J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2019.26.4.247 ISSN(Online) 2287-6081
Volume 26, Number 4 (November 2019), Pages 247–254

# ON THE HYERS-ULAM-RASSIAS STABILITY OF AN ADDITIVE-CUBIC-QUARTIC FUNCTIONAL EQUATION

Yang-Hi Lee

Abstract. In this paper, we investigate Hyers-Ulam-Rassias stability of the functional equation

$$\begin{aligned} &f(x+ky)-k^2f(x+y)+2(k^2-1)f(x)-k^2f(x-y)+f(x-ky)\\ &\quad -k^2(k^2-1)(f(y)+f(-y))=0, \end{aligned}$$

where $k$ is a fixed real number with $|k| \neq 0, 1$.

## 1. Introduction

Throughout this paper, let $V$ and $W$ be real vector spaces and $k$ a fixed real number such that $|k| \neq 0, 1$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x)-f(-x)}{2},\\ f_e(x) &:= \frac{f(x)+f(-x)}{2},\\ Af(x,y) &:= f(x+y)-f(x)-f(y),\\ Cf(x,y) &:= f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),\\ Q'f(x,y) &:= f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-24f(y),\\ D_kf(x,y) &:= f(x+ky)-k^2f(x+y)+2(k^2-1)f(x)-k^2f(x-y)+f(x-ky)\\ &\quad -k^2(k^2-1)(f(y)+f(-y)) \end{aligned} \tag{1.1}$$

for all $x, y \in V$. Every solution of functional equation $Af(x,y) = 0$, $Cf(x,y) = 0$ and $Q'f(x,y) = 0$ are called an additive mapping, a cubic mapping and a quartic mapping, respectively. If a mapping can be expressed by the sum of an additive mapping, a cubic mapping and a quartic mapping, then we call the mapping

Received by the editors March 28, 2019. Accepted November 19, 2019.
2010 *Mathematics Subject Classification.* 39B82, 39B52.
*Key words and phrases.* stability of a functional equation, additive-cubic-quartic functional equation, additive-cubic-quartic mapping.

© 2019 Korean Soc. Math. Educ.

an additive-cubic-quartic mapping. A functional equation is called an additive-cubic-quartic functional equation provided that each solution of that equation is an additive-cubic-quartic mapping and every additive-cubic-quartic mapping is a solution of that equation. Many mathematicians [2, 5, 7, 9, 11] have studied the stability of the following additive-cubic-quartic functional equation

$$11f(x+2y)+11f(x-2y) = 44f(x+y)+44f(x-y)+12f(3y)-48f(2y)+60f(y)-66f(x).$$

In 1940, Ulam [10] questioned about the stability of group homomorphisms. In 1941, Hyers [6] solved this question for Cauchy functional equation, which is a partial answer to Ulam's question. In 1978, Rassias [8] made Hyers' result generalized (Refer to Găvruta's paper [3] for a more generalized result). The concept of stability used by Rassias is called 'Hyers-Ulam-Rassias stability'.

M.E. Gordji etc. [4] investigated the stability of the functional equation $D_k f(x, y) = 0$ on the random normed spaces for the case $k$ is a fixed integer.

In this paper, we will show that the functional equation $D_r f(x, y) = 0$ is an additive-cubic-quartic functional equation when $r$ is a rational number, and also investigate Hyers-Ulam-Rassias stability of that functional equation $D_k f(x, y) = 0$ for $k$ is a real number.

## 2. Main Theorems

The following theorem is a particular case of Baker's theorem [1].

**Theorem 2.1** ([1, Theorem 1]). *Suppose that $V$ and $W$ are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalars such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

*for all $x, y \in V$, then each $f_l$ is a "generalized" polynomial mapping of "degree" at most $m - 1$.*

Baker [1] also states that if $f$ is a "generalized" polynomial mapping of "degree" at most $m - 1$, then $f$ is expressed as $f(x) = x_0 + \sum_{l=1}^{m-1} a_l^*(x)$ for $x \in V$, where $a_l^*$ is a monomial mapping of degree $l$ and $f$ has a property $f(rx) = x_0 + \sum_{l=1}^{m-1} r^l a_l^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. The monomial mapping of degree 1, 2, 3 and 4 are also called an additive mapping, a quadric mapping, a cubic mapping and a quartic mapping, respectively.

Therefore, if $f, g, h, f'$ are generalized polynomial mappings of degree at most 4 satisfying $f(rx) = rf(x)$, $g(rx) = r^2 g(x)$, $h(rx) = r^3 h(x)$ and $f'(rx) = r^4 f'(x)$ for all $x \in V$ when $r$ is a fixed rational number with $r \neq 0, \pm 1$, then $f, g, h, f'$ are an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping, respectively.

Hereafter we will use the following abbreviation for convenience:

$$\begin{aligned}\Lambda f(x) :=& \frac{1}{k^4 - k^2}\big((4k^2 - 3)D_k f_o(x, x) - 2k^2 D_k f_o(2x, x) + 2k^2 D_k f_o(x, 2x) \\ &- 2D_k f_o((k + 1)x, x) + 2D_k f_o((k - 1)x, x) - k^2 D_k f_o(2x, 2x) \\ &+ D_k f_o(x, 3x) - D_k f_o((2k + 1)x, x) + D_k f_o((2k - 1)x, x)\big).\end{aligned} \tag{2.1}$$

Now we will show that the functional equation $D_r f(x, y) = 0$ is an additive-cubic-quartic functional equation when $r$ is a rational number such that $r \neq 0, \pm 1$.

**Theorem 2.2.** *Let $r$ be a rational number such that $r \neq 0, \pm 1$. A mapping $f$ satisfies the functional equation $D_r f(x, y) = 0$ for all $x, y \in V$ if and only if $f$ is an additive-cubic-quartic mapping.*

*Proof.* Assume that a mapping $f : V \to W$ satisfies the functional equation $D_r f(x, y) = 0$ for all $x, y \in V$ and $g, h$ are the mappings defined by $g(x) = \frac{-f_o(2x)+8f_o(x)}{6}$ and $h(x) = \frac{f_o(2x)-2f_o(x)}{6}$. Then $D_r g(x, y) = 0$, $D_r h(x, y) = 0$ and $D_r f_e(x, y) = 0$ hold for all $x, y \in V$. According to Theorem 2.1, we obtain that $g$, $h$ and $f_e$ are generalized polynomial mappings of degree at most 4. From the equalities

$$f_o(4x) - 10f_o(2x) + 16f_o(x) = \Lambda f(x) \text{ and } f_e(rx) - r^4 f_e(x) = \frac{D_r f(0, x)}{2} \tag{2.2}$$

for all $x \in V$, where $\Lambda f(x)$ is the mapping defined in (2.1), we know that $g, h, f_e$ satisfy the properties $g(2x) = 2g(x)$, $h(2x) = 2^3 h(x)$ and $f_e(rx) = r^4 f_e(x)$ for all $x \in V$, respectively. As mentioned in the previous sentence above this theorem, $g, h, f_e$ are an additive mapping, a cubic mapping, and a quartic mapping, respectively. Since the equality $f = g + h + f_e$ holds, $f$ is an additive-cubic-quartic mapping.

Conversely, assume that $f$ is an additive-cubic-quartic mapping, i.e. there exist an additive mapping $g$, a cubic mapping $h$, and a quartic mapping $f'$ such that $f = g + h + f'$. Notice that the equalities $g(rx) = rg(x)$, $g(x) = -g(-x)$, $h(rx) = r^3 h(x)$, $h(x) = -h(-x)$, $f'(rx) = r^4 f'(x)$ and $f'(x) = f'(-x)$ for all $x \in V$ and $r \in \mathbb{Q}$. First

$D_r g(x, y) = 0$ is obtained from the equality

$$D_r g(x, y) = r^2 Ag(x + y, x - y) - Ag(x + ry, x - ry) - (r^2 - 1)Ag(x, x)$$

for all $x, y \in V$. Let us first prove $D_n h(x, y) = 0$ and $D_n f'(x, y) = 0$ for $n$ is a natural number. Using mathematical induction, the equalities $D_n h(x, y) = 0$ and $D_n f'(x, y) = 0$ are obtained from the equalities

$$\begin{aligned}
&D_1 h(x, y) \equiv 0 \equiv D_1 f'(x, y),\\
&D_2 f'(x, y) = Q' f'(x, y),\\
&D_2 h(x, y) = Ch(x, y) - Ch(x - y, y),\\
&D_n f'(x, y) = D_{n-1} f'(x + y, y) + D_{n-1} f'(x - y, y) - D_{n-2} f'(x, y)\\
&\qquad\qquad + (n - 1)^2 Q' f'(x, y),\\
&D_n h(x, y) = D_{n-1} h(x + y, y) + D_{n-1} h(x - y, y) - D_{n-2} h(x, y) + (n - 1)^2 D_2 h(x, y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}\backslash\{1, 2\}$. Let us now prove $D_r f'(x, y) = 0$ and $D_r h(x, y) = 0$ for any rational numbers $r$ with $r \neq 0, \pm 1$. Notice that if $r \in \mathbb{Q}$, then there exist $m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $D_{\frac{n}{m}} h(x, y) = 0$, $D_{\frac{-n}{m}} h(x, y) = 0$, $D_{\frac{n}{m}} f'(x, y) = 0$ and $D_{\frac{-n}{m}} f'(x, y) = 0$ are derived from the equalities

$$\begin{aligned}
D_{\frac{n}{m}} h(x, y) =& D_n h\left(x, \frac{y}{m}\right) - \frac{n^2}{m^2} D_m h\left(x, \frac{y}{m}\right),\\
D_{\frac{-n}{m}} h(x, y) =& D_{\frac{n}{m}} h(x, -y),\\
D_{\frac{n}{m}} f'(x, y) =& D_n f'\left(x, \frac{y}{m}\right) - \frac{n^2}{m^2} D_m f'\left(x, \frac{y}{m}\right),\\
D_{\frac{-n}{m}} f'(x, y) =& D_{\frac{n}{m}} f'(x, -y)
\end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $D_r h(x, y) = 0$ and $D_r f'(x, y) = 0$ for all $x, y \in V$. $\square$

Now we can prove the following Hyers-Ulam-Rassias stability theorem.

**Theorem 2.3.** *Let $p \neq 1, 3, 4$ be a positive real number, $X$ a real normed space, and $Y$ a real Banach space. Suppose that $f : X \to Y$ is a mapping such that*

$$\|D_k f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.3}$$

*for all $x, y \in X$. Then there exists a unique solution mapping $F$ of the functional equation $D_k F(x, y) = 0$ such that*

(2.4)

$$\|f(x)-F(x)\| \le \begin{cases} \Big[\frac{1}{2||k|^p-|k|^4|}+\frac{K}{3\cdot 2^p}\big(\frac{4}{2^p-8}-\frac{1}{2^p-2}\big)\Big]\theta\|x\|^p & \textit{if} \quad 4<p, \\ \Big[\frac{1}{2||k|^p-|k|^4|}+\frac{K}{3\cdot 2^p}\big(\frac{4}{2^p-8}-\frac{1}{2^p-2}\big)\Big]\theta\|x\|^p & \textit{if} \quad 3<p<4, \\ \Big[\frac{1}{2||k|^p-|k|^4|}+\frac{K}{6}\big(\frac{1}{8-2^p}+\frac{1}{2^p-2}\big)\Big]\theta\|x\|^p & \textit{if} \quad 1<p<3, \\ \Big[\frac{1}{2||k|^p-|k|^4|}+\frac{K}{6}\big(\frac{1}{2-2^p}-\frac{1}{8-2^p}\big)\Big]\theta\|x\|^p & \textit{if} \quad 0<p<1 \end{cases}$$

*for all* $x \in X$*, where*

$$K = \frac{12k^2+13+5k^2 2^p+3^p+2|k-1|^p+2|k+1|^p+|2k-1|^p+|2k+1|^p}{|k^4-k^2|}.$$

*Proof.* Notice that $2k^2(k^2-1)\|f(0)\| = \|D_k f(x,y)\| \le 0$, which implies $f(0)=0$. we will prove this theorem by dividing it into two cases, $|k|<1$ and $1<|k|$.

Case 1: Assume that $1<|k|$. Let $J_n f : X \to Y$ be the mappings defined by

$$J_n f(x) = \begin{cases} k^{4n} f_e(k^{-n}x) + \frac{4\cdot 8^n - 2^n}{3} f_o\big(\frac{x}{2^n}\big) - \frac{8^{n+1}-2^{n+3}}{3} f_o\big(\frac{x}{2^{n+1}}\big) & \text{if} \quad 4<p, \\ \frac{f_e(k^n x)}{k^{4n}} + \frac{4\cdot 8^n-2^n}{3} f_o\big(\frac{x}{2^n}\big) - \frac{8^{n+1}-2^{n+3}}{3} f_o\big(\frac{x}{2^{n+1}}\big) & \text{if} \quad 3<p<4, \\ \frac{f_e(k^n x)}{k^{4n}} - \frac{2^{n-1}}{3}\Big(f_o\big(\frac{x}{2^{n-1}}\big) - 8 f_o\big(\frac{x}{2^n}\big)\Big) + \frac{f_o(2^{n+1}x)-2f_o(2^n x)}{6\cdot 8^n} & \text{if} \quad 1<p<3, \\ \frac{f_e(k^n x)}{k^{4n}} + \frac{8f_o(2^n x)-f_o(2^{n+1}x)}{6\cdot 2^n} + \frac{f_o(2^{n+1}x)-2f_o(2^n x)}{6\cdot 8^n} & \text{if} \quad p<1 \end{cases}$$

for all $x \in X$ and all nonnegative integers $n$. Then, by (2.2) and the definitions of $J_n f$ and $\Lambda f$, we have the equality

(2.5)

$$J_n f(x) - J_{n+1} f(x) = \begin{cases} \frac{k^{4n}}{2} D_k f\big(0, \frac{x}{k^{n+1}}\big) + \frac{4\cdot 8^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) - \frac{2^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) & \text{if } 4<p, \\ -\frac{D_k f(0,k^n x)}{2\cdot k^{4(n+1)}} + \frac{4\cdot 8^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) - \frac{2^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) & \text{if } 3<p<4, \\ -\frac{D_k f(0,k^n x)}{2\cdot k^{4(n+1)}} - \frac{1}{48\cdot 8^n}\Lambda f\big(2^n x\big) - \frac{2^{n-1}}{3}\Lambda f\big(\frac{x}{2^{n+1}}\big) & \text{if } 1<p<3, \\ -\frac{D_k f(0,k^n x)}{2\cdot k^{4(n+1)}} + \frac{1}{12\cdot 2^n}\Lambda f\big(2^n x\big) - \frac{1}{48\cdot 8^n}\Lambda f\big(2^n x\big) & \text{if } p<1 \end{cases}$$

holds for all $x \in X$ and all nonnegative integers $n$. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we obtain that if $f : X \to Y$ is a mapping such that $D_k f(x,y) = 0$ for all $x, y \in X$, then

$$J_n f(x) = f(x) \tag{2.6}$$

for all $x \in X$ and all positive integers $n$. The inequality

$$\|\Lambda f(x)\| \le K\theta\|x\|^p \tag{2.7}$$

follows from (2.3) and the definition of $\Lambda f$. It follows from (2.5) and (2.7) that

$$\|J_nf(x)-J_{n+1}f(x)\| \leq \begin{cases} \left(\frac{k^{4n}}{2\cdot|k|^{(n+1)p}}+\frac{(4\cdot 8^n-2^n)K}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 4<p,\\ \left(\frac{|k|^{np}}{2\cdot k^{4(n+1)}}+\frac{(4\cdot 8^n-2^n)K}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 3<p<4,\\ \left(\frac{|k|^{np}}{2\cdot k^{4(n+1)}}+\frac{K2^{np}}{6\cdot 8^{n+1}}+\frac{2^nK}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 1<p<3,\\ \left(\frac{|k|^{np}}{2\cdot k^{4(n+1)}}+\frac{(4^{n+1}-1)2^{np}K}{6\cdot 8^{n+1}}\right)\theta\|x\|^p & \text{if } 0<p<1 \end{cases}$$

for all $x\in X$. Together with the equality $J_nf(x)-J_{n+m}f(x)=\sum_{i=n}^{n+m-1}(J_if(x)$ $-J_{i+1}f(x))$ for all $x\in X$, we get

(2.8)

$$\|J_nf(x)-J_{n+m}f(x)\| \leq \begin{cases} \sum_{i=n}^{n+m-1}\left(\frac{k^{4i}}{2\cdot|k|^{(i+1)p}}+\frac{(4\cdot 8^i-2^i)K}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 4<p,\\ \sum_{i=n}^{n+m-1}\left(\frac{|k|^{ip}}{2\cdot k^{4(i+1)}}+\frac{(4\cdot 8^i-2^i)K}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 3<p<4,\\ \sum_{i=n}^{n+m-1}\left(\frac{|k|^{ip}}{2\cdot k^{4(i+1)}}+\frac{K2^{ip}}{6\cdot 8^{i+1}}+\frac{2^iK}{6\cdot 2^{(i+1)p}}\right)\theta\|x\|^p & \\ & \text{if } 1<p<3,\\ \sum_{i=n}^{n+m-1}\left(\frac{|k|^{ip}}{2\cdot k^{4(i+1)}}+\frac{(4^{i+1}-1)2^{ip}K}{6\cdot 8^{i+1}}\right)\theta\|x\|^p & \text{if } p<1 \end{cases}$$

for all $x\in X$ and $n,m\in\mathbb{N}\cup\{0\}$. It follows from (2.8) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x\in X$. Since $Y$ is complete, the sequence $\{J_nf(x)\}$ converges for all $x\in X$. Hence we can define a mapping $F:X\to Y$ by

$$F(x):=\lim_{n\to\infty}J_nf(x)$$

for all $x\in X$. Moreover, letting $n=0$ and passing the limit $n\to\infty$ in (2.8) we get the inequality (2.4). For the case $1<p<3$, we easily get

$$\begin{aligned}\|D_kF(x,y)\| = &\lim_{n\to\infty}\Bigg\|\frac{D_kf_e\left(k^nx,k^ny\right)}{k^{4n}}+\frac{2^n}{6}\left(-D_kf_o\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+8D_kf_o\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right)\\ &+\frac{D_kf_o\left(2^{n+1}x,2^{n+1}y\right)-2D_kf_o\left(2^nx,2^ny\right)}{6\cdot 8^n}\Bigg\|\\ \leq&\lim_{n\to\infty}\left(\frac{|k|^{np}}{k^{4n}}+\frac{2^n(2^p+8)}{6\cdot 2^{np}}+\frac{2^{np}(2^p+2)}{6\cdot 8^n}\right)\times\theta(\|x\|^p+\|y\|^p)\\ =&0\end{aligned}$$

for all $x,y\in X$. Also we easily show that $D_kF(x,y)=0$ by the similar method for the other cases, either $p<1$ or $3<p<4$ or $4<p$.

To prove the uniqueness of $F$, let $F':X\to Y$ be another solution mapping satisfying (2.4). Instead of the condition (2.4), it is sufficient to show that there is a unique mapping that satisfies condition $\|f(x)-F(x)\|\leq\big(\frac{1}{2|k^4-|k|^p|}+\frac{K}{6|8-2^p|}+$

$\frac{K}{6|2-2^p|})\theta\|x\|^p$ simply. By (2.6), the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $1 < p < 3$, we have

$$\begin{aligned}
\|J_nf(x) - F'(x)\| &= \|J_nf(x) - J_nF'(x)\| \\
&\leq \Big\| -\frac{2^n}{6}\Big(f_o\Big(\frac{2x}{2^n}\Big) - 8f_o\Big(\frac{x}{2^n}\Big)\Big) + \frac{f_o(2^{n+1}x) - 2f_o(2^nx)}{6\cdot 8^n} + \frac{f_e(k^nx)}{k^{4n}} \\
&\quad + \frac{2^n}{6}\Big(F'_o\Big(\frac{2x}{2^n}\Big) - 8F'_o\Big(\frac{x}{2^n}\Big)\Big) - \frac{F'_o(2^{n+1}x) - 2F'_o(2^nx)}{6\cdot 8^n} - \frac{F'_e(k^nx)}{k^{4n}}\Big\| \\
&\leq \frac{2^n}{6}\Big\|(f_o - F'_o)\Big(\frac{2x}{2^n}\Big)\Big\| + \frac{2^{n+3}}{6}\Big\|(f_o - F'_o)\Big(\frac{x}{2^n}\Big)\Big\| + \frac{\|(f_o - F'_o)(2^{n+1}x)\|}{6\cdot 8^n} \\
&\quad + \frac{2\|(f_o - F'_o)(2^nx)\|}{6\cdot 8^n} + \frac{\|(f_e - F'_e)(k^nx)\|}{k^{4n}} \\
&\leq \Big(\frac{2^{n-1+p} + 2^{n+2}}{3\cdot 2^{np}} + \frac{2^{(n+1)p} + 2^{np+1}}{3\cdot 2^{3n+1}} + \frac{|k|^{np}}{k^{4n}}\Big) \\
&\quad \times \Big(\frac{K}{6|8-2^p|} + \frac{K}{6|2-2^p|} + \frac{1}{2|k^4 - |k|^p|}\Big)\theta\|x\|^p
\end{aligned}$$

for all $x \in X$ and all positive integers $n$. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, either $0 < p < 1$ or $3 < p < 4$ or $4 < p$, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

Case 2: Assume that $|k| < 1$. Let $J_nf : X \to Y$ be the mappings defined by $J_nf(x) =$

$$\begin{cases}
\frac{f_e(k^nx)}{k^{4n}} + \frac{4\cdot 8^n - 2^n}{3}f_o\big(\frac{x}{2^n}\big) - \frac{8^{n+1} - 2^{n+3}}{3}f_o\big(\frac{x}{2^{n+1}}\big) & \text{if } 4 < p, \\
k^{4n}f_e(k^{-n}x) + \frac{4\cdot 8^n - 2^n}{3}f_o\big(\frac{x}{2^n}\big) - \frac{8^{n+1} - 2^{n+3}}{3}f_o\big(\frac{x}{2^{n+1}}\big) & \text{if } 3 < p < 4, \\
k^{4n}f_e(k^{-n}x) - \frac{2^{n-1}}{3}\big(f_o\big(\frac{x}{2^{n-1}}\big) - 8f_o\big(\frac{x}{2^n}\big)\big) + \frac{f_o(2^{n+1}x) - 2f_o(2^nx)}{6\cdot 8^n} & \text{if } 1 < p < 3, \\
k^{4n}f_e(k^{-n}x) + \frac{8f_o(2^nx) - f_o(2^{n+1}x)}{6\cdot 2^n} + \frac{f_o(2^{n+1}x) - 2f_o(2^nx)}{6\cdot 8^n} & \text{if } p < 1
\end{cases}$$

for all $x \in X$ and all nonnegative integers. Then, by the definitions of $J_nf$ and $\Lambda f$, the equality

$J_nf(x) - J_{n+1}f(x) =$

$$\begin{cases}
-\frac{D_kf(0,k^nx)}{2\cdot k^{4(n+1)}} + \frac{4\cdot 8^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) - \frac{2^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) & \text{if } 4 < p, \\
\frac{k^{4n}}{2}D_kf\big(0, \frac{x}{k^{n+1}}\big) + \frac{4\cdot 8^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) - \frac{2^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) & \text{if } 3 < p < 4, \\
\frac{k^{4n}}{2}D_kf\big(0, \frac{x}{k^{n+1}}\big) - \frac{1}{48\cdot 8^n}\Lambda f\big(2^nx\big) - \frac{2^{n-1}}{3}\Lambda f\big(\frac{x}{2^{n+1}}\big) & \text{if } 1 < p < 3, \\
\frac{k^{4n}}{2}D_kf\big(0, \frac{x}{k^{n+1}}\big) + \frac{1}{12\cdot 2^n}\Lambda f\big(2^nx\big) - \frac{1}{48\cdot 8^n}\Lambda f\big(2^nx\big) & \text{if } p < 1
\end{cases}$$

holds for all $x \in X$ and all nonnegative integers $n$. Proof of the remaining part is omitted because it follows a procedure very similar to the case of $1 < |k|$ from the above equality. $\square$

## REFERENCES


1. J. Baker: A general functional equation and its stability. Proc. Natl. Acad. Sci. **133** (2005), no. 6, 1657-1664.
2. Y.J. Cho & R. Saadati: Lattictic non-archimedean random stability of ACQ functional equation. Adv. Differ. Equ. **2011**(1), 31.
3. P. Găvruta: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184** (1994), 431-436.
4. M.E. Gordji, Y.J. Cho, H. Khodaei & M. Ghanifard: Solutions and stability of generalized mixed type QCA-functional equations in random normed spaces. Annals of the Alexandru Ioan Cuza University-Mathematics **59** (2013), 299-320.
5. M.E. Gordji, S.K. Gharetapeh, C. Park & S. Zolfaghri: Stability of an additive-cubic-quartic functional equation. Adv. Differ. Equ. **2009** Article ID 395693.
6. D.H. Hyers: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.
7. J.M. Rassias, M. Arunkumar, E. Sathya & N. Mahesh Kumar: Solution and stability of an ACQ functional equation in generalized 2-normed spaces. Intern. J. Fuzzy Mathematical Archive **7** (2015), 213-224.
8. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72** (1978), 297-300.
9. R. Saadati, M.M. Zohdi & S.M. Vaezpour: Nonlinear L-random stability of an ACQ functional equation. J. Inequal. Appl. **2011** Article ID 194394.
10. S.M. Ulam: A Collection of Mathematical Problems. Interscience, New York, 1960.
11. Z. Wang, X. Li & Th.M. Rassias: Stability of an additive-cubic-quartic functional equation in multi-Banach spaces. Abstr. Appl. Anal. **2011** Article ID 536520.


DEPARTMENT OF MATHEMATICS EDUCATION, GONGJU NATIONAL UNIVERSITY OF EDUCATION, GONGJU 32553, REPUBLIC OF KOREA
*Email address*: `lyhmzi@gjue.ac.kr`