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FEW RESULTS IN CONNECTION WITH SUM AND PRODUCT THEOREMS OF RELATIVE (p, q)-φ ORDER, RELATIVE (p, q)-φ TYPE AND RELATIVE (p, q)-φ WEAK TYPE OF MEROMORPHIC FUNCTIONS WITH RESPECT TO ENTIRE FUNCTIONS

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Abstract. Orders and types of entire and meromorphic functions have been actively investigated by many authors. In the present paper, we aim at investigating some basic properties in connection with sum and product of relative (p, q)-φ order, relative (p, q)-φ type, and relative (p, q)-φ weak type of meromorphic functions with respect to entire functions where p, q are any two positive integers and $\varphi : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f (see [12]) is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds (see [1]). When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f (see [5, p.4]), playing the same role as $M_f(r)$. If f is non-constant entire function, then its Nevanlinna's characteristic function is strictly increasing and continuous and therefore there exists its inverse functions $T_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

However, throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [5, 9, 10, 11] and therefore we do not explain those in details. Now we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x =$

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$\log\left(\log^{[k-1]} x\right)$ for $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper p and q always denote positive integers.

Mainly the growth investigation of meromorphic functions has usually been done through its Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any meromorphic function with respect to an entire function, the notions of relative growth indicators [8] will come. Extending this notion, Debnath et. al. [4] introduce the definition of relative (p, q)-th order and relative (p, q)-th lower order of a meromorphic function f with respect to another entire function g respectively in the light of index-pair (detail about index-pair one may see [4, 6, 7]). For details about it, one may see [4]. Extending this notion, recently Biswas [2] introduced the definitions of relative (p, q)-φ order and the relative (p, q)-φ lower order of a meromorphic function f with respect to another entire function g as follows:

Definition 1 ([2])**.** Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. The relative (p, q)-φ order and the relative (p, q)-φ lower order of a meromorphic function f with respect to an entire function g are defined as

$$\rho_g^{(p,q)}(f, \varphi) = \limsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} \varphi(r)}$$

and

$$\lambda_g^{(p,q)}(f, \varphi) = \liminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} \varphi(r)}.$$

If we consider $\varphi(r) = r$, then the above definition reduce to the definitions of relative (p, q)-th order and relative (p, q)-th lower order of a meromorphic f with respect to an entire g, introduced by Debnath et. al. [4].

If the relative (p, q)-φ order and the relative (p, q)-φ lower order of f with respect to g are the same, then f is called a function of regular relative (p, q)-φ growth with respect to g. Otherwise, f is said to be irregular relative (p, q)-φ growth with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative (p, q)-φ type and relative (p, q)-φ lower type of entire or meromorphic functions with respect to another entire function which are as follows:

Definition 2 ([2])**.** Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ type and the relative (p,q)-φ lower type of a meromorphic function f with respect to another entire function g having non-zero finite relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ are defined as:

$$\sigma_g^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}$$

and

$$\overline{\sigma}_g^{(p,q)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}.$$

Analogously, to determine the relative growth of f having same non zero finite relative (p,q)-φ lower order with respect to g, one can introduce the definition of relative (p,q)-φ weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g of finite positive relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 3 ([2])**.** Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ weak type $\tau_g^{(p,q)}(f,\varphi)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f,\varphi)$ of a meromorphic function f with respect to another entire function g having non-zero finite relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ are defined as:

$$\tau_g^{(p,q)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}$$

and

$$\overline{\tau}_g^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

If we consider $\varphi(r) = r$, then $\sigma_g^{(p,q)}(f,r)$ and $\tau_g^{(p,q)}(f,r)$ are respectively known as relative (p,q)-th type and relative (p,q)-th weak type of f with respect to g. For details about relative (p,q)-th type, relative (p,q)-th weak type etc., one may see [3].

Here, in this paper, we aim at investigating some basic properties of relative (p,q)-φ order, relative (p,q)-φ type and relative (p,q)-φ weak type of a meromorphic function with respect to an entire function under somewhat different conditions.

Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]). *Let f be an entire function which satisfies the Property (A) then for any positive integer n and for all sufficiently large r,*

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 2 ([5, p.18]). *Let f be an entire function. Then for all sufficiently large values of r,*

$$T_f(r) \leq \log M_f(r) \leq 3T_f(2r) \ .$$

3. Main Results

In this section we present some results which will be needed in the sequel.

Theorem 4. *Let f_1, f_2 be meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular relative (p,q)-φ growth with respect to g_1. Also let g_1 have the Property (A). Then we have*

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\}.$$

The equality holds when any one of $\lambda_{g_1}^{(p,q)}(f_i, \varphi) > \lambda_{g_1}^{(p,q)}(f_j, \varphi)$ hold and at least any one of f_j is of regular relative (p,q)-φ growth with respect to g_1 where $i, j = 1, 2$ and $i \neq j$.

Proof. The result is obvious when $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = 0$. So we suppose that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) > 0$. We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k, \varphi)$ is finite for $k = 1, 2$. Now let us consider that $\max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\} = \Delta$ and f_2 be of regular relative (p,q)-φ growth with respect to g_1.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_1}^{(p,q)}(f_1, \varphi)$, we have for a sequence values of r tending to infinity that

$$T_{f_1}(r) \leq T_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q)}(f_1, \varphi) + \varepsilon\right)\log^{[q]}\varphi(r)\right]\right]$$

$$\text{(1)} \qquad i.e.,\ T_{f_1}(r) \leq T_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right].$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_{g_1}^{(p,q)}(f_2,\varphi)\ \left(=\lambda_{g_1}^{(p,q)}(f_2,\varphi)\right)$, we obtain for all sufficiently large values of r that

$$\text{(2)} \qquad T_{f_2}(r) \leq T_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q)}(f_2,\varphi)+\varepsilon\right)\log^{[q]}\varphi(r)\right]\right]$$

$$\text{(3)} \qquad i.e.,\ T_{f_2}(r) \leq T_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right].$$

Since $T_{f_1\pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, in view of (1) , (3) and Lemma 2, we obtain for a sequence values of r tending to infinity that

$$T_{f_1\pm f_2}(r) \leq 2\log M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right] + O(1)$$

$$\text{(4)} \qquad i.e.,\ T_{f_1\pm f_2}(r) \leq 3\log M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right].$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from (4) for a sequence values of r tending to infinity and $\sigma > 1$ that

$$T_{f_1\pm f_2}(r) \leq \frac{1}{3}\log\left[M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right]\right]^9$$

$$i.e.,\ T_{f_1\pm f_2}(r) \leq \frac{1}{3}\log M_{g_1}\left[\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right]^\sigma\right]$$

$$i.e.,\ T_{f_1\pm f_2}(r) \leq T_{g_1}\left[2\left[\exp^{[p]}\left[(\Delta+\varepsilon)\log^{[q]}\varphi(r)\right]\right]^\sigma\right].$$

Now we get from above by letting $\sigma \to 1^+$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p]} T_{g_1}^{-1}(T_{f_1\pm f_2}(r))}{\log^{[q]}\varphi(r)} < (\Delta+\varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}.$$

Similarly, if we consider that f_1 is of regular relative (p,q)-φ growth with respect to g_1 or both f_1 and f_2 are of regular relative (p,q)-φ growth with respect to g_1, then one can easily verify that

$$\text{(5)} \qquad \lambda_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}.$$

Further without loss of any generality, let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ and $f = f_1 \pm f_2$. Then in view of (5) we get that $\lambda_{g_1}^{(p,q)}(f,\varphi) \leq \lambda_{g_1}^{(p,q)}(f_2,\varphi)$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_2,\varphi) \leq \max\{\lambda_{g_1}^{(p,q)}(f,\varphi), \lambda_{g_1}^{(p,q)}(f_1,\varphi)\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$, therefore we have $\lambda_{g_1}^{(p,q)}(f_2,\varphi) \leq \lambda_{g_1}^{(p,q)}(f,\varphi)$ and hence $\lambda_{g_1}^{(p,q)}(f,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}$.

Therefore, $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = \lambda_{g_1}^{(p,q)}(f_i, \varphi) \mid i = 1,2$ provided $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \neq \lambda_{g_1}^{(p,q)}$ (f_2, φ). Thus the theorem is established. □

Theorem 5. *Let f_1 and f_2 be any two meromorphic functions and g_1 be an entire function such that such that $\rho_{g_1}^{(p,q)}(f_1, \varphi)$ and $\rho_{g_2}^{(p,q)}(f_1, \varphi)$ exist. Also let g_1 have the Property (A). Then we have*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) \leq \max\left\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_1}^{(p,q)}(f_2, \varphi)\right\}.$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1, \varphi) \neq \rho_{g_1}^{(p,q)}(f_2, \varphi)$.

We omit the proof of Theorem 5 as it can easily be carried out in the line of Theorem 4.

Theorem 6. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{g_1}^{(p,q)}(f_1, \varphi)$ and $\lambda_{g_2}^{(p,q)}(f_1, \varphi)$ exist. Also let $g_1 \pm g_2$ have the Property (A). Then we have*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}.$$

The equality holds when $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \neq \lambda_{g_2}^{(p,q)}(f_1, \varphi)$.

Proof. The result is obvious when $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) = \infty$. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) < \infty$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1, \varphi)$ is finite for $k = 1,2$. Further let $\Psi = \min\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_k}^{(p,q)}(f_1, \varphi)$, we have for all sufficiently large values of r that

$$T_{g_k}\left[\exp^{[p]}\left[\left(\lambda_{g_k}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\log^{[q]}\varphi(r)\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1,2 \tag{6}$$

$$i.e,\ T_{g_k}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]}\varphi(r)\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1,2$$

Since $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we obtain from above and Lemma 2 for all sufficiently large values of r that

$$T_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]}\varphi(r)\right]\right] \leq 2T_{f_1}(r) + O(1)$$

$$i.e.,\ T_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]}\varphi(r)\right]\right] < 3T_{f_1}(r).$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from above for all sufficiently large values of r and any $\sigma > 1$ that

$$\frac{1}{9}\log M_{g_1 \pm g_2}\left[\frac{\exp^{[p]}\left[(\Psi - \varepsilon)\log^{[q]}\varphi(r)\right]}{2}\right] < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2} \left[\frac{\exp^{[p]} \left[(\Psi - \varepsilon) \log^{[q]} \varphi(r) \right]}{2} \right]^{\frac{1}{9}} < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2} \left[\left(\frac{\exp^{[p]} \left[(\Psi - \varepsilon) \log^{[q]} \varphi(r) \right]}{2} \right)^{\frac{1}{\sigma}} \right] < T_{f_1}(r)$$

$$i.e.,\ T_{g_1 \pm g_2} \left[\left(\frac{\exp^{[p]} \left[(\Psi - \varepsilon) \log^{[q]} \varphi(r) \right]}{2} \right)^{\frac{1}{\sigma}} \right] < T_{f_1}(r)$$

As $\varepsilon > 0$ is arbitrary, we get from above by letting $\sigma \to 1^+$

$$\lambda^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) = \geq \Psi = \min \left\{ \lambda^{(p,q)}_{g_1}(f_1, \varphi), \lambda^{(p,q)}_{g_2}(f_1, \varphi) \right\}. \tag{7}$$

Now without loss of any generality, we may consider that $\lambda^{(p,q)}_{g_1}(f_1, \varphi) < \lambda^{(p,q)}_{g_2}(f_1, \varphi)$ and $g = g_1 \pm g_2$. Then in view of (7) we get that $\lambda^{(p,q)}_{g}(f_1, \varphi) \geq \lambda^{(p,q)}_{g_1}(f_1, \varphi)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that

$$\lambda^{(p,q)}_{g_1}(f_1, \varphi) \geq \min \left\{ \lambda^{(p,q)}_{g}(f_1, \varphi), \lambda^{(p,q)}_{g_2}(f_1, \varphi) \right\}.$$

As we assume that $\lambda^{(p,q)}_{g_1}(f_1, \varphi) < \lambda^{(p,q)}_{g_2}(f_1, \varphi)$, therefore we have $\lambda^{(p,q)}_{g_1}(f_1, \varphi) \geq \lambda^{(p,q)}_{g}(f_1, \varphi)$ and hence $\lambda^{(p,q)}_{g}(f_1, \varphi) = \lambda^{(p,q)}_{g_1}(f_1, \varphi) = \min \left\{ \lambda^{(p,q)}_{g_1}(f_1, \varphi), \lambda^{(p,q)}_{g_2}(f_1, \varphi) \right\}$. Therefore, $\lambda^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) = \lambda^{(p,q)}_{g_i}(f_1, \varphi) \mid i = 1, 2$ provided $\lambda^{(p,q)}_{g_1}(f_1, \varphi) \neq \lambda^{(p,q)}_{g_2}(f_1, \varphi)$. Thus the theorem follows. □

Theorem 7. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that f_1 is of regular relative (p, q)-φ growth with respect to at least any one of g_1 and g_2. If $g_1 \pm g_2$ have the Property (A), then we have*

$$\rho^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) \geq \min \left\{ \rho^{(p,q)}_{g_1}(f_1, \varphi), \rho^{(p,q)}_{g_2}(f_1, \varphi) \right\}.$$

The equality holds when any one of $\rho^{(p,q)}_{g_i}(f_1, \varphi) < \rho^{(p,q)}_{g_j}(f_1, \varphi)$ hold and at least f_1 is of regular relative (p, q)-φ growth with respect to any one of g_j where $i, j = 1, 2$ and $i \neq j$.

We omit the proof of Theorem 7 as it can easily be carried out in the line of Theorem 6.

Theorem 8. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \pm g_2$ have the Property (A). Then we have*

$$\begin{aligned} &\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2, \varphi) \\ \leq\ &\max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_2}^{(p,q)}(f_1, \varphi)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2, \varphi), \rho_{g_2}^{(p,q)}(f_2, \varphi)\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) Any one of $\rho_{g_i}^{(p,q)}(f_1, \varphi) < \rho_{g_j}^{(p,q)}(f_1, \varphi)$ hold and at least f_1 is of regular relative (p,q)-φ growth with respect to any one of g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) Any one of$\rho_{g_i}^{(p,q)}(f_2, \varphi) < \rho_{g_j}^{(p,q)}(f_2, \varphi)$ hold and at least f_2 is of regular relative (p,q)-φ growth with respect to any one of g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i, \varphi) < \rho_{g_1}^{(p,q)}(f_j, \varphi)$ and $\rho_{g_2}^{(p,q)}(f_i, \varphi) < \rho_{g_2}^{(p,q)}(f_j, \varphi)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 5 and Theorem 7 we get that

$$\begin{aligned} &\max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_2}^{(p,q)}(f_1, \varphi)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2, \varphi), \rho_{g_2}^{(p,q)}(f_2, \varphi)\right\}\right] \\ &= \max\left[\rho_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi), \rho_{g_1 \pm g_2}^{(p,q)}(f_2, \varphi)\right] \\ &\geq \rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2, \varphi)\ . \end{aligned} \tag{8}$$

Since $\rho_{g_1}^{(p,q)}(f_i, \varphi) < \rho_{g_1}^{(p,q)}(f_j, \varphi)$ and $\rho_{g_2}^{(p,q)}(f_i, \varphi) < \rho_{g_2}^{(p,q)}(f_j, \varphi)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\left\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_2}^{(p,q)}(f_1, \varphi)\right\} > \min\left\{\rho_{g_1}^{(p,q)}(f_2, \varphi), \rho_{g_2}^{(p,q)}(f_2, \varphi)\right\} \text{ or}$$

$$\min\left\{\rho_{g_1}^{(p,q)}(f_2, \varphi), \rho_{g_2}^{(p,q)}(f_2, \varphi)\right\} > \min\left\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_2}^{(p,q)}(f_1, \varphi)\right\} \text{ holds.}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } \rho_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) > \rho_{g_1 \pm g_2}^{(p,q)}(f_2, \varphi) \text{ or } \rho_{g_1 \pm g_2}^{(p,q)}(f_2, \varphi) > \rho_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi)$$

which is the condition for holding equality in (8).

Hence the theorem follows. □

Theorem 9. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let g_1, g_2 and $g_1 \pm g_2$ satisfy the Property (A). Then we have*

$$\begin{aligned} &\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2, \varphi) \\ &\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_2, \varphi)\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) Any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_1 *for* $i = 1,\ 2,\ j = 1, 2$ *and* $i \neq j$*; and*
(ii) Any one of $\lambda_{g_2}^{(p,q)}(f_i,\varphi) > \lambda_{g_2}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_2 *for* $i = 1,\ 2,\ j = 1, 2$ *and* $i \neq j$*.*
The equality holds when $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ *hold simultaneously for* $i = 1, 2$*;* $j = 1, 2$ *and* $i \neq j$*.*

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 4 and Theorem 6, we obtain that

$$\begin{aligned}
&\min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \\
&= \min\left[\lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi), \lambda_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi)\right] \\
&\geq \lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi). \qquad (9)
\end{aligned}$$

Since $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get that

$$\text{either } \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\} < \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\} \text{ or}$$

$$\max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\} < \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem holds, it follows from above that

$$\text{either } \lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) < \lambda_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi) \text{ or } \lambda_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi) < \lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi)$$

which is the condition for holding equality in (9).

Hence the theorem follows. □

Theorem 10. *Let* $f_1,\ f_2$ *be any two meromorphic functions and* g_1 *be any entire function such that at least* f_1 *or* f_2 *is of regular relative* (p,q)*-*φ *growth with respect to* g_1*. Also let* g_1 *satisfy the Property (A). Then we have*

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}.$$

The equality holds when any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_1 *where* $i, j = 1, 2$ *and* $i \neq j$*.*

Proof. Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, applying the same procedure as adopted in Theorem 4 we get that

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2, \varphi) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\}.$$

Now without loss of any generality, let $\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_1}^{(p,q)}(f_2, \varphi)$ and $f = f_1 \cdot f_2$. Then $\lambda_{g_1}^{(p,q)}(f, \varphi) \leq \lambda_{g_1}^{(p,q)}(f_2, \varphi)$. Further, $f_2 = \frac{f}{f_1}$ and $T_{f_1}(r) = T_{\frac{1}{f_1}}(r) + O(1)$. Therefore $T_{f_2}(r) \leq T_f(r) + T_{f_1}(r) + O(1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_2, \varphi) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f, \varphi), \lambda_{g_1}^{(p,q)}(f_1, \varphi)\right\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_1}^{(p,q)}(f_2, \varphi)$, therefore we have $\lambda_{g_1}^{(p,q)}(f_2, \varphi) \leq \lambda_{g_1}^{(p,q)}(f, \varphi)$ and hence $\lambda_{g_1}^{(p,q)}(f, \varphi)$ $= \lambda_{g_1}^{(p,q)}(f_2, \varphi) = \max\ \{\ \lambda_{g_1}^{(p,q)}(f_1, \varphi),\ \lambda_{g_1}^{(p,q)}(f_2, \varphi)\ \}$. Therefore, $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2, \varphi) = \lambda_{g_1}^{(p,q)}(f_i, \varphi) \mid i = 1, 2$ provided $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \neq \lambda_{g_1}^{(p,q)}(f_2, \varphi)$.

Hence the theorem follows. □

Next we prove the result for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 11. *Let f_1, f_2 be any two meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular relative (p,q)-φ growth with respect to g_1. Also let g_1 satisfy the Property (A). Then we have*

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}, \varphi\right) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\},$$

provided $\frac{f_1}{f_2}$ is meromorphic. The equality holds when at least f_2 is of regular relative (p,q)-φ growth with respect to g_1 and $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \neq \lambda_{g_1}^{(p,q)}(f_2, \varphi)$.

Proof. Since $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$ and $T_{\frac{f_1}{f_2}}(r) \leq T_{f_1}(r) + T_{\frac{1}{f_2}}(r)$, we get in view of Theorem 4 that

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}, \varphi\right) \leq \max\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\right\}. \tag{10}$$

Now in order to prove the equality conditions, we discuss the following two cases:

CASE I. Suppose $\frac{f_1}{f_2}\ (= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_1}^{(p,q)}(f_2, \varphi),$$

and f_2 is of regular relative (p,q)-φ growth with respect to g_1.

Now if possible, let $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}, \varphi\right) < \lambda_{g_1}^{(p,q)}(f_2, \varphi)$. Therefore from $f_1 = h \cdot f_2$ we get that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi)$ which is a contradiction. Therefore $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2}, \varphi\right) \geq$

$\lambda_{g_1}^{(p,q)}(f_2,\varphi)$ and in view of (10), we get that

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)=\lambda_{g_1}^{(p,q)}(f_2,\varphi).$$

Case II. Suppose $\frac{f_1}{f_2}\,(=h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1,\varphi)>\lambda_{g_1}^{(p,q)}(f_2,\varphi),$$

and f_2 is of regular relative (p,q)-φ growth with respect to g_1.

Now from $f_1=h\cdot f_2$ we get that either $\lambda_{g_1}^{(p,q)}(f_1,\varphi)\le\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)$ or $\lambda_{g_1}^{(p,q)}(f_1,\varphi)\le\lambda_{g_1}^{(p,q)}(f_2,\varphi)$. But according to our assumption $\lambda_{g_1}^{(p,q)}(f_1,\varphi)\nleq\lambda_{g_1}^{(p,q)}(f_2,\varphi)$. Therefore $\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)\ge\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ and in view of (10), we get that

$$\lambda_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)=\lambda_{g_1}^{(p,q)}(f_1,\varphi).$$

Hence the theorem follows. □

Now we state the following theorem which can easily be carried out in the line of Theorem 10 and Theorem 11 and therefore its proof is omitted.

Theorem 12. *Let f_1 and f_2 be any two meromorphic functions and g_1 be any entire function such that such that $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ exist. Also let g_1 satisfy the Property (A). Then we have*

$$\rho_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)\le\max\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_1}^{(p,q)}(f_2,\varphi)\right\}.$$

The equality holds when $\rho_{g_1}^{(p,q)}(f_1,\varphi)\ne\rho_{g_1}^{(p,q)}(f_2,\varphi)$. Similar results hold for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 13. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ exist. Also let $g_1\cdot g_2$ satisfy the Property (A). Then we have*

$$\lambda_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)\ge\min\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi),\lambda_{g_2}^{(p,q)}(f_1,\varphi)\right\}.$$

The equality holds when any one of $\lambda_{g_i}^{(p,q)}(f_1,\varphi)<\lambda_{g_j}^{(p,q)}(f_1,\varphi)$ hold where $i,j=1,2$ and $i\ne j$ and g_i satisfy the Property (A). Similar results hold for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (A). The equality holds when $\lambda_{g_1}^{(p,q)}(f_1,\varphi)\ne\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and g_1 satisfy the Property (A).

Proof. Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, applying the same procedure as adopted in Theorem 6 we get that

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1, \varphi) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}.$$

Now without loss of any generality, we may consider that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_2}^{(p,q)}(f_1, \varphi)$ and $g = g_1 \cdot g_2$. Then $\lambda_{g}^{(p,q)}(f_1, \varphi) \geq \lambda_{g_1}^{(p,q)}(f_1, \varphi)$. Further, $g_1 = \frac{g}{g_2}$ and and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \geq \min\left\{\lambda_{g}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_2}^{(p,q)}(f_1, \varphi)$, so we have $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \geq \lambda_{g}^{(p,q)}(f_1, \varphi)$ and hence $\lambda_{g}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \min\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}$. Therefore, $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1, \varphi) = \lambda_{g_i}^{(p,q)}(f_1, \varphi) \mid i = 1, 2$ provided $\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_2}^{(p,q)}(f_1, \varphi)$ and g_1 satisfy the Property (A). Hence the first part of the theorem follows.

Now we prove our results for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \neq \lambda_{g_2}^{(p,q)}(f_1, \varphi)$. Since $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$ and $T_{\frac{g_1}{g_2}}(r) \leq T_{g_1}(r) + T_{\frac{1}{g_2}}(r)$, we get in view of Theorem 6 that

$$\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) \geq \min\left\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\right\}. \tag{11}$$

Now in order to prove the equality conditions, we discuss the following two cases:

CASE I. Suppose $\frac{g_1}{g_2}(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1, \varphi) > \lambda_{g_2}^{(p,q)}(f_1, \varphi).$$

Now if possible, let $\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) > \lambda_{g_2}^{(p,q)}(f_1, \varphi)$. Therefore from $g_1 = h \cdot g_2$ we get that $\lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_2}^{(p,q)}(f_1, \varphi)$, which is a contradiction. Therefore $\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) \leq \lambda_{g_2}^{(p,q)}(f_1, \varphi)$ and in view of (11), we get that

$$\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) = \lambda_{g_2}^{(p,q)}(f_1, \varphi).$$

CASE II. Suppose that $\frac{g_1}{g_2}(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q)}(f_1, \varphi) < \lambda_{g_2}^{(p,q)}(f_1, \varphi).$$

Therefore from $g_1 = h \cdot g_2$, we get that either $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \geq \lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi)$ or $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \geq \lambda_{g_2}^{(p,q)}(f_1, \varphi)$. But according to our assumption $\lambda_{g_1}^{(p,q)}(f_1, \varphi) \ngeq \lambda_{g_2}^{(p,q)}(f_1, \varphi)$. Therefore $\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) \leq \lambda_{g_1}^{(p,q)}(f_1, \varphi)$ and in view of (11), we get that

$$\lambda_{\frac{g_1}{g_2}}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_1, \varphi).$$

Hence the theorem follows. □

Theorem 14. *Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ exist. Further let f_1 be of regular relative (p,q)-φ growth with respect to at least any one of g_1 and g_2. Also let $g_1 \cdot g_2$ satisfies the Property (A). Then we have*

$$\rho_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) \geq \min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\right\}.$$

The equality holds when any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ hold and at least f_1 is of regular relative (p,q)-φ growth with respect to any one of g_j where $i,j = 1,2$ and $i \neq j$ and g_i satisfies the Property (A).

Theorem 15. *Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ exist. Further let f_1 be of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2. Then we have*

$$\rho_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) \geq \min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\right\},$$

provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (A). The equality holds when at least f_1 is of regular relative (p,q)-φ growth with respect to g_2, $\rho_{g_1}^{(p,q)}(f_1,\varphi) \neq \rho_{g_2}^{(p,q)}(f_1,\varphi)$ and g_1 satisfies the Property (A).

We omit the proof of Theorem 14 and Theorem 15 as those can easily be carried out in the line of Theorem 13.

Now we state the following four theorems without their proofs as those can easily be carried out in the line of Theorem 8 and Theorem 9 respectively.

Theorem 16. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then we have*

$$\begin{aligned}&\rho_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi)\\ \leq\ & \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2,\varphi), \rho_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right],\end{aligned}$$

when the following two conditions holds:
(i) Any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ hold and at least f_1 is of regular relative (p,q)-φ growth with respect to any one of g_j and g_i satisfy the Property (A) for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ hold and at least f_2 is of regular relative (p,q)-φ growth with respect to any one of g_j and g_i satisfy the Property (A) for $i =$

1, 2, $j = 1,2$ and $i \neq j$.
The quality holds when $\rho_{g_1}^{(p,q)}(f_i,\varphi) < \rho_{g_1}^{(p,q)}(f_j,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_i,\varphi) < \rho_{g_2}^{(p,q)}(f_j,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 17. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A). Then we have*

$$\begin{aligned} & \lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) \\ \geq\ & \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) Any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ hold and at least any one of f_j is of regular relative (p,q)-φ growth with respect to g_1 for $i = 1$, 2, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\lambda_{g_2}^{(p,q)}(f_i,\varphi) > \lambda_{g_2}^{(p,q)}(f_j,\varphi)$ hold and at least any one of f_j is of regular relative (p,q)-φ growth with respect to g_2 for $i = 1$, 2, $j = 1,2$ and $i \neq j$.
The equality holds when $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 18. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$ satisfy the Property (A). Then we have*

$$\begin{aligned} & \rho_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) \\ \leq\ & \max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\right\}, \min\left\{\rho_{g_1}^{(p,q)}(f_2,\varphi), \rho_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) At least f_1 is of regular relative (p,q)-φ growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_1,\varphi) \neq \rho_{g_2}^{(p,q)}(f_1,\varphi)$; and
(ii) At least f_2 is of regular relative (p,q)-φ growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_2,\varphi) \neq \rho_{g_2}^{(p,q)}(f_2,\varphi)$.
The equality holds when $\rho_{g_1}^{(p,q)}(f_i,\varphi) < \rho_{g_1}^{(p,q)}(f_j,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_i,\varphi) < \rho_{g_2}^{(p,q)}(f_j,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 19. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$, g_1 and g_2 satisfy the Property (A). Then we have*

$$\lambda_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)$$
$$\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi),\lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\},\max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi),\lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right]$$

when the following two conditions hold:
(i) At least f_2 is of regular relative (p,q)-φ growth with respect to g_1 and $\lambda_{g_1}^{(p,q)}(f_1,\varphi) \neq \lambda_{g_1}^{(p,q)}(f_2,\varphi)$; and
(ii) At least f_2 is of regular relative (p,q)-φ growth with respect to g_2 and $\lambda_{g_2}^{(p,q)}(f_1,\varphi) \neq \lambda_{g_2}^{(p,q)}(f_2,\varphi)$.
The equality holds when $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Next we intend to find out the sum and product theorems of relative (p,q)-φ type (respectively relative (p,q)-φ lower type) and relative (p,q)-φ weak type of meromorphic function with respect to an entire function taking into consideration of the above theorems.

Theorem 20. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\rho_{g_1}^{(p,q)}(f_1,\varphi)$, $\rho_{g_1}^{(p,q)}(f_2,\varphi)$, $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_2,\varphi)$ be all non zero and finite.*
(A) If any one of $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ hold for i, $j = 1,2$; $i \neq j$, and g_1 has the Property (A), then

$$\sigma_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_i,\varphi) \text{ and } \overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\sigma}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2.$$

(B) If any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ hold and at least f_1 is of regular relative (p,q)-φ growth with respect to any one of g_j for i, $j = 1,2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (A), then

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \sigma_{g_i}^{(p,q)}(f_1,\varphi) \text{ and } \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_i}^{(p,q)}(f_1,\varphi) \mid i = 1,2.$$

(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) Any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ hold and at least f_1 is of regular relative (p,q)-φ growth with respect to any one of g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(ii) Any one of $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ hold and at least f_2 is of regular relative (p,q)-φ growth with respect to any one of g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_i,\varphi) > \rho_{g_2}^{(p,q)}(f_j,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(iv) $\rho_{g_m}^{(p,q)}(f_l,\varphi) =$

$\max\Big[\min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_2}^{(p,q)}(f_1,\varphi)\right\},\min\left\{\rho_{g_1}^{(p,q)}(f_2,\varphi),\rho_{g_2}^{(p,q)}(f_2,\varphi)\right\}\Big] \mid l,m=1,2,$
and $g_1 \pm g_2$ *has the Property (A);*
then

$$\sigma_{g_1\pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi)=\sigma_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m=1,2$$

and

$$\overline{\sigma}_{g_1\pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi)=\overline{\sigma}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m=1,2.$$

Proof. From the definition of relative (p,q)-φ type and relative (p,q)-φ lower type of meromorphic function with respect to an entire function, we have for all sufficiently large values of r that

$$T_{f_k}(r) \le T_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q)}(f_k,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{12}$$

$$T_{f_k}(r) \ge T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q)}(f_k,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right] \tag{13}$$

and for a sequence of values of r tending to infinity, we obtain that

$$T_{f_k}(r) \ge T_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q)}(f_k,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{14}$$

and

$$T_{f_k}(r) \le T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q)}(f_k,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_l}^{(p,q)}(f_k)}\right\}\right], \tag{15}$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1,\ 2$ and $l = 1, 2$.

CASE I. Suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ hold. Also let $\varepsilon\,(>0)$ be arbitrary. Since $T_{f_1\pm f_2}(r) \le T_{f_1}(r)+T_{f_2}(r)+O(1)$ for all large r, so in view of (12), we get for all sufficiently large values of r that

$$T_{f_1\pm f_2}(r) \le$$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right](1+A). \tag{16}$$

where $A = \dfrac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_2,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_2,\varphi)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]}$, and in view of $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$, and for all sufficiently large values of r, we can make the term A sufficiently small . Hence for any $\alpha = 1 + \varepsilon_1$, it follows from (16) for all sufficiently large values of r that

$$T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right]\cdot(1+\varepsilon_1)$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1)}\right\}\right]\cdot\alpha.$$

Hence making $\alpha \to 1+$, we get in view of Theorem 5, $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ and above for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{g_1}^{-1}(T_{f_1\pm f_2}(r))}{\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)}} \leq \sigma_{g_1}^{(p,q)}(f_1,\varphi)$$

$$i.e.,\ \sigma_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \leq \sigma_{g_1}^{(p,q)}(f_1,\varphi)\ . \tag{17}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ hold. Then $\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \leq \sigma_{g_1}^{(p,q)}(f_1,\varphi)$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 5 and $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$, we obtain that $\rho_{g_1}^{(p,q)}(f,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ holds. Hence in view of (17) $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \leq \sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)$. Therefore

$$\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \sigma_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_1,\varphi).$$

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_1}^{(p,q)}(f_2,\varphi)$, then one can easily verify that $\sigma_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_2,\varphi)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ hold. Also let $\varepsilon\,(>0)$ are arbitrary. Since $T_{f_1\pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, from (12) and (15), we get for a sequence of values of r tending to infinity that

$$T_{f_1\pm f_2}(r_n) \leq$$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right](1+B). \tag{18}$$

where $B = \dfrac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_2,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_2,\varphi)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]}$, and in view of $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$, we can make the term B sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (18) that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = \overline{\sigma}_{g_1}^{(p,q)}(f_1, \varphi)$ when $\rho_{g_1}^{(p,q)}(f_1, \varphi) > \rho_{g_1}^{(p,q)}(f_2, \varphi)$ hold. Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1, \varphi) < \rho_{g_1}^{(p,q)}(f_2, \varphi)$, then one can easily verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = \overline{\sigma}_{g_1}^{(p,q)}(f_2, \varphi)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

CASE III. Let us consider that $\rho_{g_1}^{(p,q)}(f_1, \varphi) < \rho_{g_2}^{(p,q)}(f_1, \varphi)$ with at least f_1 is of regular relative (p, q)-φ growth with respect to g_2. We can make the term $C = \frac{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+O(1)}{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_2}^{(p,q)}(f_1,\varphi)}\right\}\right]}$ sufficiently small by taking n sufficiently large, since $\rho_{g_1}^{(p,q)}(f_1, \varphi) < \rho_{g_2}^{(p,q)}(f_1, \varphi)$. Hence $C < \varepsilon_1$.

As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we get that

$$T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right) \leq$$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,)}(f_1,\varphi)}\right\}\right] +$$

$$T_{g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right] + O(1).$$

Therefore for any $\alpha = 1 + \varepsilon_1$, we obtain in view of $C < \varepsilon_1$, (13) and (14) for a sequence of values of r tending to infinity that

$$T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right) \leq \alpha T_{f_1}(r_n)$$

Now making $\alpha \to 1+$, we obtain from above for a sequence of values of r tending to infinity that

$$\left(\sigma_{g_1}^{(p,q)}(f_1, \varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi)} < \log^{[p-1]} T_{g_1 \pm g_2}^{-1} T_{f_1}(r_n)$$

Since $\varepsilon > 0$ is arbitrary, we find that

$$\sigma_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) \geq \sigma_{g_1}^{(p,q)}(f_1, \varphi). \tag{19}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\rho_{g_1}^{(p,q)}(f_1, \varphi) < \rho_{g_2}^{(p,q)}(f_1, \varphi)$ and at least f_1 is of regular relative (p, q)-φ growth with respect to g_2. Then $\sigma_{g}^{(p,q)}(f_1, \varphi) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) \geq \sigma_{g_1}^{(p,q)}(f_1, \varphi)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 7 and $\rho_{g_1}^{(p,q)}(f_1, \varphi) < \rho_{g_2}^{(p,q)}(f_1, \varphi)$, we obtain that $\rho_{g}^{(p,q)}(f_1, \varphi) < \rho_{g_2}^{(p,q)}(f_1, \varphi)$ as at least f_1 is of regular relative (p, q)-φ growth with respect to g_2. Hence in view of (19), $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \geq \sigma_{g}^{(p,q)}(f_1,\varphi) = \sigma_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi)$. Therefore $\sigma_{g}^{(p,q)}(f_1,\varphi) = \sigma_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \sigma_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \sigma_{g_1}^{(p,q)}(f_1,\varphi)$.

Similarly if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \sigma_{g_2}^{(p,q)}(f_1,\varphi)$.

Case IV. In this case suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_2. we can also make the term $D = \frac{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+O(1)}{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_2}^{(p,q)}(f_1,\varphi)}\right\}\right]}$ sufficiently small by taking r sufficiently large as $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$. So $D < \varepsilon_1$ for sufficiently large r. As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, therefore from (13), we get for all sufficiently large values of r that

$$T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right) \leq$$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right] +$$

$$T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right] + O(1)$$

$$i.e.,\ T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1,\varphi)}\right\}\right)$$

$$\leq (1+\varepsilon_1)\, T_{f_1}(r), \tag{20}$$

and therefore using the similar technique for as executed in the proof of Case III we get from (20) that $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)$ where $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$ and at least f_1 is of regular relative (p,q)-φ growth with respect to g_2.

Likewise if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 8 and the first part and second part of the theorem. Hence its proof is omitted. □

Theorem 21. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$, $\lambda_{g_1}^{(p,q)}(f_2,\varphi)$, $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_2}^{(p,q)}(f_2,\varphi)$ be all nonzero and finite.*

(A) Any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_1 *for* $i,\ j = 1,2$*;* $i \neq j$*, and* g_1 *has the Property (A), then*

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_i,\varphi) \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2.$$

(B) Any one of $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *hold for* $i,\ j = 1,2$*;* $i \neq j$ *and* $g_1 \pm g_2$ *has the Property (A), then*

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_i}^{(p,q)}(f_1,\varphi) \text{ and } \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_i}^{(p,q)}(f_1,\varphi) \mid i = 1,2.$$

(C) Assume the functions f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(i) Any one of $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_1 *for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(ii) Any one of $\rho_{g_2}^{(p,q)}(f_i,\varphi) > \rho_{g_2}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)*-*φ *growth with respect to* g_2 *for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(iii) $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *and* $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ *holds simultaneously for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(iv) $\lambda_{g_m}^{(p,q)}(f_l,\varphi) =$
$\min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \mid l,m = 1,2$ *and* $g_1 \pm g_2$ *has the Property (A)*
then we have

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \tau_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2$$

and

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\tau}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2.$$

Proof. For any arbitrary positive number $\varepsilon(>0)$, we have for all sufficiently large values of r that

$$T_{f_k}(r) \leq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q)}(f_k,\varphi) + \varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\right\}\right], \tag{21}$$

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q)}(f_k,\varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\right\}\right], \tag{22}$$

and for a sequence of values of r tending to infinity we obtain that

$$T_{f_k}(r) \geq T_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q)}(f_k,\varphi) - \varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\right\}\right] \tag{23}$$

and

$$(24)\qquad T_{f_k}(r) \le T_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q)}(f_k,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\right\}\right],$$

where $k = 1, 2$ and $l = 1, 2$.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_2 is of regular relative (p,q)-φ growth with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. Since $T_{f_1\pm f_2}(r) \le T_{f_1}(r)+T_{f_2}(r)+O(1)$ for all large r, we get from (21) and (24), for a sequence $\{r_n\}$ of values of r tending to infinity that

$$T_{f_1\pm f_2}(r_n) \le$$

$$(25)\qquad T_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right](1+E).$$

where $E = \dfrac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_2,\varphi)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, we can make the term E sufficiently small by taking n sufficiently large. Now with the help of Theorem 4 and using the similar technique of Case I of Theorem 20, we get from (25) that

$$(26)\qquad \tau_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \le \tau_{g_1}^{(p,q)}(f_1,\varphi).$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ and at least f_2 is of regular relative (p,q)-φ growth with respect to g_1. Then $\tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) \le \tau_{g_1}^{(p,q)}(f_1,\varphi)$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 4, $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ and at least f_2 is of regular relative (p,q)-φ growth with respect to g_1, we obtain that $\lambda_{g_1}^{(p,q)}(f,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ holds. Hence in view of (26), $\tau_{g_1}^{(p,q)}(f_1,\varphi) \le \tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)$. Therefore $\tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \tau_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1 then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1\pm f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_2,\varphi)$.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_2 is of regular relative (p,q)-φ growth with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. As $T_{f_1\pm f_2}(r) \le T_{f_1}(r)+T_{f_2}(r)+O(1)$ for all large r, we obtain from (21) for all sufficiently large values of r that

$T_{f_1 \pm f_2}(r) \leq$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right](1+F). \tag{27}$$

where $F = \frac{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_i}^{(p,q)}(f_2,\varphi)}\right\}\right]+O(1)}{T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)+\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]}$, and in view of $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$, we can make the term F sufficiently small by taking r sufficiently large and therefore for similar reasoning of Case I we get from (27) that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)$ when $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ and at least f_2 is of regular relative (p,q)-φ growth with respect to g_1.

Likewise, if we consider $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1 then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

CASE III. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore we can make the term $G = \frac{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+O(1)}{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_2}^{(p,q)}(f_1,\varphi)}\right\}\right]}$ sufficiently small by taking r sufficiently large since $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. So $G < \varepsilon_1$. Since $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we get from (22) for all sufficiently large values of r that

$$T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right) \leq$$

$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+$$

$$T_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+O(1)$$

$$i.e.,\ T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right)$$

$$\leq (1+\varepsilon_1)\,T_{f_1}(r). \tag{28}$$

Therefore in view of Theorem 6 and using the similar technique of Case III of Theorem 20, we get from (28) that

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \tau_{g_1}^{(p,q)}(f_1,\varphi). \tag{29}$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$, so $\tau_{g}^{(p,q)}(f_1,\varphi) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \tau_{g_1}^{(p,q)}(f_1,\varphi)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 6 and $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ we obtain that $\lambda_{g}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ holds. Hence in view of (29) $\tau_{g_1}^{(p,q)}(f_1,\varphi) \geq \tau_{g}^{(p,q)}(f_1,\varphi) = \tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi)$. Therefore $\tau_{g}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi)$.

Likewise, if we consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_2}^{(p,q)}(f_1,\varphi)$, then one can easily verify that $\tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_2}^{(p,q)}(f_1,\varphi)$.

Case IV. In this case further we consider $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Further we can make the term $H = \dfrac{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right]+O(1)}{T_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_2}^{(p,q)}(f_1,\varphi)}\right\}\right]}$ sufficiently small by taking n sufficiently large, since $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore $H < \varepsilon_1$ for sufficiently large n. As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we obtain from (22) and (23), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right) \leq$$
$$T_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right] +$$
$$T_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right] + O(1)$$
$$i.e.,\ T_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)-\varepsilon\right)\left[\log^{[q-1]}\varphi(r_n)\right]^{\lambda_{g_1}^{(p,q)}(f_1,\varphi)}\right\}\right)$$
$$\leq (1+\varepsilon_1)\,T_{f_1}(r), \tag{30}$$

and therefore using the similar technique for as executed in the proof of Case IV of Theorem 20, we get from (30) that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)$ when $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_2}^{(p,q)}(f_1,\varphi)$, then one can easily verify that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 9 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 4 to Theorem 7 under somewhat different conditions.

Theorem 22. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) The following condition is assumed to be satisfied:
(i) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$ holds and g_1 has the Property (A), then

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$ holds and $g_1 \pm g_2$ has the Property (A);
(ii) f_1 is of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2, then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi).$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

CASE I. Suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$ $(0 < \rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_1}^{(p,q)}(f_2,\varphi) < \infty)$. Now in view of Theorem 5 it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \leq \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) < \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi). \tag{31}$$

Let $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$. Then in view of the first part of Theorem 20 and (31) we obtain that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) = \sigma_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$. Similarly with the help of the first part of Theorem 20, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$. This proves the first part of the theorem.

CASE II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi)$ $(0 < \rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi) < \infty)$, f_1 is of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$ and $g_1 \pm g_2$ satisfy the Property (A). Therefore in view of Theorem 7, it follows that $\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi)$ and if possible let

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi). \tag{32}$$

Let us consider that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$. Then. in view of the proof of the second part of Theorem 20 and (32) we obtain that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) = \sigma_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1,\varphi) = \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ which is a contradiction. Hence $\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi)$. Also in view of the proof of second part of Theorem 20 one can derive the same conclusion for the condition $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$ and therefore the second part of the theorem is established. □

Theorem 23. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) The following conditions are assumed to be satisfied:
(i) $(f_1 \pm f_2)$ is of regular relative (p,q)-φ growth with respect to at least any one of g_1 and g_2, and g_1, g_2 , $g_1 \pm g_2$ have the Property (A);
(ii) Either $\sigma_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$;
(iv) Either $\sigma_{g_2}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2,\varphi)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) f_1 and f_2 are of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2, and $g_1 \pm g_2$ has the Property (A);
(ii) Either $\sigma_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1 \pm g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1 \pm g_2}^{(p,q)}(f_2,\varphi)$;
(iii) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$;
(iv) Either $\sigma_{g_1}^{(p,q)}(f_2,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2,\varphi)$; then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_2,\varphi).$$

We omit the proof of Theorem 23 as it is a natural consequence of Theorem 22.

Theorem 24. *Let f_1, f_2 be any two meromorphic functions and g_1,g_2 be any two entire functions.*
(A) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 is of regular relative (p,q)-φ growth with respect to g_1;

(ii) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$ *holds and* g_1 *has the Property (A), then*

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) f_1, g_1 *and* g_2 *be any three entire functions such that* $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ *exists;*
(ii) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ *holds and* $g_1 \pm g_2$ *has the Property (A), then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi).$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

CASE I. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi) < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular relative (p,q)-φ growth with respect to g_1. Now, in view of Theorem 4, it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \leq \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) < \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi). \tag{33}$$

Let $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$. Then in view of the proof of the first part of Theorem 21 and (33) we obtain that $\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2 \mp f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_2,\varphi)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$. Similarly in view of the proof of the first part of Theorem 21 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$. This proves the first part of the theorem.

CASE II. Let us consider that

$$\lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)\,(0 < \lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_1,\varphi) < \infty.$$

Therefore in view of Theorem 6, it follows that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and if possible let

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi). \tag{34}$$

Suppose $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$. Then in view of the second part of Theorem 21 and (34), we obtain that $\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_{g_1 \pm g_2 \mp g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_2}^{(p,q)}(f_1,\varphi)$ which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Analogously with the help of the second part of Theorem 21, the same conclusion can also be derived under the condition $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ and therefore the second part of the theorem is established. □

Theorem 25. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 is of regular relative (p,q)-φ growth with respect to g_1 and g_2. Also g_1, g_2, $g_1 \pm g_2$ have satisfy the Property (A);
(ii) Either $\tau_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi)$ or

$$\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1 \pm f_2,\varphi);$$

(iii) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$;
(iv) Either $\tau_{g_2}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2,\varphi)$; then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 or f_2 are of regular relative (p,q)-φ growth with respect to $g_1 \pm g_2$, and $g_1 \pm g_2$ has satisfy the Property (A);
(ii) Either $\tau_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1 \pm g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_2,\varphi)$ holds;
(iii) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ holds;
(iv) Either $\tau_{g_1}^{(p,q)}(f_2,\varphi) \neq \tau_{g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2,\varphi)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_2,\varphi).$$

We omit the proof of Theorem 25 as it is a natural consequence of Theorem 24.

Theorem 26. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\rho_{g_1}^{(p,q)}(f_1,\varphi)$, $\rho_{g_1}^{(p,q)}(f_2,\varphi)$, $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_2,\varphi)$ be all non zero and finite.*
(A) Assume the functions f_1, f_2 and g_1 satisfy the following conditions:
(i) Any one of $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ hold for i, $j = 1,2$ and $i \neq j$;
(ii) g_1 satisfies the Property (A), then

$$\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_i,\varphi) \text{ and } \overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \overline{\sigma}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2.$$

Similarly,

$$\sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \sigma_{g_1}^{(p,q)}(f_i,\varphi) \text{ and } \overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \overline{\sigma}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2$$

holds provided (i) $\frac{f_1}{f_2}$ *is meromorphic, (ii)* $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi) \mid i,\ 1,2;\ j = 1,2;\ i \neq j$ *and (iii)* g_1 *satisfy the Property (A).*

(B) Assume the functions g_1, g_2 *and* f_1 *satisfy the following conditions:*

(i) Any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *hold and at least* f_1 *is of regular relative* (p,q)*-*φ *growth with respect to any one of* g_j *for* $i,\ j = 1,2$ *and* $i \neq j$*, and* g_i *satisfies the Property (A);*

(ii) $g_1 \cdot g_2$ *satisfies the Property (A), then*

$$\sigma_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \sigma_{g_i}^{(p,q)}(f_1,\varphi) \ \text{ and } \ \overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_i}^{(p,q)}(f_1,\varphi) \mid i = 1,2.$$

Similarly,

$$\sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \sigma_{g_i}^{(p,q)}(f_1,\varphi) \ \text{ and } \ \overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{gi}^{(p,q)}(f_1,\varphi) \mid i = 1,2$$

holds provided (i) $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A), (ii) At least* f_1 *is of regular relative* (p,q)*-*φ *growth with respect to* g_2*, (iii)* $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi) \mid i = 1,2;\ j = 1,2;\ i \neq j$ *and (iv)* g_1 *satisfy the Property (A).*

(C) Assume the functions $f_1, f_2,\ g_1$ *and* g_2 *satisfy the following conditions:*

(i) $g_1 \cdot g_2$ *satisfies the Property (A);*

(ii) Any one of $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *hold and at least* f_1 *is of regular relative* (p,q)*-*φ *growth with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*

(iii) Any one of $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ *hold and at least* f_2 *is of regular relative* (p,q)*-*φ *growth with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*

(iv) $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ *and* $\rho_{g_2}^{(p,q)}(f_i,\varphi) > \rho_{g_2}^{(p,q)}(f_j,\varphi)$ *holds simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$*;*

(v) $\rho_{g_m}^{(p,q)}(f_l,\varphi) =$
$\max\Big[\min\Big\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\Big\}, \min\Big\{\rho_{g_1}^{(p,q)}(f_2,\varphi), \rho_{g_2}^{(p,q)}(f_2,\varphi)\Big\}\Big] \mid l,m = 1,2;$
then

$$\sigma_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) = \sigma_{g_m}^{(p,q)}(f_l,\varphi) \ \text{ and } \overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) = \overline{\sigma}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2\ .$$

Similarly,

$$\sigma_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \sigma_{g_m}^{(p,q)}(f_l,\varphi) \ \text{ and } \overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \overline{\sigma}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2.$$

holds provided $\frac{f_1}{f_2}$ *is meromorphic function and* $\frac{g_1}{g_2}$ *is entire function which satisfy the following conditions:*

(i) $\frac{g_1}{g_2}$ *satisfies the Property (A);*

(ii) At least f_1 *is of regular relative* (p,q)*-*φ *growth with respect to* g_2 *and* $\rho_{g_1}^{(p,q)}(f_1,\varphi)$

$\neq \rho_{g_2}^{(p,q)}(f_1,\varphi)$;
(iii) At least f_2 is of regular relative (p,q)-φ growth with respect to g_2 and $\rho_{g_1}^{(p,q)}(f_2,\varphi)$ $\neq \rho_{g_2}^{(p,q)}(f_2,\varphi)$;
(iv) $\rho_{g_1}^{(p,q)}(f_i,\varphi) < \rho_{g_1}^{(p,q)}(f_j,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_i,\varphi) < \rho_{g_2}^{(p,q)}(f_j,\varphi)$ holds simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$;
(v) $\rho_{g_m}^{(p,q)}(f_l,\varphi) =$
$\max\left[\min\left\{\rho_{g_1}^{(p,q)}(f_1,\varphi),\rho_{g_2}^{(p,q)}(f_1,\varphi)\right\},\min\left\{\rho_{g_1}^{(p,q)}(f_2,\varphi),\rho_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \mid l,m=1,2.$

Proof. Let us suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi)$, $\rho_{g_1}^{(p,q)}(f_2,\varphi)$, $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_2,\varphi)$ are all non zero and finite.

Case I. Suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$. Also let g_1 satisfy the Property (A). Since $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 20 we get that

$$\sigma_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) \leq \sigma_{g_1}^{(p,q)}(f_1,\varphi). \tag{35}$$

Further without loss of any generality, let $f=f_1\cdot f_2$ and $\rho_{g_1}^{(p,q)}(f_2,\varphi) < \rho_{g_1}^{(p,q)}(f_1,\varphi)$ $=\rho_{g_1}^{(p,q)}(f,\varphi)$. Then in view of (35), we obtain that $\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)$ $\leq \sigma_{g_1}^{(p,q)}(f_1,\varphi)$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq$ $T_f(r) + T_{f_2}(r) + O(1)$ and in this case also we obtain from (35) that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \leq$ $\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)$. Hence $\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \sigma_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)$ $=\sigma_{g_1}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_1}^{(p,q)}(f_2,\varphi)$, then one can verify that $\sigma_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) = \sigma_{g_1}^{(p,q)}(f_2,\varphi)$.

Next we may suppose that $f=\frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions.

Sub Case I_A. Let $\rho_{g_1}^{(p,q)}(f_2,\varphi) < \rho_{g_1}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 12, $\rho_{g_1}^{(p,q)}(f_2,\varphi) < \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f,\varphi)$. We have $f_1 = f\cdot f_2$. So, $\sigma_{g_1}^{(p,q)}(f_1,\varphi) =$ $\sigma_{g_1}^{(p,q)}(f,\varphi) = \sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)$.

Sub Case I_B. Let $\rho_{g_1}^{(p,q)}(f_2,\varphi) > \rho_{g_1}^{(p,q)}(f_1,\varphi)$. Therefo re in view of Theorem 12, $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_1}^{(p,q)}(f_2,\varphi) = \rho_{g_1}^{(p,q)}(f,\varphi)$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) +$ $O(1)$, So $\sigma_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \sigma_{g_1}^{(p,q)}(f_2,\varphi)$.

Case II. Let $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$. Also let g_1 satisfy the Property (A). As $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as explored in Case II of Theorem 20, one can easily verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) =$

$\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)$ and $\overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)=\overline{\sigma}_{g_1}^{(p,q)}(f_i,\varphi)\mid i=1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi)<\rho_{g_1}^{(p,q)}(f_2,\varphi)$, then one can verify that $\overline{\sigma}_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)=\overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$ and $\overline{\sigma}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)=\overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$.

Therefore the first part of theorem follows from Case I and Case II.

CASE III. Let $g_1\cdot g_2$ satisfy the Property (A) and $\rho_{g_1}^{(p,q)}(f_1,\varphi)<\rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_2. Since $T_{g_1\cdot g_2}(r)\le T_{g_1}(r)+T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 20 we get that

$$\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)\ge\sigma_{g_1}^{(p,q)}(f_1,\varphi). \tag{36}$$

Further without loss of any generality, let $g=g_1\cdot g_2$ and $\rho_g^{(p,q)}(f_1,\varphi)=\rho_{g_1}^{(p,q)}(f_1,\varphi)$ $<\rho_{g_2}^{(p,q)}(f_1,\varphi)$. Then in view of (36), we obtain that $\sigma_g^{(p,q)}(f_1,\varphi)=\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)$ $\ge\sigma_{g_1}^{(p,q)}(f_1,\varphi)$. Also $g_1=\frac{g}{g_2}$ and $T_{g_2}(r)=T_{\frac{1}{g_2}}(r)+O(1)$. Therefore $T_{g_1}(r)\le T_g(r)+T_{g_2}(r)+O(1)$ and in this case we obtain from (36) that $\sigma_{g_1}^{(p,q)}(f_1,\varphi)\ge\sigma_g^{(p,q)}(f_1,\varphi)=\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)$. Hence $\sigma_g^{(p,q)}(f_1,\varphi)=\sigma_{g_1}^{(p,q)}(f_1,\varphi)\Rightarrow\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)=\sigma_{g_1}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi)>\rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then one can verify that $\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)=\sigma_{g_2}^{(p,q)}(f_1,\varphi)$.

Next we may suppose that $g=\frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

SUB CASE III$_A$. Let $\rho_{g_1}^{(p,q)}(f_1,\varphi)<\rho_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 15, $\rho_g^{(p,q)}(f_1,\varphi)=\rho_{g_1}^{(p,q)}(f_1,\varphi)<\rho_{g_2}^{(p,q)}(f_1,\varphi)$. We have $g_1=g\cdot g_2$. So $\sigma_{g_1}^{(p,q)}(f_1,\varphi)$ $=\sigma_g^{(p,q)}(f_1,\varphi)=\sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi)$.

SUB CASE III$_B$. Let $\rho_{g_1}^{(p,q)}(f_1,\varphi)>\rho_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 15, $\rho_g^{(p,q)}(f_1,\varphi)=\rho_{g_2}^{(p,q)}(f_1,\varphi)<\rho_{g_1}^{(p,q)}(f_1,\varphi)$. Since $T_g(r)=T_{\frac{1}{g}}(r)+O(1)=T_{\frac{g_2}{g_1}}(r)+O(1)$, So $\sigma_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi)=\sigma_{g_2}^{(p,q)}(f_1,\varphi)$.

CASE IV. Suppose $g_1\cdot g_2$ satisfy the Property (A). Also let $\rho_{g_1}^{(p,q)}(f_1,\varphi)<\rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_2. As $T_{g_1\cdot g_2}(r)\le T_{g_1}(r)+T_{g_2}(r)$ for all large r, the same procedure as explored in Case IV of Theorem 20, one can easily verify that $\overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)=\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi)$ and $\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi)=\overline{\sigma}_{g_i}^{(p,q)}(f_1,\varphi)\mid i=1,2$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then one can verify that $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$ and $\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 16 and Theorem 18 and the above cases. □

Theorem 27. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$, $\lambda_{g_1}^{(p,q)}(f_2,\varphi)$, $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_2}^{(p,q)}(f_2,\varphi)$ be all non zero and finite.*
(A) Assume the functions f_1, f_2 and g_1 satisfy the following conditions:
(i) Any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ hold and at least any one of f_j is of regular relative (p,q)-φ growth with respect to g_1 for i, $j = 1,2$ and $i \neq j$;
(ii) g_1 satisfies the Property (A), then

$$\tau_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_i,\varphi) \;\; \textit{and} \;\; \overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2.$$

Similarly,

$$\tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \tau_{g_1}^{(p,q)}(f_i,\varphi) \;\; \textit{and} \;\; \overline{\tau}_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \overline{\tau}_{g_1}^{(p,q)}(f_i,\varphi) \mid i = 1,2$$

holds provided $\frac{f_1}{f_2}$ is meromorphic, at least f_2 is of regular relative (p,q)-φ growth with respect to g_1 where g_1 satisfy the Property (A) and $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ $\mid i = 1,2;\ j = 1,2;\ i \neq j$.
(B) Assume the functions g_1, g_2 and f_1 satisfy the following conditions:
(i) Any one of $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ hold for i, $j = 1,2$, $i \neq j$; and g_i satisfy the Property (A)
(ii) $g_1 \cdot g_2$ satisfy the Property (A), then

$$\tau_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_i}^{(p,q)}(f_1,\varphi) \;\; \textit{and} \;\; \overline{\tau}_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_i}^{(p,q)}(f_1,\varphi) \mid i = 1,2.$$

Similarly,

$$\tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \tau_{g_i}^{(p,q)}(f_1,\varphi) \;\; \textit{and} \;\; \overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{gi}^{(p,q)}(f_1,\varphi) \mid i = 1,2$$

holds provided $\frac{g_1}{g_2}$ is entire and satisfy the Property (A), g_1 satisfy the Property (A) and $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi) \mid i = 1,2;\ j = 1,2;\ i \neq j$.
(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
(i) $g_1 \cdot g_2$, g_1 and g_2 are satisfy the Property (A);
(ii) Any one of $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ hold and at least any one of f_j is of

regular relative (p,q)-φ *growth with respect to* g_1 *for* $i = 1, 2,\ j = 1, 2$ *and* $i \neq j$;
(iii) Any one of $\lambda_{g_2}^{(p,q)}(f_i,\varphi) > \lambda_{g_2}^{(p,q)}(f_j,\varphi)$ *hold and at least any one of* f_j *is of regular relative* (p,q)-φ *growth with respect to* g_2 *for* $i = 1, 2,\ j = 1, 2$ *and* $i \neq j$;
(iv) $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ *holds simultaneously for* $i = 1, 2;\ j = 1, 2$ *and* $i \neq j$;
(v) $\lambda_{g_m}^{(p,q)}(f_l,\varphi) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \mid l,m = 1,2$*; then*

$$\tau_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi) = \tau_{g_m}^{(p,q)}(f_l,\varphi) \text{ and } \overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi) = \overline{\tau}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2.$$

Similarly,

$$\tau_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \tau_{g_m}^{(p,q)}(f_l,\varphi) \text{ and } \overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \overline{\tau}_{g_m}^{(p,q)}(f_l,\varphi) \mid l,m = 1,2.$$

holds provided $\frac{f_1}{f_2}$ *is meromorphic and* $\frac{g_1}{g_2}$ *is entire functions which satisfy the following conditions:*
(i) $\frac{g_1}{g_2}$, g_1 *and* g_2 *satisfy the Property (A);*
(ii) At least f_2 *is of regular relative* (p,q)-φ *growth with respect to* g_1 *and* $\lambda_{g_1}^{(p,q)}(f_1,\varphi) \neq \lambda_{g_1}^{(p,q)}(f_2,\varphi)$;
(iii) At least f_2 *is of regular relative* (p,q)-φ *growth with respect to* g_2 *and* $\lambda_{g_2}^{(p,q)}(f_1,\varphi) \neq \lambda_{g_2}^{(p,q)}(f_2,\varphi)$;
(iv) $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_i}^{(p,q)}(f_2,\varphi) < \lambda_{g_j}^{(p,q)}(f_2,\varphi)$ *holds simultaneously for* $i = 1, 2;\ j = 1, 2$ *and* $i \neq j$;
(v) $\lambda_{g_m}^{(p,q)}(f_l,\varphi) = \min\left[\max\left\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\right\}, \max\left\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\right\}\right] \mid l,m = 1,2.$

Proof. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$, $\lambda_{g_1}^{(p,q)}(f_2,\varphi)$, $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and $\lambda_{g_2}^{(p,q)}(f_2,\varphi)$ are all non zero and finite.

CASE I. Suppose $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_2 is of regular relative (p,q)-φ growth with respect to g_1 and g_1 satisfy the Property (A). Since $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 21 we get that

$$\tau_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) \leq \tau_{g_1}^{(p,q)}(f_1,\varphi). \tag{37}$$

Further without loss of any generality, let $f = f_1\cdot f_2$ and $\lambda_{g_1}^{(p,q)}(f_2,\varphi) < \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f,\varphi)$. Then in view of (37), we obtain that $\tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)$

$\leq \tau_{g_1}^{(p,q)}(f_1,\varphi)$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$ and in this case we obtain from the above arguments that $\tau_{g_1}^{(p,q)}(f_1,\varphi) \leq \tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi)$. Hence $\tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \tau_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \tau_{g_1}^{(p,q)}(f_2,\varphi)$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions satisfying the conditions specified in the theorem.

Sub Case I$_A$. Let $\lambda_{g_1}^{(p,q)}(f_2,\varphi) < \lambda\ _{g_1}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 11, $\lambda_{g_1}^{(p,q)}(f_2,\varphi) < \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f,\varphi)$. We have $f_1 = f \cdot f_2$. So $\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f,\varphi) = \tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right)$.

Sub Case I$_B$. Let $\lambda_{g_1}^{(p,q)}(f_2,\varphi) > \lambda_{g_1}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 11, $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f,\varphi)$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\tau_{g_1}^{(p,q)}\left(\frac{f_1}{f_2},\varphi\right) = \tau_{g_1}^{(p,q)}(f_2,\varphi)$.

Case II. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_2 is of regular relative (p,q)-φ growth with respect to g_1 where g_1 satisfy the Property (A). As $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, so applying the same procedure as adopted in Case II of Theorem 21 we can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{gi}^{(p,q)}(f_1,\varphi) \mid i = 1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least f_1 is of regular relative (p,q)-φ growth with respect to g_1, then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$.

Therefore the first part of theorem follows Case I and Case II.

Case III. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and $g_1 \cdot g_2$ satisfy the Property (A).Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 21 we get that

$$\tau_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) \leq \tau_{g_1}^{(p,q)}(f_1,\varphi). \tag{38}$$

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\lambda_g^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Then in view of (38), we obtain that $\tau_g^{(p,q)}(f_1,\varphi) = \tau_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) \geq \tau_{g_1}^{(p,q)}(f_1,\varphi)$. Also $g_1 = \frac{g}{g_2}$ and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain from above arguments that

$\tau_{g_1}^{(p,q)}(f_1,\varphi) \geq \tau_g^{(p,q)}(f_1,\varphi) = \tau_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi)$. Hence $\tau_g^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi) \Rightarrow \tau_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}(f_1,\varphi)$.

If $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_2}^{(p,q)}(f_1,\varphi)$, then one can easily verify that $\tau_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \tau_{g_2}^{(p,q)}(f_1,\varphi)$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

SUB CASE III$_A$. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 13, $\lambda_g^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. We have $g_1 = g\cdot g_2$. So $\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_g^{(p,q)}(f_1,\varphi) = \tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi)$.

SUB CASE III$_B$. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore in view of Theorem 13, $\lambda_g^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) < \lambda_{g_1}^{(p,q)}(f_1,\varphi)$. Since $T_g(r) = T_{\frac{1}{g}}(r) + O(1) = T_{\frac{g_2}{g_1}}(r) + O(1)$, So $\tau_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \tau_{g_2}^{(p,q)}(f_1,\varphi)$.

CASE IV. Suppose $\lambda_{g_1}^{(p,q)}(f_1,\varphi) < \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and $g_1\cdot g_2$ satisfy the Property (A). Since $T_{g_1\cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, then adopting the same procedure as of Case IV of Theorem 21, we obtain that $\overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{gi}^{(p,q)}(f_1,\varphi) \mid i = 1, 2$.

Similarly if we consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_2}^{(p,q)}(f_1,\varphi)$, then one can easily verify that $\overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 17 , Theorem 19 and the above cases. □

Theorem 28. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*

(A) The following condition is assumed to be satisfied:

(i) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$ holds;

(ii) g_1 satisfies the Property (A), then

$$\rho_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:

(i) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$ holds;

(ii) f_1 is of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2. Also $g_1\cdot g_2$ satisfy the Property (A). Then we have

$$\rho_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi).$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfying the conditions of the theorem.

CASE I. Suppose that $\rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$ $(0 < \rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_1}^{(p,q)}(f_2,\varphi) < \infty)$ and g_1 satisfy the Property (A). Now in view of Theorem 12, it is easy to see that $\rho_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \le \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$. If possible let

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) < \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi). \tag{39}$$

Let $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$. Now in view of the first part of Theorem 26 and (39) we obtain that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) = \sigma_{g_1}^{(p,q)}\left(\frac{f_1 \cdot f_2}{f_2},\varphi\right) = \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi)$. Similarly with the help of the first part of Theorem 26, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$. This prove the first part of the theorem.

CASE II. Let us consider that

$$\rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi)\,(0 < \rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi) < \infty),$$

f_1 is of regular relative (p,q)-φ growth with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 14, it follows that $\rho_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) \ge \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi)$ and if possible let

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi). \tag{40}$$

Further suppose that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$. Therefore in view of the proof of the second part of Theorem 26 and (40), we obtain that $\sigma_{g_1}^{(p,q)}(f_1,\varphi) = \sigma_{\frac{g_1 \cdot g_2}{g_2}}^{(p,q)}(f_1,\varphi)$ $= \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ which is a contradiction. Hence

$$\rho_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi).$$

Likewise in view of the proof of second part of Theorem 26, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$. This proves the second part of the theorem. □

Theorem 29. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*

(A) The following conditions are assumed to be satisfied:

(i) $(f_1 \cdot f_2)$ is of regular relative (p,q)-φ growth with respect to at least any one g_1 or g_2;

(ii) $(g_1 \cdot g_2)$, g_1 and g_2 all satisfy the Property (A);

(iii) Either $\sigma_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1 \cdot f_2,\varphi)$ or $\overline{\sigma}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1 \cdot f_2,\varphi)$;

(iv) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi)$*;*
(v) Either $\sigma_{g_2}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2,\varphi)$*; then*

$$\rho_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) $(g_1\cdot g_2)$ *satisfies the Property (A);*
(ii) f_1 *and* f_2 *are of regular relative* (p,q)-φ *growth with respect to at least any one* g_1 *or* g_2*;*
(iii) Either $\sigma_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_1\cdot g_2}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_1\cdot g_2}^{(p,q)}(f_2,\varphi)$*;*
(iv) Either $\sigma_{g_1}^{(p,q)}(f_1,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_1,\varphi)$ *or* $\overline{\sigma}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_1,\varphi)$*;*
(v) Either $\sigma_{g_1}^{(p,q)}(f_2,\varphi) \neq \sigma_{g_2}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\sigma}_{g_1}^{(p,q)}(f_2,\varphi) \neq \overline{\sigma}_{g_2}^{(p,q)}(f_2,\varphi)$*; then*

$$\rho_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi) = \rho_{g_1}^{(p,q)}(f_1,\varphi) = \rho_{g_1}^{(p,q)}(f_2,\varphi) = \rho_{g_2}^{(p,q)}(f_1,\varphi) = \rho_{g_2}^{(p,q)}(f_2,\varphi).$$

We omit the proof of Theorem 29 as it is a natural consequence of Theorem 28.

Theorem 30. *Let* f_1, f_2 *be any two meromorphic functions and* g_1, g_2 *be any two entire functions.*
(A) The following conditions are assumed to be satisfied:
(i) At least any one of f_1 *or* f_2 *is of regular relative* (p,q)-φ *growth with respect to* g_1*;*
(ii) If either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$ *holds.*
(iii) g_1 *satisfies the Property (A), then*

$$\lambda_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:
(i) f_1 *is any meromorphic function and* g_1, g_2 *are any two entire functions such that* $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ *exist and* $g_1\cdot g_2$ *satisfy the Property (A);*
(ii) If either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ *holds, then*

$$\lambda_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi).$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfy the conditions of the theorem.

CASE I. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi) < \infty)$, g_1 satisfies the Property (A) and at least f_1 or f_2 be of regular relative (p,q)-φ growth with respect to g_1. Now in view of Theorem 10 it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi)$ $\leq \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1\cdot f_2,\varphi) < \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi). \tag{41}$$

Also let $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$. Then in view of the proof of first part of Theorem 27 and (41), we obtain that

$$\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_{g_1}^{(p,q)}\left(\frac{f_1 \cdot f_2}{f_2},\varphi\right) = \tau_{g_1}^{(p,q)}(f_2,\varphi)$$

which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi)$. Analogously, in view of the proof of first part of Theorem 27 and using the same technique as above, one can easily derive the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$. Hence the first part of the theorem is established.

Case II. Let us consider that

$$\lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)\,(0 < \lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_1,\varphi) < \infty$$

and $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 13, it follows that $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) \geq \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and if possible let

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi). \tag{42}$$

Further let $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$. Then in view of second part of Theorem 27 and (42), we obtain that

$$\tau_{g_1}^{(p,q)}(f_1,\varphi) = \tau_{\frac{g_1 \cdot g_2}{g_2}}^{(p,q)}(f_1,\varphi) = \tau_{g_2}^{(p,q)}(f_1,\varphi)$$

which is a contradiction. Hence $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Similarly by second part of Theorem 27, we get the same conclusion when $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ and therefore the second part of the theorem follows. □

Theorem 31. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*

(A) The following conditions are assumed to be satisfied:

(i) $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A);

(ii) At least any one of f_1 or f_2 is of regular relative (p,q)-φ growth with respect to g_1 and g_2;

(iii) Either $\tau_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1 \cdot f_2,\varphi)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1 \cdot f_2,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1 \cdot f_2,\varphi)$;

(iv) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi)$;

(v) Either $\tau_{g_2}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2,\varphi)$; then

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_2,\varphi).$$

(B) The following conditions are assumed to be satisfied:

(i) $g_1 \cdot g_2$ satisfies the Property (A);

(ii) At least any one of f_1 *or* f_2 *is of regular relative* (p,q)-φ *growth with respect to* $g_1 \cdot g_2$*;*
(iii) Either $\tau_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1\cdot g_2}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1\cdot g_2}^{(p,q)}(f_2,\varphi)$ *holds;*
(iv) Either $\tau_{g_1}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2}^{(p,q)}(f_1,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1,\varphi)$ *holds;*
(v) If either $\tau_{g_1}^{(p,q)}(f_2,\varphi) \neq \tau_{g_2}^{(p,q)}(f_2,\varphi)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_2,\varphi) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2,\varphi)$ *holds, then*

$$\lambda_{g_1\cdot g_2}^{(p,q)}(f_1\cdot f_2,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_2,\varphi).$$

We omit the proof of Theorem 31 as it is a natural consequence of Theorem 30.

Remark 32. If we take $\frac{f_1}{f_2}$ instead of $f_1 \cdot f_2$ and $\frac{g_1}{g_2}$ instead of $g_1 \cdot g_2$ where $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire function, and the other conditions of Theorem 28, Theorem 29, Theorem 30 and Theorem 31 remain the same, then conclusion of Theorem 28, Theorem 29, Theorem 30 and Theorem 31 remains valid.

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