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GENERALIZED VECTOR MINTY'S LEMMA

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Abstract. In this paper, the author defines a new generalized (η, δ, α)-pseudomonotone mapping and considers the equivalence of Stampacchia-type vector variational-like inequality problems and Minty-type vector variational-like inequality problems for generalized (η, δ, α)-pseudomonotone mappings in Banach spaces, called the generalized vector Minty's lemma.

1. Introduction

In the last more than 30 years, variational inequalities for numerical functions, which were originated from Hartman and Stampacchia [8], have made much developments in the theory and applications. In particular, Minty's lemma [2, 6, 7, 10, 14, 15, 18] has been shown to be an important tool in the variational field including variational inequality problems, obstacle problems, confined plasmas, free-boundary problems, elasticity problems and stochastic optimal control problems when the operator is monotone and the domain is convex. The classical Minty's inequality and Minty's lemma offered the regularity results of the solution for a generalized nonhomogeneous boundary value problem [2] and, when the operator is a gradient, also a minimum principle for convex optimization problems [6].

On the other hand, vector variational inequality is closely related to vector optimization problem. Giannessi [7] established the equivalence between a differentiable convex vector optimization problem and a vector variational inequality. Lee et al. [15] showed that vector variational inequality can be an efficient tool for studying vector optimization problems. Moreover, using a vector variational-like inequality,

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Lee *et al.* [14] proved existence theorems for solutions of nondifferentiable invex optimization problems. Recently, Banbagallo [3] obtained the continuity of solutions to-time-dependent nonlinear variational and quasi-variational inequalities which express many dynamic equilibrium problems by using Minty's lemma of the notion of the Mosco's convergence.

In 1999, Lee and Lee [11] firstly obtained a vector version of Minty's lemma for set-valued mappings in Banach spaces using Nadler's result [17] and considered the existences of solutions for Stampacchia-type vector varinational-like inequalities and Minty-type vector variational-like inequalities for set-valued mappings in Banach spaces under a certain pseudomonotonicity condition and a certain new hemicontinuity condition.

In 2000, Lee et al. [13] obtained a vector version of Behera and Panda's generalization of Minty's lemma.

In 2002, Lee et al. [12] introduced (η, θ, δ)-pseudomonotone-type set-valued mappings and showed the existence of solutions to Stampacchia-type scalar variational-type inequality problems and Minty-type scalar variational-type inequality problems for (η, θ, δ)-pseudomonotone-type hemicontinuous set-valued mappings in nonreflexive Banach spaces.

In 2006, Bai et al. [1] introduced a relaxed η-α-pseudomonotone single-valued mapping and considered a scalar-type of Minty lemma for relaxed η-α-pseudomonotone mappings in reflexive Banach spaces.

In 2008, Chinaie et al. [5] introduced two kinds of η-f-pseudomonotone-type set-valued mappings and considered two vector versions of Minty's lemma and obtained existences of solutions to three kinds of vector variational-like inequalities for two kinds of η-f-pseudomonotone-type set-valued mappings in normed spaces.

In 2009, Kazmi and Khan [9] considered a generalized system in real Banach spaces and established some existence theorems for generalized system without monotone concept but Brouwer's fixed point theorem. Further they extended the concept of C-strong pseudomonotonicity for extending Minty's lemma for a generalized system.

Inspired by the previous work, we defined a new generalized (η, δ, α)-pseudomonotone mapping and considered the equivalence of Stampacchia-type vector variational-like inequality problems and Minty-type vector variational-like inequality problems for generalized (η, δ, α)-pseudomonotone mappings, called the generalized vector Minty's lemma.

2. Generalized Vector Minty's lemma

Definition 2.1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces and $CP(Y)$ be the collection of all nonempty compact subsets of Y with the Hausdorff metric D induced by $\|\cdot\|_Y$.

Let $T : X \to CP(Y)$ be a set-valued mapping with nonempty compact set values. T is said to be *uniformly continuous* if for any given positive number ε, there exists a positive number δ such that for any $x, y \in X$ with $\|x - y\| < \delta$, $D(Tx, Ty) < \varepsilon$ holds.

Definition 2.2. A mapping $f : X \to Y$ from a vector space X to a vector space Y with a convex cone P in Y is said to be P-*convex* if $f(t \cdot x + (1-t) \cdot y) \leq_P t \cdot f(x) + (1-t) f(y)$ for $x, y \in X$ and $t \in [0,1]$, where $y \leq_P x$ means $x - y \in P$ and $y \nleq_{intP} x$ means $x - y \notin intP$, the interior of P.

Definition 2.3. Let K and C be subsets of real Banach spaces X and Y, respectively. Let $\eta : K \times K \to X$, $\delta : K \times K \to Y$ be mappings and $F : K \times C \to 2^K$, $G : K \to 2^C$ be set-valued mappings. Let $P : K \to 2^Y$ be a proper closed convex cone-valued mapping with a nonempty interior cone. A mapping $T : K \to L(X, Y)$ is said to be *generalized* (η, α, δ)*-pseudomonotone* with respect to F and G, if for $x \in K$ there exist $z \in G(x)$ and $h \in F(x, z)$ such that

$$\langle Th, \eta(y, x)\rangle + \delta(y, x) \nleq_{intP(x)} 0 \quad for \;\; y \in K$$

implies

$$\langle Tk, \eta(y, x)\rangle + \delta(y, x) - \alpha(x, y) \nleq_{intP(x)} 0$$

for $x, y \in K,\ w \in G(y)$ and $k \in F(y, w)$
where $\alpha : K \times K \to Y$ is a mapping such that $\lim_{t \to 0^+} \frac{\alpha(x, ty + (1-t)x)}{t} = 0$.

Letting $Y = \mathbb{R}, \delta \equiv 0, P(x) = \mathbb{R}_+$ for $x \in K$ and defining $G : K \to 2^C$, $F : K \times C \to 2^K$ by $G(x) = \{x\}$, $F(x, y) = \{x\}$ for $x \in K$ and a function $\beta : K \to \mathbb{R}$ by, for each $x \in K$. $\beta(x) = \alpha(y, y + x)$ for $y \in K$ with $\beta(\lambda x) = \lambda^p \beta(x)$ for $\lambda > 0$ and $p > 1$ in Definition 2.3, we have the following definition of the relaxed η-β-pseudomonotonicity.

Definition 2.4 ([1])**.** Let $\eta : K \times K \to X$ be a mapping and $\beta : K \to \mathbb{R}$ be a function such that $\beta(tx) = t^p \cdot \beta(x)$ for $t > 0$, $x \in K$ and $p > 1$.
A mapping $T : K \to X^*$ is said to be *relaxed* η-β*-pseudomonotone* if for any $x, y \in K$, $\langle Ty, \eta(x, y)\rangle \geq 0$ implies $\langle Tx, \eta(x, y)\rangle \geq \beta(x - y)$.

Lemma 2.1 ([17]). *Let $(X, ||\cdot||)$ be a normed vector space and D be a Hausdorff metric on the collection $C(X)$ of all closed and bounded subsets of X, induced by a metric d in terms of $d(x,y) = ||x-y||$, which is defined by*

$$D(A,B) = \max\left(\sup_{x\in A}\inf_{y\in B}||x-y||, \sup_{y\in B}\inf_{x\in A}||x-y||\right)$$

for A and B in $C(X)$. If A and B are compact sets in X, then for each $x \in A$, there exists $y \in B$ such that

$$||x-y|| \leq D(A,B).$$

Lemma 2.2 ([4]). *Let (X,P) be an ordered Banach space with a closed convex pointed cone whose interior is nonempty. Then for $x,y,z \in X$, we have the following:*

$$if\ \ z \nleq_{intP} x \text{ and } y \leq_P x,\ then\ \ z \nleq_{intP} y.$$

Lemma 2.3 ([16]). *Let K and C be nonempty subsets of real topological vector spaces X and Y, respectively and Z a real topological vector space. Let $F : K \times C \to 2^Z$ and $G : K \to 2^C$ be set-valued mappings. If F and G are upper semi-continuous and compact-valued, then a set-valued mapping $H : K \to 2^Z$ defined by $H(x) = \bigcup_{z\in G(x)} F(x,z) = F(x, G(x))$ is also upper semi-continuous and compact-valued.*

The classical Minty's lemma is stated as following:

Theorem 2.1. *Let X be a real reflexive Banach space and K a nonempty closed convex subset of X with the dual X^*. Let $T : K \to X^*$ be a monotone and hemicontinuous mapping. Then the following are equivalent:*

(a) (*Stampacchia-type*) *there exists an $x_0 \in K$ such that*

$$\langle Tx_0, y - x_0\rangle \geq 0 \quad \text{for all} \quad y \in K.$$

(b) (*Minty-type*) *there exists an $x_0 \in K$ such that*

$$\langle Ty, y - x_0\rangle \geq 0 \quad \text{for all} \quad y \in K.$$

Now we consider our generalized vector Minty's lemma.

Theorem 2.2. *Let X and Y be Banach spaces and K be a nonempty closed and convex subset of X. Let $\eta : K \times K \to X$ and $\delta : K \times K \to Y$ be mappings. Let $P : K \to 2^Y$ be a proper closed convex cone-valued mapping with a nonempty interior cone and $B = \bigcap_{x\in K} P(x)$. Let $F : K \times C \to 2^K$ and $G : K \to 2^C$ be compact*

set-valued mappings, where C is a given subset of Y. Let $H : K \to 2^{L(X,Y)}$ be a nonempty set-valued mapping defined by $H(x) = \bigcup_{z\in G(x)} F(x,z) = F(x, G(x))$ for $x \in K$. Let $T : K \to L(X,Y)$ be a mapping and $g : K \to Y$ be a mapping defined by $g(x) = \langle Th, \eta(x,y)\rangle$ for $h, x, y \in K$.
Suppose that the following conditions hold;

(i) *$\eta(x,y) + \eta(y,x) = 0$ and $\delta(x,y) + \delta(y,x) = 0$ for all $x, y \in K$.*
(ii) *g is B-convex and δ is B-convex in the first argument.*
(iii) *F and G are upper semi-continuous and H is D-uniformly continuous.*

If T is continuous and generalized (η,δ,α)-pseudomonotone with respect to F and G, then the following vector variational-like problems are equivalent:

(A) *(Stampacchia-type) There exist $x_0 \in K$, $z_0 \in G(x_0)$ and $k_0 \in F(x_0, z_0)$ such that*

$$\langle Tk_0, \eta(y,x_0)\rangle + \delta(y,x_0) \not\leq_{intP(x_0)} 0 \text{ for } y \in K. \tag{2.1}$$

(B) *(Minty-type) There exists $x_0 \in K$ such that*

$$\langle Th, \eta(y,x_0)\rangle + \delta(y,x_0) - \alpha(x_0,y) \leq_{intP(x_0)} 0 \text{ for } y \in K,\ z \in G(y),\ h \in F(y,z). \tag{2.2}$$

Proof. Since the sufficiency is clear from the definition of T, we show the necessity. Suppose that the vector variational-like inequality problem (2.2) holds. For $y \in K$, letting $y_t = ty + (1-t)x_0$ for $t \in (0,1)$, we have $y_t \in K$. Thus for $h_t \in H(y_t) = F(y_t, G(y_t))$,

$$\langle Th_t, \eta(y_t,x_0)\rangle + \delta(y_t,x_0) - \alpha(x_0,y_t) \not\leq_{intP(x_0)} 0. \tag{2.3}$$

From the B-convexity of δ in the first argument and condition (i), we have

$$\begin{aligned} \delta(y_t,x_0) &\leq_P t\cdot\delta(y,x_0) + (1-t)\delta(x_0,x_0) \\ &= t\cdot\delta(y,x_0) \end{aligned} \tag{2.4}$$

Also, from the B- convexity of g and condition (i), we have

$$\begin{aligned} \langle Th_t, \eta(y_t,x_0)\rangle &= \langle Th_t, \eta(ty+(1-t)x_0, x_0)\rangle \\ &\leq_P t\cdot\langle Th_t, \eta(y,x_0)\rangle + (1-t)\cdot\langle Th_t, \eta(x_0,x_0)\rangle \\ &= t\cdot\langle Th_t, \eta(y,x_0)\rangle. \end{aligned} \tag{2.5}$$

By using Lemma 2.2, from (2.3), (2.4) and (2.5),
we have

$$\langle Th_t, \eta(y, x_0)\rangle + \delta(y, x_0) - \frac{\alpha(x_0, y_t)}{t} \nleq_{intP(x_0)} 0. \tag{2.6}$$

On the other hand, by Lemma 2.3, $H(y_t)$ and $H(x_0)$ are compact and by Lemma 2.1, for each $h_t \in H(y_t)$, we can choose $k_t \in H(x_0)$ satisfying

$$\|h_t - k_t\| \leq D(H(y_t), H(x_0)).$$

Since the net $\langle k_t\rangle$ has a convergent subnet $\langle k_{t_s}\rangle$ by the compactness of $H(x_0)$, we assume the subnet $\langle k_{t_s}\rangle$ as a net $\langle k_t\rangle$ converging to a limit $k_0 \in H(x_0)$ as $t \to 0^+$.

Since $\|y_t - x_0\| = t \cdot \|y - x_0\| \to 0$ as $t \to 0^+$, by the D-uniform continuity of H, $D(H(y_t), H(x_0)) \to 0$ as $t \to 0^+$.

Thus

$$\|h_t - k_0\| \leq \|h_t - k_t\| + \|k_t - k_0\| \leq D(H(y_t), H(x_0)) + \|k_t - k_0\| \to 0 \ \ as \ \ t \to 0^+.$$

So by the continuity of T, we have

$$\begin{aligned} \|\langle Th_t, \eta(y, x_0)\rangle - \langle Tk_0, \eta(y, x_0)\rangle\| &= \|\langle Th_t - Tk_0, \eta(y, x_0)\rangle\| \\ &\leq \|Th_t - Tk_0\|\|\eta(y, x_0)\| \\ &\to 0 \ as \ t \to 0^+ \end{aligned}$$

Since $Y \setminus (-intP(x_0))$ is closed and $\lim\limits_{t\to 0^+} \frac{\alpha(x_0,y_t)}{t} = 0$, from (2.6) we have

$$\langle Tk_0, \eta(y, x_0)\rangle + \delta(y, x_0) \in Y \setminus (-intP(x_0)).$$

Since $k_0 \in H(x_0) = \bigcup\limits_{z\in G(x_0)} F(x_0, z) = F(x_0, G(x_0))$, for some $z_0 \in G(x_0)$, $k_0 \in F(x_0, z_0)$. Consequently, we have for some $x_0 \in K$, $z_0 \in G(x_0)$, and $k_0 \in F(x_0, z_0)$,

$$\langle Tk_0, \eta(y, x_0)\rangle + \delta(y, x_0) \nleq_{intP(x_0)} 0 \ for \ y \in K.$$

□

We obtain the following scalar Minty's type lemma as a corollary.

Corollary 2.1 ([1, Theorem 3.1])**.** *Let K be a nonempty closed convex subset of a real reflexive Banach space X. Let $\eta : K \times K \to X$ be a mapping and $\alpha : K \to \mathbb{R}$ be a function with $\alpha(tz) = t^p\alpha(z)$ for $t > 0, z \in K$ and $p > 1$. Let $T : K \to X^*$ be an $\eta-$hemicontinuous and relaxed η-α-pseudomonotone mapping.*

Assume that

(a) *$\eta(x, x) = 0$ for all $x \in K$*
(b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex.*

Then the followings are equivalent :

(i) (*Stampacchia-type*) *there exists a* $x \in K$ *satisfying* $\langle Tx, \eta(y,x)\rangle \geq 0$ *for* $y \in K$,

(ii) (*Minty-type*) *there exists a* $x \in K$ *satisfying* $\langle Ty, \eta(y,x)\rangle \geq \alpha(y-x)$ *for* $y \in K$.

Remark. Our result also extends and improves some results in [11, 13].

References

1. M.R. Bai, S.Z. Zhoua & G.Y. Nib: Variational-like inequalities with relaxed η-α-pseudomonotone mappings in Banach spaces *Appl. Math. Lett.* **19** (2006), 547-554.
2. C. Baiocchi & A. Capelo: *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems.* John Wiley and Sons Ltd., New York (1984).
3. A. Barbagallo: Regularity results for evolutionary nonlinear variational and quasi-variational inequalities with applicaions to dynamic equilibrium problems. *J. Glob. Optim.* **40** (2008), 29-39: DOI/O. 1007/s 10898-007-9194-5.
4. G.Y. Chen & X.Q. Yang: The vector complementarity problem and its equivalence with the minimal element. *J. Math. Anal. Appl.* **153** (1990), 136-158.
5. M. Chinaie, T. Jabarootian, M. Rezaie & J. Zafarani: Minty's lemma and vector variational-like inequalities. *J. Glob. Optim.* **40** (2008), 463-473: DOI 10. 1007/s 10898-007-9177-6.
6. F. Giannessi: On connections among separation, penalization and regularization for variational inequalities with point-to-set-operators. *Rendiconti del Circolo Matematico di Palermo, Series* II, *Suppl.* **48** (1997), 11-18.
7. ______: *On Minty variational principle, In New Trends in Mathematical Programming.* Kluwer Academic Publishes, Dordrecht (1997).
8. P. Hartman & G. Stampacchia: On some nonlinear elliptic differential functional equations. *Acta Math.* **115** (1996), 271-310.
9. K.R. Kazmi & S.A. Khan: Existence of solutions to a generalized system. *J. Optim. Theory Appl.* **142** (2009), 355-361, DOI 10. 1007/s 10957-009-9530-7.
10. D. Kinderlehrer & G. Stampacchia. *An introduction to Variational Inequalities.* Academic Press, New York (1980).
11. B.-S. Lee & G.M. Lee: A vector version of Minty's lemma and application. *Appl. Math. Lett.* **12** (1999), 43-50.
12. B.-S. Lee, G.M. Lee & S.-J. Lee: Variational-type inequalities for (η, θ, δ)-pseudo-monotone-type set-valued mappings in nonreflexive Banach spaces. *Appl. Math. Lett.* **15** (2002), 109-114.

13. B.-S. Lee & S.-J. Lee: A vector extension of Bechera and Panda's generalization of Minty's lemma. *Indian. J. Pure Appl. Math.* **31** (2000), no. 11, 1483-1489.
14. G.M. Lee, D.S. Kim & H. Kuk: Existence of solutions for vector optimization problems. *J. Math. Anal. Appl.* **220** (1998), 90-98.
15. G.M. Lee, D.S. Kim, B.S. Lee & N.D. Yen: Vector variational inequality as a tool for studying vector optimization problems. *Nonlinear Anal. Th. Meth. Appl.* **34** (1998), 745-765.
16. L.-J. Lin & Z.-T. Yu: On some equilibrium problems for multimaps. *J. Comp. Appl. Math.* **129** (2001), 171-183.
17. S.B. Nadler Jr.: Multi-valued contraction mappings. *Pacific J. Math.* **30** (1969), no. 2, 475-488.
18. G. Stampacchia: *Variational Inequalities. In : A. Ghizzetti* (ed.), *Theory and Applications of Monotone Operators.* PP. 101-192. Proc. NATO ADV. Study Institute, Venice (1968), Oderisi Gubbio (1969).

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