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SOME ĆIRIC TYPE FIXED POINT RESULTS IN NON-ARCHIMEDEAN MODULAR METRIC SPACES

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Abstract. In this paper, we establish some Ćiric type fixed point theorems in $\alpha-$complete and orbitally $T-$complete non-Archimedean modular metric spaces. Meanwhile, we present an illustrative example to emphasis the realized improvements. These obtained results extend and improve certain well known results in the literature.

1. Introduction

Modular metric spaces are a natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. Modular metric spaces were introduced in [4, 5]. The introduction of this new concept is justified by the physical interpretation of the modular. Roughly, whereas a metric on a set represents nonnegative finite distances between any two points of the set, a modular on a set attributes a nonnegative (possibly, infinite valued) "field of (generalized) velocities": to each "time" $\lambda > 0$ (the absolute value of) an average velocity $\omega_\lambda(x, y)$ is associated in such a way that in order to cover the "distance" between points $x, y \in X$ it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$. But in this paper, we look at these spaces as the nonlinear version of the classical modular spaces introduced by Nakano [18] on vector spaces and modular function spaces introduced by Musielak [17] and Orlicz [19].

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In recent years, many researchers studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (for instance lithium polymetachrylate). An interesting model for these fluids, is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case that p is a function [6].

We remark that the usual approach in dealing with the Dirichlet energy problem [7, 10] is to convert the energy functional, naturally defined by a modular, to a convoluted and complicated problem which involves the Luxemburg norm.

In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. Recently, there was a strong interest to study the existence of fixed points in the setting of modular function spaces after the first paper [13] was published in 1990. For more on metric fixed point theory, the reader may consult the book [11] and for modular function spaces the book [16].

In this paper we establish some Ćiric type fixed point theorems in $\alpha-$complete and orbitally $T-$complete non-Archimedean modular metric spaces. Meanwhile, we present an illustrative example to emphasis the realized improvements. These obtained results extend and improve certain well known results in the literature.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function, for semplicity we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 ([4, 5])**.** A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a *modular metric* on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If in the Definition 1.1 we use the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then ω is said to be a *pseudomodular metric* on X. A modular metric ω on X is called *regular* if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, ω is called *convex* if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y).$$

Remark 1.2. If ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0, +\infty)$ for all $x, y \in X$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 1.3 ([4, 5])**.** Let ω be a pseudomodular on X and $x_0 \in X$ fixed. Consider the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \quad \text{such that} \quad \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X_ω^* are called *modular spaces* (around x_0).

It is clear that $X_\omega \subset X_\omega^*$ but this inclusion may be proper in general. Let ω be a modular on X, from [4, 5], we deduce that the modular space X_ω can be equipped with a (nontrivial) metric, induced by ω and defined by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\} \quad \text{for all} \quad x, y \in X_\omega.$$

If ω is a convex modular on X, according to [4, 5] the two modular spaces coincide, that is $X_\omega^* = X_\omega$ and this common set can be endowed with the metric d_ω^* defined by

$$d_\omega^*(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\} \quad \text{for all} \quad x, y \in X_\omega.$$

These distances will be called *Luxemburg distances.*

Example 2.1 presented by Abdou and Khamsi [1] is an important motivation for developing the theory of modular metric spaces. Other examples may be found in [4, 5].

Definition 1.4. Let X_ω be a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Then

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-*convergent* to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to +\infty$. x will be called the ω-limit of (x_n).
(2) $(x_n)_{n\in N}$ is called ω-*Cauchy* if $\omega_1(x_m, x_n) \to 0$, as $m, n \to +\infty$.
(3) M is called ω-*closed* if the ω-limit of a ω-convergent sequence of M always belong to M.

(4) M is called ω-*complete* if any ω-Cauchy sequence in M is ω-convergent to a point of M.
(5) M is called ω-*bounded* if we have $\delta_\omega(M) = \sup\{\omega_1(x, y); x, y \in M\} < +\infty$.

Recently Paknazar et al. [20] introduced the following type modular metric space.

Definition 1.5. If in the definition 1.1, we replace (iii) by

$$(iv)\ \omega_{\max\{\lambda,\mu\}}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$$

for all $\lambda, \mu > 0$ and $x, y, z \in X$

then X_ω is called non-Archimedean modular metric space. Since (iv) implies (iii), so every non- Archimedean modular metric space is a modular metric space.

Definition 1.6 ([22])**.** Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-*admissible mapping* if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.7 ([21])**.** Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-*admissible mapping* with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Definition 1.8 ([8])**.** Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, +\infty)$ and $T : X \to X$ be functions. We say that T is an α-η-*continuous mapping* on (X, d) if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ as $n \to +\infty$

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad \text{for all}\, n \in \mathbb{N} \Longrightarrow\ Tx_n \to Tx.$$

Definition 1.9. Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α^m-*admissible mapping* if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(T^m x, T^m y) \geq 1 \text{ for all } m \in \mathbb{N}.$$

Definition 1.10. Let X_ω be a non-Archimedean modular metric space. Let $\alpha, \eta : X_\omega \times X_\omega \to [0, +\infty)$ be two functions and let $T : X_\omega \to X_\omega$ be a mapping. We say that T is an α-*continuous mapping* on X_ω, if for given $x \in X_\omega$ and sequence $\{x_n\}$ with $\omega_1(x_n, x) \to 0$ as $n \to +\infty$

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all}\, n \in \mathbb{N} \Longrightarrow\ \omega_1(Tx_n, Tx) \to 0.$$

If $\eta(x, y) = 1$ for all $x, y \in X_\omega$, then T is called $\alpha - \omega$-*continuous*.

Definition 1.11. Let X_ω be a non-Archimedean modular metric space and $\alpha : X \times X \to [0, +\infty)$. The non-Archimedean modular space X_ω is said to be α-*complete* if and only if every $\omega-$Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, $\omega-$converges in X_ω.

Example 1.12. Let $X = [0, +\infty)$ and $\omega_\lambda(x, y) = \frac{1}{\lambda}|x - y|$ be a modular metric on X_ω. Assume that $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ are defined by

$$Tx = \begin{cases} x^8, & \text{if } x \in [0, 1] \\ \\ 15, & \text{if } (1, +\infty) \end{cases}, \quad \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then T is an α-continuous mapping. But clearly T is not $\omega-$continuous.

2. Main Results

In this section we establish some Ćiric type fixed point theorems in the setting of α-complete non-Archimedean modular metric space.

Theorem 2.1. *Let X_ω be an α-complete non-Archimedean modular metric space with ω regular. Let T be an $\alpha-$continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ with and for some $k \in (0, 1)$ we have*

(2.1)

$$\min\{\omega_1(Tx, Ty), \omega_1(x, Tx), \omega_1(y, Ty)\} - \min\{\omega_1(x, Ty), \omega_1(y, Tx)\} \leq k\omega_1(x, y)$$

then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. We construct an iterative sequence $\{x_n\}$ as follows

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

If there exists a positive integer n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T that completes the proof. Throughout the proof, we assume that $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \cdots$. Letting $x = x_n$ and $y = x_{n+1}$ in (3.3) we obtain the inequality

$$\begin{aligned} & \min\{\omega_1(Tx_n, Tx_{n+1}), \omega_1(x_n, Tx_n), \omega_1(x_{n+1}, Tx_{n+1})\} \\ - \quad & \min\{\omega_1(x_n, Tx_{n+1}), \omega_1(x_{n+1}, Tx_n)\} \leq k\omega_1(x_n, x_{n+1}) \end{aligned}$$

which implies that

$$\min\{\omega_1(x_n, x_{n+1}), \omega_1(x_{n+1}, x_{n+2})\} \leq k\omega_1(x_n, x_{n+1}). \tag{2.3}$$

Since we assume $k \in [0,1)$, the inequality (2.3) implies that

$$\omega_1(x_{n+1}, x_{n+2}) \leq k\omega_1(x_n, x_{n+1})$$

for every $n = 0, 1, 2, \cdots$. Thus we get

$$\omega_1(x_{n+1}, x_{n+2}) \leq k\omega_1(x_n, x_{n+1}) \leq k^2\omega_1(x_{n-1}, x_n) \leq \cdots \leq k^{n+1}\omega_1(x_0, x_1). \tag{2.4}$$

We claim that $\{x_n\}$ is a ω-Cauchy sequence. We assume that $n \leq m$. Then by using (2.4) we have

$$\begin{aligned}
\omega_1(x_n, x_m) &= \omega_{\max\{1,1,\ldots,1\}}(x_n, x_m) \\
&\leq \omega_1(x_n, x_{n+1}) + \omega_1(x_{n+1}, x_{n+2}) + \cdots + \omega_1(x_{m-1}, x_m) \\
&\leq [k^n + k^{n-2} + \cdots + k^{m-1}]\omega_1(x_0, x_1) \\
&= k^m\omega_1(x_0, x_1) \sum_{i=0}^{m-n-1} k^i \\
&\leq k^m\omega_1(x_0, x_1) \sum_{i=0}^{\infty} k^i \\
&\leq k^m\omega_1(x_0, x_1) \frac{1}{1-k},
\end{aligned}$$

since $k < 1$. Letting $n \to \infty$ in the inequality above, we derive that

$$\lim_{n\to\infty} \omega_1(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a ω-Cauchy sequence. Since X_ω is $\alpha-$complete then there exists $z \in X$ such that

$$\lim_{n\to\infty} \omega_1(x_n, z) = 0. \tag{2.5}$$

Since T is $\alpha-$continuous, so

$$\lim_{n\to\infty} \omega_1(x_n, Tz) = \omega_1(Tx_{n-1}, Tz) = 0. \tag{2.6}$$

Regarding the uniqueness, we derive that $Tz = z$. □

Example 2.2. Let $X = (-\infty, -2) \cup [-1, 1] \cup (2, +\infty)$. We endow X with the metric

$$\omega_\lambda(x, y) = \begin{cases} \frac{1}{\lambda}\max\{|x|, |y|\}, & \text{if } x \neq y \\ 0, & x = y \end{cases}$$

be a non-Archimedean modular metric on X. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by,

$$Tx = \begin{cases} \sqrt{\sqrt{2x^2 - 1}}, & \text{if } x \in (-\infty, -3] \\ \sqrt[5]{x^3 - 1}, & \text{if } x \in (-3, -2) \\ \frac{1}{4}x^2, & \text{if } x \in [-1, 1] \\ 6, & \text{if } x \in (1, 4) \\ 10, & \text{if } x \in (4, \infty) \end{cases}$$

$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$. Clearly X_ω is not a $\omega-$complete modular metric space. But it is an α-complete modular metric space. In fact if $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. Now since $([-1, 1], \omega)$ is a $\omega-$complete modular metric space, then the sequence $\{x_n\}$ converges in $[-1, 1] \subseteq X$. Let $\alpha(x, y) \geq 1$, then $x, y \in [-1, 1]$. On the other hand $Tw \in [-1, 1]$ for all $w \in [-1, 1]$. Then $\alpha(Tx, Ty) \geq 1$. That is T is an α-admissible mapping. Let $\{x_n\}$ be a sequence, such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. So $\{Tx_n\} \subseteq [-1, 1]$. Now, since T is continuous on $[-1, 1]$, then $Tx_n \to Tx$ as $n \to \infty$. That is T is an α-continuous mapping. Clearly $\alpha(0, T0) \geq 1$. Let $\alpha(x, y) \geq 1$. Then $x, y \in [-1, 1]$.

$$\begin{aligned} &\min\{\omega_1(Tx, Ty), \omega_1(x, Tx), \omega_1(y, Ty)\} - \min\{\omega_1(x, Ty), \omega_1(y, Tx)\} \\ &= \min\{\max\{\frac{1}{4}x^2, \frac{1}{4}y^2\}, \max\{x, \frac{1}{4}x^2\}, \max\{y, \frac{1}{4}y^2\}\} \\ &\quad - \min\{\max\{x, \frac{1}{4}y^2\}, \max\{y, \frac{1}{4}x^2\}\} \\ &\leq \min\{\max\{\frac{1}{4}x^2, \frac{1}{4}y^2\}, \max\{x, \frac{1}{4}x^2\}, \max\{y, \frac{1}{4}y^2\}\} \\ &\leq \min\{\max\{\frac{1}{4}x^2, \frac{1}{4}y^2\}, \max\{x, \frac{1}{4}x^2\}, \max\{y, \frac{1}{4}y^2\}\} \\ &\leq \max\{\frac{1}{4}x^2, \frac{1}{4}y^2\} \\ &\leq \frac{1}{4}\max\{|x|, |y|\} \\ &\leq \frac{1}{4}\omega_1(x, y). \end{aligned}$$

Hence all conditions of theorem 2.1 hold and T has a fixed point.

Theorem 2.3. *Let X_ω be an α-complete non-Archimedean modular metric space with ω regular. Let T be an $\alpha-$continuous and α^m-admissible self-mapping on X_ω and $\varepsilon > 0$. Suppose that there exists a point $x_0 \in X$ such that $\omega_1(x_0, T^n(x_0)) < \varepsilon$ for some $n \in \mathbb{N}$ and $\alpha(x_0, T(x_0)) \geq 1$. If for all $x, y \in X_\omega$ with $0 < \omega_1(x, y) < \varepsilon$ and for some $k \in (0, 1)$ we have,*

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(y)), \omega_1(T(y), y)\} \leq k\omega_1(x, y), \tag{2.7}$$

then, T has a periodic point.

Proof. Set $M = \{n \in \mathbb{N} : \omega_1(x, T^n(x)) < \varepsilon : \quad \text{for} \quad x \in X\}$. By the assumption of the theorem $M \neq \emptyset$. Let $m = \min M$ and $x \in X$ such that $\omega_1(x, T^m(x)) < \varepsilon$.

Suppose that $m = 1$, that is, $\omega_1(x, T(x)) < \varepsilon$. By applying (3.4), one can get

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(T(x))), \omega_1(T(T(x)), T(x))\} \leq k\omega_1(x, T(x)).$$

The case $\omega_1(x, T(x)) \leq k\omega_1(x, T(x))$ provides a contraction due to the fact that $k < 1$. Thus, $\omega_1(T(x), T(T(x))) = \omega_1(T(x), T^2(x)) \leq k\omega_1(x, T(x))$. As in the proof of theorem 2.1, one can consider the iterative sequence $x_{n+1} = T(x_n)$ and $\alpha(x_0, T(x_0)) \geq 1$, and observe that $Tz = z$ for some $z \in X$.

Suppose $m \geq 2$. This is equivalent to stating that the condition

$$\omega_1(T(y), y) \geq \varepsilon \tag{2.8}$$

holds for each $y \in X$. Then, from $\omega_1(x, T^m(x)) < \varepsilon$ and (3.4) it follows that

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(T^m(x))), \omega_1(T(T^m(x)), T^m(x))\} \leq k\omega_1(x, T^m(x)).$$

Since $T^m(x) \in X$, one has $\omega_1(T(T^m(x)), T^m(x)) = \omega_1(T(w), w)$ when we rename $T^m(x) = w$. Regarding (2.8), we obtain $\omega_1(T(w), w) = \omega_1(T(T^m(x)), T^m(x)) \geq \varepsilon$ and $\omega_1(T(x), x) \geq \varepsilon$. Thus,

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(T^m(x))), \omega_1(T(T^m(x)), T^m(x))\} = \omega_1(T(x), T^{m+1}(x)).$$

In particular,

$$\omega_1(T(x), T^{m+1}(x)) \leq k\omega_1(x, T^m(x)).$$

Recursively, one can get

$$\omega_1(T^2(x), T^{m+2}(x)) \leq \omega_1(T(x), T^{m+1}(x)) \leq k^2\omega_1(x, T^m(x)).$$

Proceeding in this way, for each $s \in \mathbb{N}$, one can obtain

$$\omega_1(T^s(x), T^{m+s}(x)) \leq \omega_1(T^{s-1}(x), T^{m+s-1}(x)) \leq \cdots \leq k^s\omega_1(x, T^m(x)).$$

Thus, for the recursive sequence $x_{n+1} = T^m(x_n)$ where $x_0 = x$,

$$\begin{aligned}\omega_1(x_n, x_{n+1}) &= \omega_1(T^{nm}(x_0), T^{(n+1)m}(x_0)) \\ &= \omega_1(T^{nm}(x_0), T^{m+nm}(x_0)) \\ &\leq k^{nm}\omega_1(x_0, T^m(x_0)).\end{aligned}$$

So we can write,

$$\begin{aligned}\omega_1(x_n, x_{n+s}) &\leq [\omega_1(x_n, x_{n+1}) + \omega_1(x_{n+1}, x_{n+2}) + \cdots + \omega_1(x_{n+s-1}, x_{n+s})] \\ &= \left[k^{nm} + k^{(n+1)m} + ... + k^{(n+p-1)m}\right] p(x_0, T^m(x_0) \qquad (2.9) \\ &= k^{nm}\left[1 + k^m + ... + k^{(s-1)m}\right] \omega_1(x_0, T^m(x_0)) \\ &\leq \frac{k^{nm}}{1-k^m}\omega_1(x_0, T^m(x_0))\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose a natural number n_0 such that $\frac{k^{nm}}{1-k^m}\omega_1(x_0, T^m(x_0)) < \varepsilon$ for all $n, m > n_0$. Thus, for any $s \in \mathbb{N}$

$$\omega_1(x_n, x_{n+s}) < \varepsilon \qquad (2.10)$$

for all $n > n_0$. So $\{x_n\}$ is an ω-Cauchy sequence in X_ω.

On the other hand we know that $\alpha(x_0, T^m(x_0)) \geq 1$. Since T is α^m-admissible mapping we deduce that $\alpha(x_1, x_2) = \alpha(T^m(x_0), T^{2m}(x_0)) \geq 1$. Continuing this process, we get

$$\alpha(x_n, x_{n+1}) = \alpha(T^m(x_0), T^{(n+1)m}(x_0)) \geq 1 \qquad (2.11)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Now since X_ω is $\alpha-$complete then there exists $z \in z$ such that $\omega_1(x_n, z) = 0$. Since T is $\alpha-$continuous then

$$\lim_{n\to\infty} \omega_1(Tx_n, Tz) = 0. \qquad (2.12)$$

Also since T is $\alpha-$admissible mapping, then (2.11) implies

$$\alpha(Tx_n, Tx_{n+1}) \geq 1. \qquad (2.13)$$

Again since T is $\alpha-$continuous then (2.12) and (2.13) implies,

$$\lim_{n\to\infty} \omega_1(T^2x_n, T^2z) = 0.$$

Continuing this process, we get

$$\lim_{n\to\infty} \omega_1(T^mx_n, T^mz) = \lim_{n\to\infty} \omega_1(x_{n+1}, T^mz) = 0.$$

So $T^mz = z$. □

Theorem 2.4. *Let X_ω be an α-complete non-Archimedean modular metric space with ω regular. Let T be an $\alpha-$continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ and for some $k \in (0,1)$ we have*

$$\begin{aligned} &\min\{[\omega_1(x, T(x))]^2, \omega_1(x,y)\omega_1(T(x), T(y)), [\omega_1(T(y), y)]^2\} \\ &\leq k\omega_1(x, T(x))\omega_1(T(y), y), \end{aligned} \tag{2.14}$$

then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X_ω by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Hence we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is α-admissible mapping with respect to η and $\alpha(x_0, Tx_0) \geq 1$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$. Continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Then from (3.5) we get

$$\begin{aligned} &\min\{[\omega_1(x_{n-1}, T(x_{n-1}))]^2, \omega_1(x_{n-1}, x_n)\omega_1(T(x_{n-1}), T(x_n)), [\omega_1(T(x_n), x_n)]^2\} \\ &\leq k\omega_1(x_{n-1}, T(x_{n-1}))\omega_1(T(x_n), x_n). \end{aligned} \tag{2.15}$$

Since $k < 1$, the case $\omega_1(x_{n-1}, x_n)\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n)\omega_1(x_n, x_{n+1})$ yields contradiction. Thus, one gets

$$\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n).$$

Recursively, one can observe that

$$\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n) \leq k^2\omega_1(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n\omega_1(x_0, T(x_0)).$$

By a routine calculation performed as in the proof of theorem 2.1, one can show that T has a fixed point. □

Theorem 2.5. *Let X be a non-empty set endowed with two modular metrics ω and ρ. Let T be a mapping of X into itself. Suppose that*

(i) X_ω is α-complete non-Archimedean modular metric space with ω regular,
(ii) $\omega_1(x, y) \leq \rho_1(x, y)$ for all $x, y \in X$,
(iii) T is $\alpha-$continuous with respect to ω,

(*iv*) *T satisfies:*

$$\begin{aligned}(2.16)\qquad &\min\{[\rho_1(T(x),T(y))]^2, \rho_1(x,y)\rho_1(T(x),T(y)), [\rho(y,T(y))]^2\}\\ &\leq k\rho_1(x,T(x))\rho_1(y,Ty)\end{aligned}$$

for all $x, y \in X$, where $0 \leq k < 1$.

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X_ω by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. As in the proof of theorem 2.1 we have

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Then by applying (3.6) one can get

$$\begin{aligned}(2.17)\qquad &\min\{[\rho_1(T(x_{n-1}),T(x_n))]^2, \rho_1(x_{n-1},x_n)\rho_1(T(x_{n-1}),T(x_n)), [\rho_1(x_n,T(x_n))]^2\}\\ &\leq k\rho_1(x_{n-1},T(x_{n-1}))\rho_1(x_n,T(x_n)).\end{aligned}$$

Because of the inequality

$$k\rho_1(x_{n-1},T(x_{n-1}))\rho_1(x_n,T(x_n)) \leq k\rho_1(x_{n-1},T(x_{n-1}))\rho_1(x_n,T(x_n)),$$

the expression in (2.17) is equivalent to $\rho_1(x_n, x_{n+1}) \leq k\rho_1(x_{n-1}, x_n)$. Recursively one can obtain

$$(2.18)\qquad \rho_1(x_n, x_{n+1}) \leq k\rho_1(x_{n-1}, x_n) \leq \cdots \leq k^n\rho_1(x_0, x_1).$$

Therefore we obtain that,

$$(2.19)\qquad \rho_1(x_n, x_{n+s}) \leq \frac{k^n}{1-k}\rho_1(x_0, x_1).$$

for any $s \in \mathbb{N}$. Taking (*ii*) of the theorem into the account, one can get

$$(2.20)\qquad \omega_1(x_n, x_{n+p}) \leq \frac{k^n}{1-k}\rho_1(x_0, x_1).$$

Thus, $\{x_n\}$ is a ω-Cauchy sequence with respect to ω. As in the proof of theorem 2.1 we deduce that T has a fixed point. □

Theorem 2.6. *Let X_ω be an α-complete non-Archimedean modular metric space with ω regular. Let T be an $\alpha-$continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ and for some $k \in (0,1)$ we have*

$$\begin{aligned}(2.21)\qquad &\min\{\omega_1(x,T(x)), \omega_1(T(x),T(y)), \omega_1(T(y),y)\}\\ &-\min\{\omega_1(x,T(y)), \omega_1(T(x),y)\} \leq k\omega_1(x,y)\end{aligned}$$

then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X_ω by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. As in the proof of theorem 2.1 we have

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Then by using (3.7), one can obtain,

$$\begin{aligned}
&\min\{\omega_1(x_{n-1}, T(x_{n-1})), \omega_1(T(x_{n-1}), x_n), \omega_1(x_n, T(x_n))\} \\
&\quad - \min\{\omega_1(x_{n-1}, T(x_n)), \omega_1(T(x_{n-1}), x_n)\} \\
&= \min\{\omega_1(x_n, x_{n+1}), \omega_1(x_{n-1}, x_n)\} \\
&\leq k\omega_1(x_{n-1}, x_n).
\end{aligned}$$

Thus,

$$\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n).$$

Recursively, one can observe that

$$\omega_1(x_n, x_{n+1}) \leq k\omega_1(x_{n-1}, x_n) \leq k^2\omega_1(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n\omega_1(x_0, x_1).$$

As in the proof of theorem 2.1 we deduce that T has a fixed point. □

3. Some Results on T-orbitally Complete Modular Metric Spaces

In [3] introduced the notions of orbitally continuous self mappings and orbitally $T-$complete metric spaces. Now we extend these notions to non-Archimedean modular metric space.

Definition 3.1. Let X_ω be a non-Archimedean modular metric space.

- A map $T : X \to X$ is called *orbitally continuous* if

$$\lim_{i\to\infty} \omega_1(z, T^{n_i}x) = 0 \Rightarrow \lim_{i\to\infty} \omega_1(Tz, TT^{n_i}x) = 0.$$

For each $x \in X_\omega$ we put $O(x) = \{x, Tx, T^2x, \cdots,\}$ where $O(x)$ is orbit of x. We say X_ω is *orbitally T-complete non-Archimedean modular metric space* if $O(x)$, is $\omega-$complete for every $x \in X_\omega$.

Now we are ready to prove some results in the setting of non-Archimedean modular metric spaces.

Theorem 3.2. *Let $T : X_\omega \to X_\omega$ be an orbitally continuous mapping on a non-Archimedean modular metric space X_ω with ω regular. Suppose that T satisfies the condition*

$$\min\{\omega_1(x,T(x)),\omega_1(T(x),T(y)),\omega_1(T(y),y)\} < \omega_1(x,y) \tag{3.1}$$

for all $x, y \in X,\ x \neq y$. If the sequence $\{T^n(x_0)\}$ has a cluster point $z \in X$ for some $x_0 \in X$, then z is a fixed point of T.

Proof. Suppose $T^m(x_0) = T^{m-1}(x_0)$ for some $m \in \mathbb{N}$, then $T^n(x_0) = T^m(x_0) = z$ for all $n \geq m$. It is clear that z is a required point.

Suppose $T^m(x_0) \neq T^{m-1}(x_0)$ for all $m \in \mathbb{N}$. Since $\{T^n(x_0)\}$ has a cluster point $z \in X$, one can write $\lim_{i\to\infty} \omega_1(T^{n_i}(x_0), z) = 0$. By replacing x and y with $T^{n-1}(x_0)$ and $T^n(x_0)$, respectively, in (3.1),

$$\begin{aligned} &\min\{\omega_1(T^{n-1}(x_0), T(T^{n-1}(x_0))), \omega_1(T(T^{n-1}(x_0)), T(T^n(x_0))), \\ &\qquad \omega_1(T(T^n(x_0)), T^n(x_0))\} \\ &< \omega_1(T^{n-1}(x_0), T^n(x_0)). \end{aligned} \tag{3.2}$$

The inequality $\omega_1(T^{n-1}(x_0), T^n(x_0)) < \omega_1(T^{n-1}(x_0), T^n(x_0))$ does not hold. Thus (3.2) is equivalent to $\omega_1(T^n(x_0), T^{n+1}(x_0)) < \omega_1(T^{n-1}(x_0), T^n(x_0))$ which shows that the sequence

$$\{\omega_1(T^n(x_0), T^{n+1}(x_0))\}_1^\infty \tag{3.3}$$

is decreasing and bounded below. Hence $\{\omega_1(T^n(x_0), T^{n+1}(x_0))\}_1^\infty$ is convergent. By T-orbital continuity,

$$\lim_{i\to\infty} \omega_1(T^{n_i+1}(x_0), T(z)) = 0.$$

Then we have

$$\omega_1(T^{n_i+1}(x_0), T^{n_i}(x_0)) \leq \omega_1(T^{n_i}(x_0), z) + \omega_1(z, T(z)) + \omega_1(T^{n_i+1}(x_0), T(z))$$

and

$$\omega_1(z, T(z)) \leq \omega_1(z, T^{n_i}(x_0)) + \omega_1(T^{n_i+1}(x_0), T^{n_i}(x_0)) + \omega_1(T^{n_i+1}(x_0), T(z)).$$

Now by taking limit as $i \to \infty$ in the above inequalities we get

$$\lim_{i\to\infty} \omega_1(T^{n_i+1}(x_0), T^{n_i}(x_0)) = \omega_1(z, T(z)). \tag{3.4}$$

Using $\{\omega_1(T^{n_i}(x_0), T^{n_i+1}(x_0))\}_1^\infty \subset \{\omega_1(T^n(x_0), T^{n+1}(x_0))\}_1^\infty$ and (3.4), we have

$$\lim_{n\to\infty} \omega_1(T^{n+1}(x_0), T^n(x_0)) = \omega_1(z, T(z)). \tag{3.5}$$

Considering the fact $\{\omega_1(T^{n_i+1}(x_0), T^{n_i+2}(x_0))\}_1^\infty \subset \{\omega_1(T^n(x_0), T^{n+1}(x_0))\}_1^\infty$ together with $\lim_{i\to\infty} T^{n_i+1}(x_0) = Tz$, $\lim_{i\to\infty} T^{n_i+2}(x_0) = T^2z$ and (3.5) show that

$$\omega_1(Tz, T^2z) = \omega_1(z, Tz). \tag{3.6}$$

Assume that $Tz \neq z$, that is $\omega_1(z, Tz) > 0$. So one can replace x and y with z and Tz, respectively in(3.1) to obtain

$$\min\{\omega_1(z, T(z)), \omega_1(T(z), T(T(z))), \omega_1(T(T(z)), T(z))\} < \omega_1(z, T(z)). \tag{3.7}$$

which yields that $\omega_1(Tz, T^2z) < \omega_1(z, Tz)$. But this contradicts (3.6). Similarly if $\omega_1(Tz, z) > 0$ we can show the contradiction. Thus $Tz = z$. □

Theorem 3.3. *Let X_ω be an orbitally T- complete non-Archimedean modular metric space with ω regular. Let T be an orbitally continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ and for some $k \in (0, 1)$ we have*

$$\min\{\omega_1(Tx, Ty), \omega_1(x, Tx), \omega_1(y, Ty)\} - \min\{\omega_1(x, Ty), \omega_1(y, Tx)\} \leq k\omega_1(x, y)$$

then T has a fixed point.

Proof. Define, $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w) \\ 0, & \text{otherwise} \end{cases}$$

where $O(w)$ is an orbit of a point $w \in X_\omega$. Then X_ω is an α-complete non-Archimedean modular metric space. Indeed if $\{x_n\}$ be an $\omega-$Cauchy sequence where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq O(w)$. Now since X_ω is an orbitally T-complete modular metric space, then $\{x_n\}$ converges. That is X_ω is an α-complete modular metric space. Also suppose that $\alpha(x, y) \geq 1$, then $x, y \in O(w)$. Hence $Tx, Ty \in O(w)$. That is $\alpha(Tx, Ty) \geq 1$. Thus T is an α-admissible mapping. Now we show that T is α-continuous. In fact if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. So $x_n \in O(w)$ for all $n \in \mathbb{N}$. Then there exists sequence $(k_i)_{i\in\mathbb{N}}$ of positive integer such that $x_n = T^{k_i}w \to x$ as $i \to \infty$. Now since T is an orbitally continuous map, then $Tx_n = T(T^{k_i}w) \to Tx$ as $i \to \infty$ as required. Hence all conditions of theorem 2.1 hold and T has a fixed point. □

Similarly we can deduce the following theorems.

Theorem 3.4. *Let X_ω be an orbitally T-complete non-Archimedean modular metric space with ω regular. Let T be an orbitally continuous on X_ω and $\varepsilon > 0$. Suppose that there exists a point $x_0 \in X$ such that $\omega_1(x_0, T^n(x_0)) < \varepsilon$ for some $n \in \mathbb{N}$ and $\alpha(x_0, T(x_0)) \geq 1$. If for all $x, y \in X_\omega$ with $0 < \omega_1(x, y) < \varepsilon$ and for some $k \in (0, 1)$ we have*

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(y)), \omega_1(T(y), y)\} \leq k\omega_1(x, y),$$

then T has a periodic point.

Theorem 3.5. *Let X_ω be an orbitally T-complete non-Archimedean modular metric space with ω regular. Let T be an orbitally continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ and for some $k \in (0, 1)$ we have*

$$\min\{[\omega_1(x, T(x))]^2, \omega_1(x, y)\omega_1(T(x), T(y)), [\omega_1(T(y), y)]^2\} \leq k\omega_1(x, T(x))\omega_1(T(y), y),$$

then T has a fixed point.

Theorem 3.6. *Let X be a non-empty set endowed with two modular metrics ω and ρ. Let T be a mapping of X into itself. Suppose that*

(*i*) *X_ω is orbitally T-complete non-Archimedean modular metric space with ω regular,*

(*ii*) *$\omega_1(x, y) \leq \rho_1(x, y)$ for all $x, y \in X$,*

(*iii*) *T is orbitally continuous with respect to ω,*

(*iv*) *T satisfies:*

$$\min\{[\rho_1(T(x), T(y))]^2, \rho_1(x, y)\rho_1(T(x), T(y)), [\rho(y, T(y))]^2\}$$
$$\leq k\rho_1(x, T(x))\rho_1(y, Ty)$$

for all $x, y \in X$, where $0 \leq k < 1$.

Then T has a fixed point in X.

Theorem 3.7. *Let X_ω be an orbitally T-complete non-Archimedean modular metric space with ω regular. Let T be an orbitally continuous self-mapping on X_ω and there exist $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$. If for all $x, y \in X_\omega$ and for some $k \in (0, 1)$ we have*

$$\min\{\omega_1(x, T(x)), \omega_1(T(x), T(y)), \omega_1(T(y), y)\}$$
$$-\min\{\omega_1(x, T(y)), \omega_1(T(x), y)\} \leq k\omega_1(x, y)$$

then T has a fixed point.

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