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MULTIDIMENSIONAL COINCIDENCE POINT RESULTS FOR CONTRACTION MAPPING PRINCIPLE

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Abstract. The main objective of this article is to establish some coincidence point theorem for g-non-decreasing mappings under contraction mapping principle on a partially ordered metric space. Furthermore, we constitute multidimensional results as a simple consequences of our unidimensional coincidence point theorem. Our results improve and generalize various known results.

1. Introduction

The notion of multidimensional fixed/coincidence point was introduced by Roldan et al. in [15], which is an extension of Berzig and Samet's notion given in [3]. However, they used permutations of variables and distinguished between the first and the last variables. For more details one can refer [1, $4-13$, $16-23$].

In this article, we obtain some coincidence point theorem for g-non-decreasing mappings under generalized (ψ, θ, φ)-contraction on a partially ordered metric space. Furthermore, we constitute multidimensional results as a simple consequences of our unidimensional coincidence point theorem. We improve and generalize the results of Alsulami [2], Razani and Parvaneh [14], Su [21] and many other famous results in the literature.

2. Preliminaries

In order to establish our main results, we recall the following notions. For simplicity, we denote $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$ and X is a non-empty set. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A and B are nonempty subsets of Λ_n such that $A \cup B = \Lambda_n$ and

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$A \cap B = \emptyset$. We will denote $\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$ and $\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}$. Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For simplicity, we denote $g(x)$ by gx where $x \in X$.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notations:

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{2.1}$$

Consider on the product space X^n the following partial order: for $Y = (y_1, y_2, ..., y_i, ..., y_n)$, $V = (v_1, v_2, ..., v_i, ..., v_n) \in X^n$,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i. \tag{2.2}$$

Notice that $\sqsubseteq$ depends on A and B. We say that two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 2.1 ([12, 15, 18]). A point $(x_1, x_2, ..., x_n) \in X^n$ is called a $\Upsilon-$*coincidence point* of the mappings $F : X^n \to X$ and $g : X \to X$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(n)}) = gx_i, \text{ for all } i \in \Lambda_n.$$

If g is the identity mapping on X, then $(x_1, x_2, ..., x_n) \in X^n$ is called a $\Upsilon-$*fixed point* of the mapping F.

The previous definition extends the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_3 = (3, 2, 1)$,

(iii) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [3] use $A = \{1, 2, ..., m\}$, $B = \{m + 1, ..., n\}$ and arbitrary mappings.

Definition 2.2 ([15])**.** Let $(X, \preceq)$ be a partially ordered space. We say that F has the *mixed* $(g, \preceq)$*-monotone property* if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all x_1, x_2, ..., x_n, y, $z \in X$ and all i,

$$gy \preceq gz \Rightarrow F(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n) \preceq_i F(x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_n).$$

Definition 2.3 ([18, 22])**.** Let (X, d) be a metric space and define Δ_n, $\rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (y_1, y_2, ..., y_n)$, $V = (v_1, v_2, ..., v_n) \in X^n$, by

$$\Delta_n(Y,\ V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i) \text{ and } \rho_n(Y,\ V) = \max_{1\leq i\leq n} d(y_i,\ v_i).$$

Then Δ_n and ρ_n are metric on X^n and (X, d) is complete if and only if (X^n, Δ_n). Similarly (X, d) is complete if and only if(X^n, ρ_n) are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k,\ Y) &\to 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty), \\ \text{and } \rho_n(Y^k,\ Y) &\to 0 \text{ (as } k \to \infty) \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n, \end{aligned}$$

where $Y^k = (y_1^k, y_2^k, ..., y_n^k)$ and $Y = (y_1, y_2, ..., y_n) \in X^n$.

Definition 2.4 ([15])**.** We will say that two mappings T, $g : X \to X$ are *commuting* if $gTx = Tgx$ for all $x \in X$. We will say that $F : X^n \to X$ and $g : X \to X$ are commuting if $gF(x_1, x_2, ..., x_n) = F(gx_1, gx_2, ..., gx_n)$ for all $x_1, x_2, ..., x_n \in X$.

Definition 2.5 ([13])**.** Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. We will say that (F, g) is a $(O, \Upsilon)-$*compatible pair* if, for all $i \in \Lambda_n$,

$$\lim_{m\to\infty} d(gF(x_m^{\sigma_i(1)},\ x_m^{\sigma_i(2)},\ ...,\ x_m^{\sigma_i(n)}),\ F(gx_m^{\sigma_i(1)},\ gx_m^{\sigma_i(2)},\ ...,\ gx_m^{\sigma_i(n)})) = 0,$$

whenever $\{x_m^1\}$, $\{x_m^2\}$, ..., $\{x_m^n\}$ are sequences in X such that $\{gx_m^1\}$, $\{gx_m^2\}$, ..., $\{gx_m^n\}$ are $\preceq -$monotone and

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)},\ x_m^{\sigma_i(2)},\ ...,\ x_m^{\sigma_i(n)}) = \lim_{n\to\infty} gx_m^i \in X, \text{ for all } i \in \Lambda_n.$$

Lemma 2.1 ([7, 13, 18, 22, 23])**.** Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$

be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Define F_Υ, $G : X^n \to X^n$, for all y_1, y_2, ..., $y_n \in X$, by

$$F_\Upsilon(y_1,\ y_2,\ ...,\ y_n) = \begin{pmatrix} F(y_{\sigma_1(1)},\ y_{\sigma_1(2)},\ ...,\ y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)},\ y_{\sigma_2(2)},\ ...,\ y_{\sigma_2(n)}), \\ ...,\ F(y_{\sigma_n(1)},\ y_{\sigma_n(2)},\ ...,\ y_{\sigma_n(n)}) \end{pmatrix}, \tag{2.3}$$

and

$$G(y_1,\ y_2,\ ...,\ y_n) = (gy_1,\ gy_2,\ ...,\ gy_n). \tag{2.4}$$

(1) If F has the mixed $(g,\ \preceq)-$monotone property, then F_Υ is monotone $(G,\ \sqsubseteq)-$non-decreasing.

(2) If F is $d-$continuous, then F_Υ is Δ_n-continuous and ρ_n-continuous.

(3) If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.

(4) A point $(y_1,\ y_2,\ ...,\ y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1,\ y_2,\ ...,\ y_n)$ is a fixed point of F_Υ.

(5) A point $(y_1,\ y_2,\ ...,\ y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1,\ y_2,\ ...,\ y_n)$ is a coincidence point of F_Υ and G.

(6) If $(X,\ d,\ \preceq)$ is regular, then $(X^n,\ \Delta_n,\ \sqsubseteq)$ and $(X^n,\ \rho_n,\ \sqsubseteq)$ are also regular.

(7) If there exist y_0^1, y_0^2, ..., $y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $G(Y_0) \sqsubseteq F_\Upsilon(Y_0)$.

(8) If F and g are $(O,\ \Upsilon)$-compatible, then F_Υ and G are O-compatible.

Definition 2.6 ([21])**.** A generalized altering distance function is a function $\psi : [0,\ +\infty) \to [0,\ +\infty)$ which satisfies the following conditions:
(i_ψ) ψ is non-decreasing,
(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

3. Main Results

Theorem 3.1. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and T, $g : X \to X$ be two mappings such that T is $(g,\ \preceq)-$non-decreasing and $T(X) \subseteq g(X)$. Assume there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\psi(d(Tx,\ Ty)) \leq \varphi(d(gx,\ gy)), \tag{3.1}$$

for all $x,\ y \in X$ with $gx \preceq gy$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. There exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$. Also assume that one of the following conditions holds.

(a) $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is compatible,

(b) $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(c) $(X,\ d)$ is complete, g is continuous and monotone non-decreasing, the pair $(T,\ g)$ is compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.

Then T and g have a coincidence point.

Proof. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. As T is $(g,\ \preceq)$-non-decreasing and so $Tx_0 \preceq Tx_1$. Continuing in this manner, we get a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$-non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{3.2}$$

Let $\zeta_n = d(gx_n,\ gx_{n+1})$. By using contractive condition (3.1), we have

$$\psi(d(gx_{n+1},\ gx_{n+2})) = \psi(d(Tx_n,\ Tx_{n+1})) \leq \varphi(d(gx_n,\ gx_{n+1})), \tag{3.3}$$

which, by the fact that $\psi(t) > \varphi(t)$ for all $t > 0$, implies

$$\psi(d(gx_{n+1},\ gx_{n+2})) < \psi(d(gx_n,\ gx_{n+1})).$$

It follows, from the monotonicity of ψ, that

$$d(gx_{n+1},\ gx_{n+2}) < d(gx_n,\ gx_{n+1}),\ \text{that is, } \zeta_{n+1} < \zeta_n.$$

Hence the sequence $\{\zeta_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = \zeta. \tag{3.4}$$

We claim that $\zeta = 0$. If possible, suppose $\zeta > 0$. Taking $n \to \infty$ in (3.3), by using (3.4) and the property of ψ and φ, we obtain

$$\psi(\zeta) \leq \lim_{n\to\infty} \psi(d(gx_{n+1},\ gx_{n+2})) \leq \lim_{n\to\infty} \varphi(d(gx_n,\ gx_{n+1})) \leq \varphi(\zeta),$$

which contradicts the fact that $\psi(t) > \varphi(t)$ for all $t > 0$. Thus, by (3.4), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = 0. \tag{3.5}$$

Now, we claim that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X. If possible, suppose $\{gx_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and

$$d(gx_{n(k)},\ gx_{m(k)}) \geq \varepsilon,\ \text{for } n(k) > m(k) > k.$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(gx_{n(k)-1},\ gx_{m(k)}) < \varepsilon.$$

By using triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq d(gx_{n(k)},\ gx_{m(k)}) \\ &\leq d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)}) \\ &\leq d(gx_{n(k)},\ gx_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (3.5), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \varepsilon. \tag{3.6}$$

By using triangle inequality, we have

$$\begin{aligned} & d(gx_{n(k)+1},\ gx_{m(k)+1}) \\ \leq\ & d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities and by using (3.5) and (3.6), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \varepsilon. \tag{3.7}$$

As $n(k) > m(k)$ and $gx_{n(k)} \succeq gx_{m(k)}$, so by using contractive condition (3.1), we have

$$\psi(d(gx_{n(k)+1},\ gx_{m(k)+1})) = \psi(d(Tx_{n(k)},\ Tx_{m(k)})) \leq \varphi(d(gx_{n(k)},\ gx_{m(k)})).$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, φ and (3.6), (3.7), we have

$$\psi(\varepsilon) \leq \varphi(\varepsilon),$$

which contradicts the fact that $\varepsilon > 0$. This shows that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X.

Now, we claim that T and g have a coincidence point between cases $(a) - (c)$.

Suppose that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is compatible. Since $(X,\ d)$ is complete, there exists $z \in X$ such that $\{gx_n\} \to z$. It follows, from (3.2), that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Since the pair $(T,\ g)$ is compatible, we conclude that $d(gz, Tz) = \lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$, that is, z is a coincidence point of T and g.

Suppose now (b) holds, that is, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X), d)$, so there exist $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$. Then $\{gx_n\} \to gz$. Now, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we get $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (3.1), we have

$$\psi(d(gx_{n+1},\ Tz)) = \psi(d(Tx_n,\ Tz)) \leq \varphi(d(gx_n,\ gz)).$$

Taking $n \to \infty$ in the above inequality and by using (ii_ψ), we get $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g.

Suppose now that (c) holds, that is, (X, d) is complete, g is continuous and monotone non-decreasing, the pair (T, g) is compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. It follows, from (3.2), that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair (T, g) is compatible and $\{ggx_n\} \to gz$, it means that $\{Tgx_n\} \to gz$.

Also, since $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, therefore we get $gx_n \preceq z$, which, by the monotonicity of g, implies $ggx_n \preceq gz$. Using the contractive condition (3.1), we get

$$\psi(d(Tgx_n,\ Tz)) \leq \varphi(d(ggx_n,\ gz)).$$

Taking $n \to \infty$ in the above inequality and by using (ii_ψ), we get $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g. □

If we take $\psi(t) = t$ and $\varphi(t) = kt$ with $k < 1$ in Theorem 3.1, we get the following result:

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered metric space and $T, g : X \to X$ be two mappings such that T is $(g, \preceq)-$non-decreasing and $T(X) \subseteq g(X)$ satisfying*

$$d(Tx,\ Ty) \leq kd(gx,\ gy),$$

for all $x, y \in X$ with $gx \preceq gy$, where $k < 1$. There exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$. Suppose one of the following conditions $(a) - (c)$ of Theorem 3.1 holds. Then T and g have a coincidence point.

Example 3.1. Let $X = \mathbb{R}$ be a metric space with the metric $d : X^2 \to [0, +\infty)$ defined by $d(x, y) = |x - y|$, for all $x, y \in X$, with the natural ordering of real numbers $\leq$. Let $T, g : X \to X$ be defined as

$$Tx = \frac{x^2}{2} \text{ and } gx = x^2, \text{ for all } x \in X.$$

Clearly the contractive condition of Theorem 3.1 satisfies with $\psi(t) = t$ and $\varphi(t) = t/2$ for all $t \geq 0$. Furthermore, all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a coincidence point of T and g.

4. Multidimensional Coincidence Point Results

Next we give an $n-$dimensional coincidence point theorem for mixed monotone mappings. For brevity, $(y_1, y_2, ..., y_n)$, $(v_1, v_2, ..., v_n)$ and $(y_0^1, y_0^2, ..., y_0^n)$ will be denoted by Y, V and Y_0 respectively.

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings and $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Suppose that the following properties are fulfilled:*

(i) $F(X^n) \subseteq g(X)$,

(ii) F has the mixed g-monotone property,

(iii) there exist $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$,

(iv) there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

$$\psi(d(F(y_1, y_2, ..., y_n), F(v_1, v_2, ..., v_n))) \leq \varphi\left(\max_{1 \leq i \leq n} d(gy_i, gv_i)\right), \tag{4.1}$$

for which $y_i, v_i \in X$ such that $gy_i \preceq_i gv_i$ for all $i \in \Lambda_n$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Also assume that one of the following conditions holds,

(a) (X, d) is complete, F and g are continuous and the pair (F, g) is (O, Υ)-compatible,

(b) $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular,

(c) (X, d) is complete, g is continuous and monotone non-decreasing, the pair (F, g) is (O, Υ)-compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then F and g have a $\Upsilon-$coincidence point.

Proof. For fixed $i \in A$, we have $gy_{\sigma_i(t)} \preceq_t gv_{\sigma_i(t)}$ for $t \in \Lambda_n$. From (4.1), we have

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ \leq\ & \varphi\left(\max_{1\leq i\leq n} d(gy_i,\ gv_i)\right), \end{aligned} \tag{4.2}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $gy_{\sigma_i(t)} \succeq_t gv_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (4.1) that

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ =\ & \psi(d(F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}),\ F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}))) \\ \leq\ & \varphi\left(\max_{1\leq i\leq n} d(gy_i,\ gv_i)\right), \end{aligned} \tag{4.3}$$

for all $i \in B$. By (2.2), (2.3), (2.4), (4.2), (4.3) and by the monotonicity of ψ, we have

$$\psi(\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V))) \leq \varphi(\rho_n(G(Y),\ G(V))),$$

for all $Y,\ V \in X^n$ with $G(Y) \sqsubseteq G(V)$. It is only require to apply Theorem 3.1 for the mappings $T = F_\Upsilon$ and $g = G$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$ and using all items of Lemma 2.1. □

Theorem 4.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings and $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Suppose that the following properties are fulfilled:*

(i) $F(X^n) \subseteq g(X)$,

(ii) F has the mixed g-monotone property,

(iii) there exist $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ verifying $gy_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for all $i \in \Lambda_n$,

(iv) there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that

$$\begin{aligned} & \psi\left(\frac{1}{n}\sum_{i=1}^{n} d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))\right) \\ \leq\ & \varphi\left(\frac{1}{n}\sum_{i=1}^{n} d(gy_i,\ gv_i)\right), \end{aligned} \tag{4.4}$$

for which $y_i,\ v_i \in X$ *such that* $gy_i \preceq_i gv_i$ *for all* $i \in \Lambda_n$*, where* $\psi(t) > \varphi(t)$ *for all* $t > 0$ *and* $\varphi(0) = 0$*. Also assume that one of the following conditions holds,*

(*a*) $(X,\ d)$ *is complete,* F *and* g *are continuous and the pair* $(F,\ g)$ *is* $(O,\ \Upsilon)$*-compatible,*

(*b*) $(g(X),\ d)$ *is complete and* $(X,\ d,\ \preceq)$ *is non-decreasing-regular,*

(*c*) $(X,\ d)$ *is complete,* g *is continuous and monotone non-decreasing, the pair* $(F,\ g)$ *is* $(O,\ \Upsilon)$*-compatible and* $(X,\ d,\ \preceq)$ *is non-decreasing-regular.*

Then F *and* g *have a* $\Upsilon-$*coincidence point.*

Proof. It is straightforward that the contractive condition (4.4) means that

$$\psi(\Delta_n(F_\Upsilon(Y),\ F_\Upsilon(V))) \leq \varphi(\Delta_n(G(Y),\ G(V))),$$

for all $Y,\ V \in X^n$ with $G(Y) \sqsubseteq G(V)$. Thus Theorem 3.1 is applicable for the mappings $T = F_\Upsilon$ and $g = G$ in the ordered metric space $(X^n,\ \Delta_n,\ \sqsubseteq)$ with the help of Lemma 2.1. □

We may also state the results similar to Corollary 3.2 for Theorem 4.1 and Theorem 4.2.

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