-configuration in $\mathbb{P}^n$. J. Pure Appl. Algebra **219** (2015), 2124-2133.
6. Y.S. Shin: Some Examples of the Union of Two Linear Star-configurations in $\mathbb{P}^2$ Having Generic Hilbert Function. J. Chungcheong Math. Soc. **26** (2013), no. 2, 403-409.
7. Y.S. Shin: The Hilbert function of the Artinian quotient of codimension 3. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **25** (2018), no. 4, 337-343

DEPARTMENT OF MATHEMATICS, SUNGSHIN WOMEN'S UNIVERSITY, SEOUL 02844, KOREA
Email address: `ysshin@sungshin.ac.kr`