J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2019.26.4.267 ISSN(Online) 2287-6081
Volume 26, Number 4 (November 2019), Pages 267–276

A FIXED POINT APPROACH TO THE STABILITY OF AN ADDITIVE-CUBIC-QUARTIC FUNCTIONAL EQUATION

Yang-Hi Lee

Abstract. In this paper, we investigate the stability of an additive-cubic-quartic functional equation

$$f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-12f(y)-12f(-y)=0$$

by applying the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

In 1940, Ulam [17] questioned the stability of group homomorphisms, and the following year Hyers [11] gave an affirmative answer to this problem for additive mappings between Banach spaces. Hyers' result has motivated many mathematicians to deal with this problem (cf. [8, 14]).

Throughout this paper, let V and W be real vector spaces and Y a real Banach space. For a given mapping $f: V \to W$, we use the following abbreviations

$$Af(x,y) := f(x+y)-f(x)-f(y),$$
$$Cf(x,y) := f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),$$
$$Q'f(x,y) := f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-24f(y)$$

for all $x, y \in V$. Solution of the functional equations $Af(x,y)=0$, $Cf(x,y)=0$ and $Q'f(x,y)=0$ are called an additive mapping, a cubic mapping, and a quartic mapping, respectively. A mapping f is called an additive-cubic-quartic mapping if f is represented by sum of an additive mapping, a cubic mapping, and a quartic mapping. A functional equation is called an additive-cubic-quartic functional equation provided that each solution of that equation is an additive-cubic-quartic mapping and every additive-cubic-quartic mapping is a solution of that equation.

Received by the editors June 12, 2019. Accepted November 19, 2019.
2010 *Mathematics Subject Classification.* 39B82, 39B52.
Key words and phrases. stability, additive-cubic-quartic functional equation, fixed point theory.

M. E. Gordji et al. [9] investigated the additive-cubic-quartic functional equation

$$\begin{aligned} f(x+ky)+&f(x-ky)-k^2f(x+y)-k^2f(x-y)\\ &-(k^2-1)(k^2f(y)+k^2f(-y)-2f(x))=0, \end{aligned}$$

where $k \neq 0, \pm 1$ is an integer, J. M. Rassias [13] investigated the additive-cubic-quartic functional equation

$$\begin{aligned} &11f(x+2y+2w)+11f(x-2y-2w)-44f(x+y+w)-44f(x-y-w)\\ &-12f(3y+3w)+48f(2y+2w)-60f(y+w)+66f(x)=0, \end{aligned}$$

and many mathematicians [6, 10, 16, 18] investigated the additive-cubic-quartic functional equation

$$\begin{aligned} &11f(x+2y)+11f(x-2y)\\ &\quad=44f(x+y)+44f(x-y)+12f(3y)-48f(2y)+60f(y)-66f(x). \end{aligned}$$

Now we consider the following functional equation

$$\begin{aligned} f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)&\\ -12f(y)-12f(-y)=0.& \end{aligned} \tag{1.1}$$

The mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^4 + bx^3 + cx$ is a solution of this functional equation, where a, b, c are real constants.

In this paper, we will show that the functional equation (1.1) is an additive-cubic-quartic functional equation and we introduce a strictly contractive mapping which allows us to use the fixed point theory for proving the stability of the functional equation (1.1) in the sense of L. Cădariu and V. Radu [4, 5]. Namely, starting from the given mapping f that approximately satisfies the functional equation (1.1), a solution F of the functional equation (1.1) is explicitly constructed by the formula

$$F(x)=\lim_{n\to\infty}\left(\sum_{i=0}^{n} {}_nC_i\frac{(-1)^{n-i}10^i}{16^n}f_o(2^{2n-i}x)+\frac{f_e(2^nx)}{16^n}\right)$$

or

$$F(x)=\lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i\left(10^i(-16)^{n-i}f_o\left(\frac{x}{2^{2n-i}}\right)+10^i(-96)^{n-i}f_e\left(\frac{x}{2^{2n-i}}\right)\right),$$

which approximates the mapping f.

2. Main Results

Recall the following result of Margolis and Diaz's fixed point theory.

Theorem 2.1 ([7, 15]). *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:
(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, for a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}, \\ Df(x, y) &:= f(x + 2y) - 4f(x + y) + 6f(x) - 4f(x - y) + f(x - 2y) \\ &\quad - 12f(y) - 12f(-y) \end{aligned}$$

for all $x, y \in V$. As we stated in the previous section, a solution of $Af = 0$, $Cf = 0$, and $Q'f = 0$ is called an additive, a cubic, and a quartic mapping, respectively. Now we will show that f is an additive-cubic-quartic mapping if f is a solution of the functional equation $Df(x, y) = 0$ for all $x, y \in V$.

Lee and Jung [12] proved the following lemma from Baker's theorem [2].

Lemma 2.2 ([12, Corollary 2.2]). *Let V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $r \in \mathbb{Q} - \{0, \pm 1\}$. Suppose that $n_1, \ldots, n_m$ are natural numbers, and $c_{l_i}, d_{l_i}, \alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j \beta_l - \alpha_l \beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If a mapping $f : V \to W$ satisfies the equality $f(rx) = r^k f(x)$ for all $x \in V$ and the inequality*

$$f(\alpha_0 x + \beta_0 y) + \sum_{l=1}^{m} \sum_{i=1}^{n_l} c_{l_i} f(d_{l_i}(\alpha_l x + \beta_l y)) = 0$$

for all $x, y \in V$, then f is a monomial mapping of degree k.

Theorem 2.3. *A mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$ if and only if f is an additive-cubic-quartic mapping.*

Proof. Define the mappings f_1 and f_2 by $f_1(x) := \frac{-f_o(2x)+8f_o(x)}{6}$ and $f_2(x) := \frac{f_o(2x)-2f_o(x)}{6}$. If a mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$, then f_1, f_2, f_e satisfy the equalities $Df_1(x,y) = 0$, $Df_2(x,y) = 0$, and $Df_e(x,y) = 0$ for all $x, y \in V$. We can obtain the equalities $f_1(2x) = 2f_1(x)$, $f_2(2x) = 2^3 f_2(x)$, $f_e(2x) = 2^4 f_e(x)$ from the equalities $f_e(2x) - 16f_e(x) = \frac{Df_e(0,x)}{2}$ and $f_o(4x) - 10f_o(2x) + 16f_o(x) = Df_o(2x,x) + 4Df_o(x,x)$ for all $x \in V$. According to Lemma 2.2, f_1, f_2, f_e are an additive mapping, a cubic mapping, and a quartic mapping, respectively. Since the equality $f = f_1 + f_2 + f_e$ holds, f is an additive-cubic-quartic mapping.

Conversely, assume that f_1, f_2, f_3 are mappings satisfying the equalities $f := f_1 + f_2 + f_3$, $Af_1(x,y) = 0$, $Cf_2(x,y) = 0$, and $Q'f_3(x,y) = 0$ for all $x, y \in V$. Then the equalities $f_1(x) = -f_1(-x)$, $f_2(x) = -f_2(-x)$, $f_3(x) = f_3(-x)$, $f_1(2x) = 2f_1(x)$, $f_2(2x) = 8f_2(x)$, and $f_3(2x) = 16f_3(x)$ hold for all $x \in V$. From the above equalities, we obtain the equalities

$$\begin{aligned} Df(x,y) =& Df_1(x,y) + Df_2(x,y) + Df_3(x,y) \\ =& -Af_1(x+2y, x-2y) + 4Af_1(x+y, x-y) + Cf_2(x,y) \\ & - Cf_2(x-y, y) + Q'f_3(x,y) \\ =& 0 \end{aligned}$$

as we desired. □

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1.1) by using the fixed point theory.

Theorem 2.4. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.1}$$

holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(2x, 2y) \le (\sqrt{41} - 5)L\varphi(x,y) \tag{2.2}$$

for all $x, y \in V$*, then there exists a unique solution* $F : V \to Y$ *of* (1.1) *satisfying the inequality*

$$\|f(x) - F(x)\| \le \frac{\Phi(x)}{32(1-L)} \tag{2.3}$$

for all $x \in V$*, where* $\Phi(x) = \varphi(2x, x) + \varphi(-2x, -x) + 4\varphi(x, x) + 4\varphi(-x, -x) + \varphi(0, x) + \varphi(0, -x)$*. In particular,* F *is represented by*

$$F(x) = \lim_{n\to\infty} \left(\sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}10^i}{16^n} f_o(2^{2n-i}x) + \frac{f_e(2^n x)}{16^n} \right) \tag{2.4}$$

for all $x \in V$*.*

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g, h) = \inf \left\{ K \in \mathbb{R}_+ \middle| \, \|g(x) - h(x)\| \le K\Phi(x) \text{ for all } x \in V \right\}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := -\frac{g(4x)}{32} + \frac{g(-4x)}{32} + \frac{11g(2x)}{32} - \frac{9g(-2x)}{32}$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}10^i}{16^n} g_o(2^{2n-i}x) + \frac{g_e(2^n x)}{16^n}$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| \le & \frac{1}{32}\|g(4x) - h(4x)\| + \frac{1}{32}\|g(-4x) - h(-4x)\| \\ & + \frac{11}{32}\|g(2x) - h(2x)\| + \frac{9}{32}\|g(-2x) - h(-2x)\| \\ \le & K(\frac{\Phi(4x)}{32} + \frac{\Phi(-4x)}{32} + \frac{11}{32}\Phi(2x) + \frac{9}{32}\Phi(-2x)) \\ \le & K(\frac{1}{16}\Phi(4x) + \frac{10}{16}\Phi(2x)) \end{aligned}$$

$$\begin{aligned}
&\le K\Big(\frac{\sqrt{41}-5}{16}L\Phi(2x)+\frac{10}{16}\Phi(2x)\Big)\\
&\le K\Big(\frac{(\sqrt{41}-5)^2}{16}L^2\Phi(x)+\frac{10(\sqrt{41}-5)}{16}L\Phi(x)\Big)\\
&\le K\frac{(\sqrt{41}-5)^2+10(\sqrt{41}-5)}{16}L\Phi(x)\\
&\le LK\Phi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) we see that

$$\begin{aligned}
&\|f(x)-Jf(x)\|\\
&=\frac{\|Df(2x,x)-Df(-2x,-x)+4Df(x,x)-4Df(-x,-x)-Df(0,x)\|}{32}\\
&\le\frac{\varphi(2x,x)+\varphi(-2x,-x)+4\varphi(x,x)+4\varphi(-x,-x)+\varphi(0,x)}{32}\\
&\le\frac{\Phi(x)}{32}
\end{aligned}$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{1}{32} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (2.4) for all $x \in V$. Notice that

$$d(f,F) \le \frac{1}{1-L}d(f,Jf) \le \frac{1}{32(1-L)},$$

which implies (2.3). By the definition of F, together with (2.1) and (2.2), we have

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty}\|DJ^n f(x,y)\|\\
&= \lim_{n\to\infty}\Big\|\sum_{i=0}^{n}{}_nC_i\frac{(-1)^{n-i}(10)^i}{16^n}Df_o(2^{2n-i}x,2^{2n-i}y)\\
&\quad+\frac{Df_e(2^n x,2^n y)}{16^n}\Big\|\\
&\le \lim_{n\to\infty}\sum_{i=0}^{n}\frac{{}_nC_i}{2}\frac{10^i}{16^n}\big(\varphi(2^{2n-i}x,2^{2n-i}y)+\varphi(-2^{2n-i}x,-2^{2n-i}y)\big)\\
&\quad+\lim_{n\to\infty}\frac{\big(\varphi(2^n x,2^n y)+\varphi(-2^n x,-2^n y)\big)}{2\cdot 16^n}
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty}\bigg(\sum_{i=0}^{n} {}_nC_i \frac{10^i}{16^n}(\sqrt{41}-5)^{n-i}L^{n-i}\big(\varphi(2^n x, 2^n y)+\varphi(-2^n x,-2^n y)\big) \\
&\qquad +\frac{1}{16^n}\big(\varphi(2^n x, 2^n y)+\varphi(-2^n x,-2^n y)\big)\bigg) \\
&\le \lim_{n\to\infty}\left(\sum_{i=0}^{n} {}_nC_i \frac{(\sqrt{41}-5)^{n-i}10^i}{16^n}+\frac{1}{16^n}\right)\big(\varphi(2^n x, 2^n y)+\varphi(-2^n x,-2^n y)\big) \\
&\le \lim_{n\to\infty}\left(\frac{((\sqrt{41}-5)+10)^n}{16^n}+\frac{1}{16^n}\right)\big(\varphi(2^n x, 2^n y)+\varphi(-2^n x,-2^n y)\big) \\
&\le \lim_{n\to\infty}\left(\frac{(\sqrt{41}+5)^n}{16^n}+\frac{1}{16^n}\right)\big(\varphi(2^n x, 2^n y)+\varphi(-2^n x,-2^n y)\big) \\
&\le \lim_{n\to\infty}\left(\frac{(\sqrt{41}+5)^n(\sqrt{41}-5)^n}{16^n}+\frac{(\sqrt{41}-5)^n}{16^n}\right)L^n\big(\varphi(x,y)+\varphi(-x,-y)\big) \\
&\le \lim_{n\to\infty} 2L^n\big(\varphi(x,y)+\varphi(-x,-y)\big) \\
&= 0
\end{aligned}
$$

for all $x, y \in V$ i.e., F is a solution of the functional equation (1.1). Notice that if F is a solution of the functional equation (1.1), then the equality $F(x) - JF(x) = \frac{DF(2x,x)-DF(-2x,-x)+4DFx,x)-4DF(-x,-x)-DF(0,x)}{32}$ implies that F is a fixed point of J. □

Theorem 2.5. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0, \infty)$ such that the inequality* (2.1) *holds for all $x, y \in V$ and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property*

$$L\varphi(2x, 2y) \ge 16\varphi(x, y) \tag{2.5}$$

for all $x, y \in V$, then there exists a unique solution $F : V \to Y$ of (1.1) *satisfying the inequality*

$$\|f(x) - F(x)\| \le \frac{\Psi(x)}{1-L} \tag{2.6}$$

for all $x \in V$, where $\Psi(x)$ is given by

$$\Psi(x) := 4\varphi\left(\frac{x}{4},\frac{x}{4}\right)+\varphi\left(\frac{x}{2},\frac{x}{4}\right)+4\varphi\left(\frac{-x}{4},\frac{-x}{4}\right)+\varphi\left(\frac{-x}{2},\frac{-x}{4}\right).$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\sum_{i=0}^{n} {}_nC_i\left(10^i(-16)^{n-i} f_o\left(\frac{x}{2^{2n-i}}\right)+10^i(-96)^{n-i} f_e\left(\frac{x}{2^{2n-i}}\right)\right) \tag{2.7}$$

for all $x \in V$.

Proof. Let the set S be the set as in the proof of Theorem 2.4. We give a generalized metric on S by

$$d(g,h) = \inf\left\{K \in \mathbb{R}_+ \middle| \, \|g(x) - h(x)\| \le K\Psi(x) \text{ for all } \ x \in V\right\}.$$

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := 10g\left(\frac{x}{2}\right) + 40g\left(\frac{x}{4}\right) + 56g\left(\frac{-x}{4}\right)$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i\left(10^i(-16)^{n-i}g_o\left(\frac{x}{2^{2n-i}}\right) + 10^i(-96)^{n-i}g_e\left(\frac{x}{2^{2n-i}}\right)\right)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le& 10\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| \\ &+ 40\left\|g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right)\right\| + 56\left\|g\left(\frac{-x}{4}\right) - h\left(\frac{-x}{4}\right)\right\| \\ \le& 96K\Psi\left(\frac{x}{4}\right) + 10K\Psi\left(\frac{x}{2}\right) \\ \le& L^2\frac{6}{16}K\Psi(x) + \frac{10}{16}LK\Psi(x) \\ \le& LK\Psi(x)\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g,h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) we see that

$$\begin{aligned}\|f(x) - Jf(x)\| &= \left\|Df\left(\frac{x}{2}, \frac{x}{4}\right) + 4Df\left(\frac{x}{4}, \frac{x}{4}\right)\right\| \\ &\le \varphi\left(\frac{x}{2}, \frac{x}{4}\right) + \varphi\left(\frac{-x}{2}, \frac{-x}{4}\right) + 4\varphi\left(\frac{x}{4}, \frac{x}{4}\right) + 4\varphi\left(\frac{-x}{4}, \frac{-x}{4}\right) \\ &\le \Psi(x)\end{aligned}$$

for all $x \in V$. It means that $d(f, Jf) \le 1 < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f,g) < \infty\}$, which is represented by (2.7) for all $x \in V$. Notice that

$$d(f,F) \le \frac{1}{1-L} d(f,Jf) \le \frac{1}{1-L},$$

which implies (2.6). By the definition of F, together with (2.1) and (2.5), we have

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty} \|DJ^n f(x,y)\| \\
&= \lim_{n\to\infty} \Big\| \sum_{i=0}^{n} {}_nC_i 10^i (-16)^{n-i} Df_o\Big(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\Big) \\
&\quad + 10^i(-96)^{n-i} Df_e\Big(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\Big)\Big)\Big\| \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 10^i 96^{n-i} \Big(\varphi\Big(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\Big) + \varphi\Big(\frac{-x}{2^{2n-i}}, \frac{-y}{2^{2n-i}}\Big)\Big) \\
&\le \lim_{n\to\infty} \Big(\sum_{i=0}^{n} {}_nC_i 10^i 6^{n-i} L^{n-i} \Big(\varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) + \varphi\Big(\frac{-x}{2^n}, \frac{-y}{2^n}\Big)\Big) \\
&\le \lim_{n\to\infty} (10+6)^n \Big(\varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) + \varphi\Big(\frac{-x}{2^n}, \frac{-y}{2^n}\Big)\Big) \\
&\le \lim_{n\to\infty} L^n \big(\varphi(x,y) + \varphi(-x,-y)\big) \\
&= 0
\end{aligned}$$

for all $x, y \in V$ i.e., F is a solution of the functional equation (1.1). Notice that if F is a solution of the functional equation (1.1), then the equality $F(x) - JF(x) = DF(\frac{x}{2}, \frac{x}{4}) + 4DF(\frac{x}{4}, \frac{x}{4})$ implies that F is a fixed point of J. □

References

1. T. Aoki: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Japan **2** (1950), 64-66.
2. J. Baker: A general functional equation and its stability. Proc. Natl. Acad. Sci. **133** (2005), no. 6, 1657-1664.
3. L. Cădariu & V. Radu: Fixed points and the stability of Jensen's functional equation. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Art. 4.
4. ______: Fixed points and the stability of quadratic functional equations. An. Univ. Timisoara Ser. Mat.-Inform. **41** (2003), 25-48.
5. ______: On the stability of the Cauchy functional equation: a fixed point approach in Iteration Theory. Grazer Mathematische Berichte, Karl-Franzens-Universitäet, Graz, Graz, Austria **346** (2004), 43-52.

6. Y.J. Cho & R. Saadati: Lattictic non-archimedean random stability of ACQ functional equation. Adv. Differ. Equ. **2011**, no. 1, 31.
7. J.B. Diaz & B. Margolis: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull. Amer. Math. Soc. **74** (1968), 305-309.
8. P. Găvruta: A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184** (1994), 431-436.
9. M.E. Gordji, Y.J. Cho, H. Khodaei & M. Ghanifard: Solutions and stability of generalized mixedtType QCA-functional equations in random normed spaces. Annals of the Alexandru Ioan Cuza University-Mathematics **59** (2013), no. 2, 299-320.
10. M.E. Gordji, S.K. Gharetapeh, C. Park & S. Zolfaghri: Stability of an additive-cubic-quartic functional equation. Adv. Differ. Equ. **2009** Article ID 395693.
11. D.H. Hyers: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
12. Y.-H. Lee & S.-M. Jung: Generalized Hyers-Ulam stability of some cubic-quadratic-additive type functional equations. (2000), submitted.
13. J M. Rassias, M. Arunkumar, E. Sathya & N. Mahesh Kumar: Solution and Stability of an ACQ Functional Equation in Generalized 2-Normed Spaces. Intern. J. Fuzzy Mathematical Archive **7** (2015), 213-224.
14. Th.M. Rassias: On the stability of the linear mapping in Banach spaces. Proc. Amer. Math. Soc. **72** (1978), 297-300.
15. I.A. Rus: Principles and Applications of Fixed Point Theory. Ed. Dacia, Cluj-Napoca 1979(in Romanian).
16. R. Saadati, M.M. Zohdi & S.M. Vaezpour: Nonlinear L-random stability of an ACQ functional equation. J. Inequal. Appl. **2011** Article ID 194394.
17. S.M. Ulam: A Collection of Mathematical Problems. Interscience, New York, 1960.
18. Z. Wang, X. Li & Th.M. Rassias: Stability of an additive-cubic-quartic functional equation in multi-Banach spaces. Abstr. Appl. Anal. **2011** Article ID 536520.

DEPARTMENT OF MATHEMATICS EDUCATION, GONGJU NATIONAL UNIVERSITY OF EDUCATION, GONGJU 32553, REPUBLIC OF KOREA
Email address: `lyhmzi@gjue.ac.kr`