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ON CYCLIC POLYGROUPS OF ORDER LESS THAN SIX AND PERIOD TWO

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Abstract. Cyclic hypergroups are of great importance due to their applications to many field in mathematics. In this paper, we classify all polygroups of order less than six where each of its non-identity elements is a generator.

1. Introduction

In 1934, F. Marty [9] introduced algebraic hyperstructures which constitutes a generalization of the well-known algebraic structures at the eighth Congress of Scandinavian Mathematicians, where he generalized the notion of a group to that of a hypergroup. A hyperstructure (or hypergroupoid) is a non-empty set together with a hyperoperation defined on it. Several books have been written till now on hyperstructures [3, 4, 5, 12]. Cyclic semihypergroups have been studied by Desalvo and Freni [7], Vougiouklis [11], Leoreanu [8]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications. Mousavi et al. [10] introduced a strongly regular relation on a hypergroup such that in a particular case the quotient is a cyclic group. Al Tahan and Davvaz [1] presented a link between hyperstructures and the infinite non abelian group, braid group.

Now, in this paper, we classify all polygroups of order four and period two. The paper is organized as follows: After an introduction, Section 2 presents some basic definitions that are used throughout this paper. Section 3 presents some new properties of single power cyclic polygroups and finds all single power cyclic polygroups

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of order less than six where each of its non-identity elements is a generator of period two. Section 4 presents an algorithm as alternative way for finding all single power cyclic polygroups of order less than six where each of its non-identity elements is a generator of period two.

2. Basic Concepts and Definitions

This section explains some basic notions and definitions that have been used in this paper.

Definition 2.1. Let H be a non-empty set. A mapping $\cdot : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H, is called a *hyperoperation* on H. The couple $(H, \cdot)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we denote

$$A \cdot B = \bigcup_{\substack{a\in A\\ b\in B}} a \cdot b,\ A \cdot x = A \cdot \{x\} \text{ and } x \cdot B = \{x\} \cdot B.$$

Definition 2.2. A hypergroupoid $(H, \cdot)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, that is

$$\bigcup_{u\in y\cdot z} x \cdot u = \bigcup_{v\in x\cdot y} v \cdot z.$$

Definition 2.3. A *hypergroup* is a semihypergroup $(H, \cdot)$ such that $H \cdot x = x \cdot H = H$ for all $x \in H$, which is called *reproduction axiom*, it means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \cdot u$ and $y \in v \cdot x$.

Definition 2.4 ([2, 6])**.** A *polygroup* is a system $< P, \cdot, e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, $\cdot$ maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
(2) $e \cdot x = x \cdot e = x$,
(3) $x \in y \cdot z \Longrightarrow y \in x \cdot z^{-1}, \quad z \in y^{-1} \cdot x$.

The element e is called identity element. The following elementary facts about polygroups follow easily from the axioms: $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ where $A^{-1} = \{a^{-1} | \ a \in A\}$.

Definition 2.5. A polygroup P is *cyclic* if there exists $p \in P$ and $s \in \mathbb{N}$ such that $P = p \cup p^2 \cup \cdots \cup p^s \cup \cdots$.

If $P = p \cup p^2 \cup \cdots \cup p^s$ then P is a *cyclic polygroup with finite period.* Otherwise, P is called a *cyclic polygroup with infinite period.* Here, $p^s = \underbrace{p \cdot p \cdot \cdots \cdot p}_{s\ times}$.

Definition 2.6. A polygroup P is a *single-power cyclic polygroup* if there exists $p \in P$ and $s \in \mathbb{N}$ such that $P = p \cup p^2 \cup \cdots \cup p^s \cup \cdots$ and $p \cup p^2 \cup \cdots \cup p^{m-1} \subset p^m$, for all $m \in \mathbb{N}$.

Definition 2.7. Let $< P_1, \cdot, e_1, ^{-1} >$ and $< P_2, \star, e_2, ^{-1} >$ be polygroups. A mapping ϕ from P_1 into P_2 is said to be a *strong homomorphism* if for all $x, y \in P_1$,

(1) $\phi(x \cdot y) = \phi(x) \star \phi(y)$,
(2) $\phi(e_1) = e_2$.

Clearly, a strong homomorphism ϕ is an *isomorphism* if ϕ is one to one and onto. We write $P_1 \cong P_2$ if P_1 is isomorphic to P_2.

Proposition 2.8. *A polygroup P in which every element has order 2 (i.e., $x^{-1} = x$ for all x) is commutative.*

Proof. We will show that for every $x, y \in P$, $x \cdot y = y \cdot x$. Let $t \in x \cdot y$, then $x \in t \cdot y^{-1} = t \cdot y$ and $y \in t^{-1} \cdot x = t \cdot x$ and $t \in y \cdot x^{-1} = y \cdot x$ and so $x \cdot y \subseteq y \cdot x$. In a similar way it can be shown that $y \cdot x \subseteq x \cdot y$. Therefore $x \cdot y = y \cdot x$. □

Corollary 2.9. *A polygroup P in which $x^2 = P$ for all non-identity element x, is commutative.*

Proof. In this case for every $x \in P$ we have $x^{-1} = x$. Now by Proposition 2.8 it follows that P is commutative. □

3. Single Power Cyclic Polygroups of Order less than Six where Each of its Non-identity Elements is a Generator of Period Two

In this section we find all single power cyclic polygroups of order less than six in which every non-identity element has period two. From order one we have just one polygroup $P = \{e\}$. Also, from order two we have one polygroup:

$\cdot$	e	a
e	e	a
a	a	e, a

and from order three, by Theorem 4.4.5 of [6], we have one polygroup:

$\cdot_1$	e	a	b
e	e	a	b
a	a	e, a, b	a, b
b	b	a, b	e, a, b

Now, for order four we need some lemma and propositions.

Lemma 3.1. *A polygroup P in which every element has order 2 (i.e., $x^{-1} = x$ for all x), if $x \neq y$ then $e \notin x \cdot y$.*

Proof. If $e \in x \cdot y$ then $x \in e \cdot y^{-1} = e \cdot y = y$ that is contradiction with $x \neq y$. □

Proposition 3.2. *A polygroup $P = \{e, a, b, c\}$ in which $x^2 = P$ for all non-identity element x, we have:*

(i) $a \in a \cdot b = b \cdot a$ *and* $a \in a \cdot c = c \cdot a$,
(ii) $b \in b \cdot c =\in c \cdot b$ *and* $b \in a \cdot b = b \cdot a$,
(iii) $c \in a \cdot c = c \cdot a$ *and* $c \in b \cdot c = c \cdot b$.

Proof. For every $x, y \neq e$ we have $x \in y \cdot y = P$. Therefore $y \in x \cdot y^{-1} = x \cdot y$ and $y \in y^{-1} \cdot x = y \cdot x$. □

Proposition 3.3. *A polygroup $P = \{e, a, b, c\}$ in which $x^2 = P$ for all non-identity element $x \in P$, we have:*

(i) if $a \in b \cdot c = c \cdot b$ *then* $b \in a \cdot c = c \cdot a$ *and* $c \in b \cdot a = a \cdot b$,
(ii) if $b \in a \cdot c = c \cdot a$ *then* $a \in b \cdot c = c \cdot b$ *and* $c \in a \cdot b = b \cdot a$,
(iii) if $c \in a \cdot b = b \cdot a$ *then* $a \in c \cdot b = b \cdot c$ *and* $b \in a \cdot c = c \cdot a$.

Proof. For every non-identity $x \neq y \neq z$. if $x \in y \cdot z$ then $y \in x \cdot z^{-1} = x \cdot z$ and $z \in y^{-1} \cdot x = y \cdot x$. □

Theorem 3.4. *Let $\cdot$ be a commutative hyperoperation on a polygroup $P = \{e, a, b, c\}$. Then P is associative if the following are satisfied:*

(1) $a \cdot (a \cdot b) = (a \cdot a) \cdot b$,
(2) $a \cdot (a \cdot c) = (a \cdot a) \cdot c$,
(3) $a \cdot (b \cdot b) = (a \cdot b) \cdot b$,

(4) $a \cdot (b \cdot c) = (a \cdot b) \cdot c,$
(5) $a \cdot (c \cdot b) = (a \cdot c) \cdot b,$
(6) $a \cdot (c \cdot c) = (a \cdot c) \cdot c,$
(7) $b \cdot (b \cdot c) = (b \cdot b) \cdot c,$
(8) $b \cdot (c \cdot c) = (b \cdot c) \cdot c.$

Proof. Since $(P, \cdot)$ is commutative, it follows that for every $x, y, z \in P$ we have $x \cdot (x \cdot x) = (x \cdot x) \cdot x$ and $z \cdot (y \cdot z) = (y \cdot z) \cdot z = (z \cdot y) \cdot z$. By using (4), (5) we have $b \cdot (a \cdot c) = (b \cdot a) \cdot c$. Also by (4) we have $c \cdot (b \cdot a) = (c \cdot b) \cdot a$. Finally for every $x \in P$ we have $x \cdot e = e \cdot x = x$. □

Theorem 3.5. *There are 2 polygroups of order four which every non-identity has period two.*

Proof. Suppose that $P = \{e, a, b, c\}$. The first row, column and diagonal are fix. By Corollary 2.9, we must determine $a \cdot b$, $a \cdot c$ and $b \cdot c$. Now, by Lemma 3.1, Proposition 3.2 and Proposition 3.3 we have the following two cases:

$\cdot_1$	e	a	b	c
e	e	a	b	c
a	a	P	a, b	a, c
b	b	a, b	P	b, c
c	a	a, c	b, c	P

$\cdot_2$	e	a	b	c
e	e	a	b	c
a	a	P	a, b, c	a, b, c
b	b	a, b, c	P	a, b, c
c	a	a, b, c	a, b, c	P

Finally, by Theorem 3.5 the associativity of them is proved:

$a \cdot_1 (a \cdot_1 b) = P$	$(a \cdot_1 a) \cdot_1 b = P$
$a \cdot_1 (a \cdot_1 c) = P$	$(a \cdot_1 a) \cdot_1 c = P$
$a \cdot_1 (b \cdot_1 b) = P$	$(a \cdot_1 b) \cdot_1 b = P$
$a \cdot_1 (b \cdot_1 c) = \{a, b, c\}$	$(a \cdot_1 b) \cdot_1 c = \{a, b, c\}$
$a \cdot_1 (c \cdot_1 b) = \{a, b, c\}$	$(a \cdot_1 c) \cdot_1 b = \{a, b, c\}$
$a \cdot_1 (c \cdot_1 c) = P$	$(a \cdot_1 c) \cdot_1 c = P$
$b \cdot_1 (b \cdot_1 c) = P$	$(b \cdot_1 b) \cdot_1 c = P$
$b \cdot_1 (c \cdot_1 c) = P$	$(b \cdot_1 c) \cdot_1 c = P$

$a \cdot_2 (a \cdot_2 b) = P$	$(a \cdot_2 a) \cdot_2 b = P$
$a \cdot_2 (a \cdot_2 c) = P$	$(a \cdot_2 a) \cdot_2 c = P$
$a \cdot_2 (b \cdot_2 b) = P$	$(a \cdot_2 b) \cdot_2 b = P$
$a \cdot_2 (b \cdot_2 c) = P$	$(a \cdot_2 b) \cdot_2 c = P$
$a \cdot_2 (c \cdot_2 b) = P$	$(a \cdot_2 c) \cdot_2 b = P$
$a \cdot_2 (c \cdot_2 c) = P$	$(a \cdot_2 c) \cdot_2 c = P$
$b \cdot_2 (b \cdot_2 c) = P$	$(b \cdot_2 b) \cdot_2 c = P$
$b \cdot_2 (c \cdot_2 c) = P$	$(b \cdot_2 c) \cdot_2 c = P$

□

Finally, for order five we apply as following:

Proposition 3.6. *A polygroup $P = \{e, a, b, c, d\}$ in which $x^2 = P$ for all non-identity element x, we have:*

(i) $a \in a \cdot b = b \cdot a$, $a \in a \cdot c = c \cdot a$ *and* $a \in a \cdot d = d \cdot a$,

(ii) $b \in b \cdot c = c \cdot b$, $b \in a \cdot b = b \cdot a$ *and* $b \in d \cdot b = b \cdot d$,

(iii) $c \in a \cdot c = c \cdot a$, $c \in b \cdot c = c \cdot b$ *and* $c \in c \cdot d = d \cdot c$.

(iv) $d \in a \cdot d = d \cdot a$, $d \in b \cdot d = d \cdot b$ *and* $d \in d \cdot c = c \cdot d$.

Proof. It is similar to the proof of Proposition 3.2. □

Proposition 3.7. *A polygroup $P = \{e, a, b, c, d\}$ in which $x^2 = P$ for all non-identity element $x \in P$, we have:*

(i) *if* $a \in b \cdot c = c \cdot b$ *then* $b \in a \cdot c = c \cdot a$ *and* $c \in b \cdot a = a \cdot b$,

(ii) *if* $a \in b \cdot d = d \cdot b$ *then* $b \in a \cdot d = d \cdot a$ *and* $d \in a \cdot b = b \cdot a$,

(iii) *if* $a \in c \cdot d = d \cdot c$ *then* $c \in a \cdot d = d \cdot a$ *and* $d \in a \cdot c = c \cdot a$,

(iv) *if* $b \in a \cdot c = c \cdot a$ *then* $a \in b \cdot c = c \cdot b$ *and* $c \in a \cdot b = b \cdot a$,

(v) *if* $b \in c \cdot d = d \cdot c$ *then* $c \in b \cdot d = d \cdot b$ *and* $d \in c \cdot b = b \cdot c$,

(vi) *if* $b \in a \cdot d = d \cdot a$ *then* $a \in b \cdot d = d \cdot b$ *and* $d \in a \cdot b = b \cdot a$,

(vii) *if* $c \in a \cdot b = b \cdot a$ *then* $a \in c \cdot b = b \cdot c$ *and* $b \in a \cdot c = c \cdot a$,

(viii) *if* $c \in a \cdot d = d \cdot a$ *then* $a \in c \cdot d = d \cdot c$ *and* $d \in a \cdot c = c \cdot a$,

(ix) *if* $c \in d \cdot b = b \cdot d$ *then* $b \in c \cdot d = d \cdot c$ *and* $d \in b \cdot c = c \cdot b$,

(x) *if* $d \in a \cdot b = b \cdot a$ *then* $a \in d \cdot b = b \cdot d$ *and* $b \in a \cdot d = d \cdot a$,

(xi) *if* $d \in a \cdot c = c \cdot a$ *then* $a \in c \cdot d = d \cdot c$ *and* $c \in a \cdot d = d \cdot a$,

(xii) *if* $d \in c \cdot b = b \cdot c$ *then* $b \in c \cdot d = d \cdot c$ *and* $c \in b \cdot d = d \cdot b$,

Proof. It is similar to the proof of Proposition 3.3. □

Corollary 3.8. *A polygroup $P = \{e, a, b, c, d\}$ in which $x^2 = P$ for all non-identity element $x \in P$, we have:*

(i) $a \cdot b = b \cdot a \in \{\{a,b\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\}\}$
(ii) $a \cdot c = c \cdot a \in \{\{a,c\},\{a,b,c\},\{a,c,d\},\{a,b,c,d\}\}$
(iii) $a \cdot d = d \cdot a \in \{\{a,d\},\{a,b,d\},\{a,c,d\},\{a,b,c,d\}\}$
(iv) $b \cdot c = c \cdot b \in \{\{b,c\},\{a,b,c\},\{b,c,d\},\{a,b,c,d\}\}$
(v) $b \cdot d = d \cdot b \in \{\{b,d\},\{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}$
(vi) $c \cdot d = d \cdot c \in \{\{c,d\},\{b,c,d\},\{a,c,d\},\{a,b,c,d\}\}$

Proof. It follows from Propositions 3.6 and 3.7 and Lemma 3.1. □

Theorem 3.9. *Let $\cdot$ be a commutative hyperoperation on a polygroup $P = \{e,a,b,c,d\}$. Then P is associative if the following are satisfied:*

(1) $a \cdot (a \cdot b) = (a \cdot a) \cdot b$,
(2) $a \cdot (a \cdot c) = (a \cdot a) \cdot c$,
(3) $a \cdot (a \cdot d) = (a \cdot a) \cdot d$,
(4) $a \cdot (b \cdot b) = (a \cdot b) \cdot b$,
(5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
(6) $a \cdot (b \cdot d) = (a \cdot b) \cdot d$,
(7) $a \cdot (c \cdot b) = (a \cdot c) \cdot b$.
(8) $a \cdot (c \cdot c) = (a \cdot c) \cdot c$,
(9) $a \cdot (c \cdot d) = (a \cdot c) \cdot d$,
(10) $a \cdot (d \cdot b) = (a \cdot d) \cdot b$,
(11) $a \cdot (d \cdot c) = (a \cdot d) \cdot c$,
(12) $a \cdot (d \cdot d) = (a \cdot d) \cdot d$,
(13) $b \cdot (a \cdot c) = (b \cdot a) \cdot c$,
(14) $b \cdot (a \cdot d) = (b \cdot a) \cdot d$,
(15) $b \cdot (b \cdot c) = (b \cdot b) \cdot c$.
(16) $b \cdot (b \cdot d) = (b \cdot b) \cdot d$,
(17) $b \cdot (b \cdot c) = (b \cdot b) \cdot c$,
(18) $b \cdot (c \cdot d) = (b \cdot c) \cdot d$,
(19) $b \cdot (d \cdot c) = (b \cdot d) \cdot c$,
(20) $b \cdot (d \cdot d) = (b \cdot d) \cdot d$,
(21) $c \cdot (b \cdot d) = (c \cdot b) \cdot d$,
(22) $c \cdot (c \cdot d) = (c \cdot c) \cdot d$,
(23) $c \cdot (d \cdot d) = (c \cdot d) \cdot d$,

Proof. It is similar to the proof of Theorem 3.4. □

Theorem 3.10. *There are 16 polygroups of order five which every non-identity has period two.*

Proof. Suppose that $P = \{e, a, b, c, d\}$. The first row, column and diagonal are fix. By Corollary 2.9, we must determine $a \cdot b$, $a \cdot c$, $a \cdot d$, $b \cdot c$, $b \cdot d$ and $c \cdot d$. Now, by Lemma 3.1, Propositions 3.6, 3.7 and Corollary 3.8 we have the following 16 cases:

$\cdot_1$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a, b	a, c	a, d
b	b	a, b	P	b, c	b, d
c	c	a, c	b, c	P	c, d
d	d	a, d	b, d	c, d	P

$\cdot_2$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a, b	a, c	a, d
b	b	a, b	P	b, c, d	b, c, d
c	c	a, c	b, c, d	P	b, c, d
d	d	a, d	b, c, d	b, c, d	P

$\cdot_3$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a, b	a, c, d	a, c, d
b	b	a, b	P	b, c	b, d
c	c	a, c, d	b, c	P	a, c, d
d	d	a, c, d	b, d	a, c, d	P

$\cdot_4$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a, b	a, c, d	a, c, d
b	b	a, b	P	b, c, d	b, c, d
c	c	a, c, d	b, c, d	P	a, b, c, d
d	d	a, c, d	b, c, d	a, b, c, d	P

$\cdot_5$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a, b, c	a, b, c	a, d
b	b	a, b, c	P	a, b, c	b, d
c	c	a, b, c	a, b, c	P	c, d
d	d	a, d	b, d	c, d	P

$\cdot_6$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c	a,b,c	a,d
b	b	a,b,c	P	a,b,c,d	b,c,d
c	c	a,b,c	a,b,c,d	P	b,c,d
d	d	a,d	b,c,d	b,c,d	P

$\cdot_7$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c	a,b,c,d	a,c,d
b	b	a,b,c	P	a,b,c	b,d
c	c	a,b,c,d	a,b,c	P	a,c,d
d	d	a,c,d	b,d	a,c,d	P

$\cdot_8$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c	a,b,c,d	a,c,d
b	b	a,b,c	P	a,b,c,d	b,c,d
c	c	a,b,c,d	a,b,c,d	P	a,b,c,d
d	d	a,c,d	b,c,d	a,b,c,d	P

$\cdot_9$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,d	a,c	a,b,d
b	b	a,b,d	P	b,c	a,b,d
c	c	a,c	b,c	P	c,d
d	d	a,b,d	a,b,d	c,d	P

$\cdot_{10}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,d	a,c	a,b,d
b	b	a,b,d	P	b,c,d	a,b,c,d
c	c	a,c	b,c,d	P	b,c,d
d	d	a,b,d	a,b,c,d	b,c,d	P

$\cdot_{11}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,d	a,c,d	a,b,c,d
b	b	a,b,d	P	b,c	a,b,d
c	c	a,c,d	b,c	P	a,c,d
d	d	a,b,c,d	a,b,d	a,c,d	P

$\cdot_{12}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,d	a,c,d	a,b,c,d
b	b	a,b,d	P	b,c,d	a,b,c,d
c	c	a,c,d	b,c,d	P	a,b,c,d
d	d	a,b,c,d	a,b,c,d	a,b,c,d	P

$\cdot_{13}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c,d	a,b,c	a,b,d
b	b	a,b,c,d	P	a,b,c	a,b,d
c	c	a,b,c	a,b,c	P	c,d
d	d	a,b,d	a,b,d	c,d	P

$\cdot_{14}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c,d	a,b,c	a,b,d
b	b	a,b,c,d	P	a,b,c,d	a,b,c,d
c	c	a,b,c	a,b,c,d	P	b,c,d
d	d	a,b,d	a,b,c,d	b,c,d	P

$\cdot_{15}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c,d	a,b,c,d	a,b,c,d
b	b	a,b,c,d	P	a,b,c	a,b,d
c	c	a,b,c,d	a,b,c	P	a,c,d
d	d	a,b,c,d	a,b,d	a,c,d	P

$\cdot_{16}$	e	a	b	c	d
e	e	a	b	c	d
a	a	P	a,b,c,d	a,b,c,d	a,b,c,d
b	b	a,b,c,d	P	a,b,c,d	a,b,c,d
c	c	a,b,c,d	a,b,c,d	P	a,b,c,d
d	d	a,b,c,d	a,b,c,d	a,b,c,d	P

Finally, by Theorem 3.9 the associativity of them is proved. For example we show the associativity of $\cdot_1$:

$a \cdot_1 (a \cdot_1 b) = P$	$(a \cdot_1 a) \cdot_1 b = P$
$a \cdot_1 (a \cdot_1 c) = P$	$(a \cdot_1 a) \cdot_1 c = P$
$a \cdot_1 (a \cdot_1 d) = P$	$(a \cdot_1 a) \cdot_1 d = P$
$a \cdot_1 (b \cdot_1 b) = P$	$(a \cdot_1 b) \cdot_1 b = P$
$a \cdot_1 (b \cdot_1 c) = \{a, b, c\}$	$(a \cdot_1 b) \cdot_1 c = \{a, b, c\}$
$a \cdot_1 (b \cdot_1 d) = \{a, b, d\}$	$(a \cdot_1 b) \cdot_1 d = \{a, b, d\}$
$a \cdot_1 (c \cdot_1 b) = \{a, b, c\}$	$(a \cdot_1 c) \cdot_1 b = \{a, b, c\}$
$a \cdot_1 (c \cdot_1 c) = P$	$(a \cdot_1 c) \cdot_1 c = P$
$a \cdot_1 (c \cdot_1 d) = \{a, c, d\}$	$(a \cdot_1 c) \cdot_1 d = \{a, c, d\}$
$a \cdot_1 (d \cdot_1 b) = \{a, b, d\}$	$(a \cdot_1 d) \cdot_1 b = \{a, b, d\}$
$a \cdot_1 (d \cdot_1 c) = \{a, c, d\}$	$(a \cdot_1 d) \cdot_1 c = \{a, c, d\}$
$a \cdot_1 (d \cdot_1 d) = P$	$(a \cdot_1 d) \cdot_1 d = P$
$b \cdot_1 (a \cdot_1 c) = \{a, b, c\}$	$(b \cdot_1 a) \cdot_1 c = \{a, b, c\}$
$b \cdot_1 (a \cdot_1 d) = \{a, b, d\}$	$(b \cdot_1 a) \cdot_1 d = \{a, b, d\}$
$b \cdot_1 (b \cdot_1 c) = P$	$(b \cdot_1 b) \cdot_1 c = P$
$b \cdot_1 (b \cdot_1 d) = P$	$(b \cdot_1 b) \cdot_1 d = P$
$b \cdot_1 (c \cdot_1 c) = P$	$(b \cdot_1 c) \cdot_1 c = P$
$b \cdot_1 (c \cdot_1 d) = \{b, c, d\}$	$(b \cdot_1 c) \cdot_1 d = \{b, c, d\}$
$b \cdot_1 (d \cdot_1 c) = \{b, c, d\}$	$(b \cdot_1 d) \cdot_1 c = \{b, c, d\}$
$b \cdot_1 (d \cdot_1 d) = P$	$(b \cdot_1 d) \cdot_1 d = P$
$c \cdot_1 (b \cdot_1 d) = \{b, c, d\}$	$(c \cdot_1 b) \cdot_1 d = \{b, c, d\}$
$c \cdot_1 (c \cdot_1 d) = P$	$(c \cdot_1 c) \cdot_1 d = P$
$c \cdot_1 (d \cdot_1 d) = P$	$(c \cdot_1 d) \cdot_1 d = P$

$\square$

4. Appendix

In this section, we explain the previous conclusions by a computer programm. For order four without consideration of lemmas and propositions (we just use the commutativity and associativity) we present the following algorithms:

Require: procedure SolveP4()

$P := 1, 2, 3, 4$ ▷ $e = 1, a = 2, b = 3, c = 4$.

$M := PowerSet(P) - \{\{\}\}$ ▷ M is the space of all possibility for elements aij. $|M| = 2^4 - 1 = 15$.

$INV := [[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2], [1, 4, 2, 3], [1, 4, 3, 2]]$ ▷ INV is the list of all permutation of elements of P such that fix $e = 1$. So $|INV| = 3! = 6$.

$PolygroupList := []$ ▷ $PolygroupList$ is a list that we add all founded Polys into it.
for f in INV **do**
  for $(a23, a24, a34, a32, a42, a43)$ in M **do**
    Set $aii := P$, for every $1 < i < 5$.
    Set $a1i := i$, $ai1 := i$, for every $0 < i < 5$.
    $A := (aij)$ ▷ A, a matrix 4 by 4, is the table of operation.
    if $IsPolygroup(A, f)$ **then**
      Add A to $PolygroupList$.
    end if
  end for
end for
return $PolygroupList$

For order five we consider to propositions, lemma and corollaries. Also this fact that inverse function just can be identity function:

Require: procedure SolveP5()
$P := \{1, 2, 3, 4, 5\}$ ▷ e=1, a=2, b=3, c=4, d=5.
$M := PowerSet(P) - \{$non-empty subsets that contains element e, singleton subsets, empty-set$\}$ ▷ M is the space of all possibility for elements aij. $|M| = 11$.
$INV := [[1, 2, 3, 4, 5]]$ ▷ INV is the $^{-1}$ function.
$PolygroupList := []$ ▷ $PolygroupList$ is a list that we add all founded Polygroups into it.
for $(a23, a24, a25, a34, a35, a45)$ in M **do**
  Set $aii := P$, for every $1 < i \leq 5$.
  Set $a1i := \{i\}$, $ai1 := \{i\}$, for every $1 \leq i \leq 5$.
  $A := (aij)$ ▷ A, a matrix 5 by 5, is the table of operation.
  if $IsPolygroup(A, f)$ **then**
    Add A to $PolygroupList$.
  end if
end for
return $PolygroupList$

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