J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 10, Number 2 (May 2003), Pages 127–132

ON INVARIANT APPROXIMATION OF NON-EXPANSIVE MAPPINGS

MEENU SHARMA AND T. D. NARANG

ABSTRACT. The object of this paper is to extend and generalize the work of Brosowski [Fixpunktsatze in der approximationstheorie. *Mathematica Cluj* **11** (1969), 195–200], Hicks & Humphries [A note on fixed point theorems. *J. Approx. Theory* **34** (1982), 221–225], Khan & Khan [An extension of Brosowski-Meinardus theorem on invariant approximation. *Approx. Theory Appl.* **11** (1995), 1–5] and Singh [An application of a fixed point theorem to approximation theory. *J. Approx. Theory* **25** (1979), 89–90; Application of fixed point theorem in approximation theory. In: *Applied nonlinear analysis* (pp. 389–394). Academic Press, 1979] in metric spaces having convex structure, and in metric linear spaces having strictly monotone metric.

Many results in approximation theory using fixed point theorems are known in the literature. Meinardus [6] and Brosowski [2] established some interesting results on invariant approximation in normed linear spaces in terms of fixed point theory. Later various researchers obtained generalizations of their results (*e.g.*, Khan & Khan [5] and the references cited therein). The object of this paper is to extend and generalize the work of Brosowski [2], Hicks & Humphries [4], Khan & Khan [5] and Singh [8, 9] in metric spaces having convex structure (a notion introduced by Takahashi [10]) and in metric linear spaces having strictly monotone metric (a notion introduced by Guseman & Peters [3]).

To start with we recall a few definitions.

Definition 1. Let (X, d) be a metric space and C a subset of X. A mapping $T : C \to X$ is said to be *non-expansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. The set $F(T) = \{x \in X : T(x) = x\}$ is called the *fixed point set* of a mapping T and a point of $F(T)$ is called a T*-invariant point* in X.

Received by the editors August 26, 2002 and, in revised form, March 18, 2003.
2000 *Mathematics Subject Classification.* 41A65 (46B20).
Key words and phrases. starshape, starshapedness.
The first author is thankful to U. G. C., India for awarding the Teacher-Fellowship.

Definition 2. For a metric space (X, d) and a non-empty subset C of X, an element $c_x \in C$ is called a *best approximant to x* from C if $d(x, c_x) \leq d(x, c)$ for every $c \in C$ *i. e.*, $d(x, c_x) = \inf_{c \in C} d(x, c) \equiv d(x, C)$.

The set of all best approximants to $x \in X$ from C is denoted by $P_C(x)$.

Definition 3 (Takahashi [10]). For a metric space (X, d) and a closed unit interval $I = [0, 1]$, a continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if, for all $x, y \in X, \lambda \in I$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure W is called a *convex metric space*.

Clearly a Banach space or any convex subset of it is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$. More generally, if X is a linear space with a translation invariant metric d satisfying

$$d(\lambda x + (1 - \lambda) y, 0) \leq \lambda d(x, 0) + (1 - \lambda) d(y, 0),$$

then X is a convex metric space. Takahashi [10] has shown that there are many convex metric spaces which can not be embedded in any normed linear space.

Definition 4 (Takahashi [10]).

(i) A non-empty subset K of a convex metric space (X, d) is said to be *starshaped* if it is starshaped with respect to one of its elements *i. e.*, if there exists a $u \in K$ such that $W(x, u, \lambda) \in K$ for every $0 \leq \lambda \leq 1$ and $x \in K$. Such a u is called a *starcentre* of K.

(ii) A non-empty subset K of a convex metric space (X, d) is said to be *convex* if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Clearly, a convex set is starshaped with respect to each of its points.

Definition 5 (Narang [7]). A convex metric space (X, d) is said to be *strictly convex* if for every $x, y \in X$ and $r > 0$, $d(x, p) \leq r$, $d(y, p) \leq r$ imply $d(W(x, y, \lambda), p) < r$ unless $x = y$, where p is arbitrary but fixed point of X.

Definition 6. A subset S of a linear space X is *starshaped* if there exists u in S such that $\lambda x + (1 - \lambda) u \in S$ whenever $\lambda \in [0, 1]$ and $x \in S$.

Definition 7. Let (X, d) be a metric linear space. The metric d for X is said to be *strictly monotone* (*cf.* Guseman & Peters [3]) if $x \neq 0$ and $0 \leq \lambda < 1$ imply $d(\lambda x, 0) < d(x, 0)$.

Definition 8. A metric linear space (X, d) is said to satisfy the $*$*-convex property* if

$$(*) \qquad d(\lambda x + (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$$

for every $x, y, z \in X$ and $0 \leq \lambda \leq 1$.

Remark. Clearly, every normed linear space satisfies the property $(*)$.

In metric linear spaces satisfying the property $(*)$, we have the following lemma which we shall be using in the proof of Theorem 1.

Lemma 1. *Let (X, d) be a metric linear space satisfying the property $(*)$, C a subset of X and $x \in X$. Then $P_C(x) \subset \partial C \cap C$, where ∂C is the boundary of C.*

Proof. Let $y \in P_C(x)$. For each $n \in \mathbb{N}$, let $\lambda_n = n/(n+1)$. Since

$$d(y, \lambda_n y + (1 - \lambda_n)x) \leq (1 - \lambda_n)d(x, y)$$

for all $n \in \mathbb{N}$, $\lim_{n\to\infty}[\lambda_n y + (1 - \lambda_n)x] = y$.

So each neighborhood of y contains at least one $\lambda_n y + (1 - \lambda_n)x$. Also,

$$d(x, \lambda_n y + (1 - \lambda_n)x) \leq \lambda_n d(y, x) < d(y, x) \text{ for all } n \in \mathbb{N}$$

implies that $\lambda_n y + (1 - \lambda_n)x \notin C$ for any $n \in \mathbb{N}$ *i. e.*, y is not an interior point of C and so $y \in \partial C$. Also $y \in P_C(x)$ implies $y \in C$. Thus, $y \in \partial C \cap C$ and hence $P_C(x) \subset \partial C \cap C$. □

Using Lemma 1, we prove the following result on invariant approximation in metric linear spaces.

Theorem 1. *Let (X, d) be a metric linear space which satisfies the property $(*)$ with strictly monotone metric d and C a subset of X. Let T be a non-expansive mapping on $P_C(x) \cup \{x\}$ where x is a T-invariant point. Then there is a $x_0 \in P_C(x)$ which is also a T-invariant point provided*

(a) $T : \partial C \to C$

(b) *$P_C(x)$ is nonempty, starshaped and compact.*

Proof. Let p be a starcentre of $P_C(x)$. Then $\lambda y + (1-\lambda)p \in P_C(x)$ for every $y \in P_C(x)$ and $0 \le \lambda \le 1$. We claim that $T : P_C(x) \to P_C(x)$.

Suppose (X, d) satisfies the property $(*)$. Then by the Lemma, $P_C(x) \subset \partial C \cap C$. So for $y \in P_C(x)$, we get $Ty \in C$ as $T : \partial C \to C$. Consider

$$\begin{aligned} d(x, Ty) &= d(Tx, Ty) \qquad \text{(as } x \text{ is a } T\text{-invariant point)} \\ &\le d(x, y) \qquad \text{(as } T \text{ is non-expansive on } P_C(x) \cup \{x\}) \\ &= d(x, C) \\ &\le d(x, Ty). \end{aligned}$$

This gives $d(x, Ty) = d(x, C)$ *i. e.*, $Ty \in P_C(x)$ for $y \in P_C(x)$ and hence $T : P_C(x) \to P_C(x)$.

Let κ_n, $0 \le \kappa_n < 1$ and $n \in \mathbb{N}$ be a sequence of real numbers such that $\kappa_n \to 1$ as $n \to \infty$. Define $T_n : P_C(x) \to P_C(x)$ by $T_n y = \kappa_n Ty + (1-\kappa_n)p$ for every $y \in P_C(x)$. Then we get

$$\begin{aligned} d(T_n x, T_n y) &= d(\kappa_n Tx + (1-\kappa_n)p, \kappa_n Ty + (1-\kappa_n)p) \\ &= d(\kappa_n Tx, \kappa_n Ty) \\ &= d(\kappa_n(Tx - Ty), 0) \\ &< d(Tx - Ty, 0) \qquad \text{(as } d \text{ is strictly monotone)} \\ &= d(Tx, Ty) \\ &\le d(x, y). \qquad \text{(as } T \text{ is non-expansive on } P_C(x) \cup \{x\}) \end{aligned}$$

Hence T_n is non-expansive on $P_C(x) \cup \{x\}$ for each $n \in \mathbb{N}$. Since $P_C(x)$ is compact and starshaped, T_n has a unique fixed point, say, x_n for each $n \in \mathbb{N}$ (*cf.* Guseman & Peters [3, Theorem 2]), *i. e.*, $T_n x_n = x_n$ for each n.

Since $P_C(x)$ is compact, $\langle x_n \rangle$ has a convergent subsequence $\langle x_{n_i} \rangle \to x_0 \in P_C(x)$. We claim that $Tx_0 = x_0$.

Consider $x_{n_i} = T_{n_i} x_{n_i} = \kappa_{n_i} T x_{n_i} + (1-\kappa_{n_i})p$. Taking limit as $n_i \to \infty$, we get $x_0 = Tx_0$ *i. e.*, $x_0 \in P_C(x)$ is a T-invariant point. □

Since every normed linear space is a metric linear space with the property $(*)$ and the metric induced by the norm is strictly monotone, we have:

Corollary 1 (Brosowski [2]). *Let T be a non-expansive linear operator on a normed linear space X. Let C be a T-invariant subset of X and x a Tinvariant point. If the* *set of best C-approximants to x is non-empty, compact and convex, then it contains a T-invariant point.*

Corollary 2 (Guseman & Peters [3]). *Let T be a non-expansive mapping on a normed linear space X. Let C be a T-invariant subset of X and x_0 a T-invariant point in X. If the set D of best C-approximants to x_0 is non-empty, compact and starshaped, then it contains a T-invariant point.*

Corollary 3 (Hicks & Humphries [4]). *Let X be a normed linear space and $T : X \to X$ a mapping. Let C be a subset of X such that C is T-invariant and let x_0 be a T-invariant point in X. If D, the set of best C-approximants to x_0 is non-empty, compact and starshaped and T is*

(i) *continuous on D*

(ii) $\|x-y\| \leq d(x_0, C) \Rightarrow \|Tx - Ty\| \leq \|x-y\|$ *for* $x, y \in D \cup \{x\}$,

then it contains a T-invariant point which is a best approximant to x_0 in C.

Note. The continuity of T on D follows from (ii)

Corollary 4 (Khan & Khan [5]). *Let T be a non-expansive operator on a normed linear space X. Let C be a subset of X and x a T-invariant point. There is a y in $P_C(x)$, which is also a T-invariant point, provided*

(a) $T : C \to C$

(b) *$P_C(x)$ is non-empty, compact and convex.*

Since each p-norm generates a translation invariant metric d satisfying the property $(*)$ and is strictly monotone, we have:

Corollary 5 (Meinardus [6]). *Let $(E, \| \|_P)$ be a p-normed linear space, $T : E \to E$ a non-expansive mapping with a fixed point $u \in E$ and C a closed T-invariant subset of E such that T is compact on C. If $P_C(u)$ is starshaped, then there exists an element in $P_C(u)$ which is also a fixed point of T.*

In strictly convex metric spaces, we have the following result on invariant approximation:

Theorem 2. *Let (X, d) be a strictly convex metric space and T a non-expansive mapping on $P_C(x) \cup \{x\}$ where x is a T-invariant point. If C is a subset of X, $T : \partial C \to C$ and $P_C(x)$ is non-empty and starshaped with starcentre q, then $P_C(x) = \{q\}$ with $Tq = q$.*

Proof. Let $p \neq q \in P_C(x)$. Then $d(x,p) = d(x,q) = d(x,C)$. Since $p \neq q$, strict convexity of the space implies $d(x, W(p,q,\lambda)) < \text{dist}(x,C)$ and so $W(p,q,\lambda) \notin P_C(x)$, $0 \leq \lambda \leq 1$. Starshapedness of $P_C(x)$ therefore implies $p = q$, *i. e.*, $P_C(x) = \{q\}$. Since $T : \partial C \to C$ and X is convex, $T : P_C(x) \to P_C(x)$ (*cf.* Al-Thagafi [1, Lemma 3.2]). Hence $Tq \in P_C(x) = \{q\}$, *i. e.*, $Tq = q$. □

REFERENCES

1. M. A. Al-Thagafi: Best approximation and fixed points in strong M-starshaped metric spaces. *Internat. J. Math. Math. Sci.* **18** (1995), no. 3, 613–616. MR **96f**:47111
2. B. Brosowski: Fixpunktsätze in der Approximationstheorie. *Mathematica (Cluj)* **11(34)** (1969), 195–220. MR **43**#3712
3. L. F. Guseman, Jr. & B. C. Peters, Jr.: Nonexpansive mappings on compact subsets of metric linear spaces. *Proc. Amer. Math. Soc.* **47** (1975), 383–386. MR **50**#5558
4. T. L. Hicks & M. D. Humphries: A note on fixed-point theorems. *J. Approx. Theory* **34** (1982), no. 3, 221–225. MR **83k**:47041
5. L. A. Khan & A. R. Khan: An extension of Brosowski-Meinardus theorem on invariant approximation. *Approx. Theory Appl. (N. S.)* **11** (1995), no. 4, 1–5. MR **97h**:41065
6. G. Meinardus: Invarianz bei linearen Approximationen. *Arch. Rational Mech. Anal.* **14** (1963), 301–303. MR **27**#6075
7. T. D. Narang: Best approximation and strict convexity of metric spaces. *Arch. Math. (Brno)* **17** (1981), no. 2, 87–90. MR **84f**:41037
8. S. P. Singh: An application of a fixed-point theorem to approximation theory. *J. Approx. Theory* **25** (1979), no. 1, 89–90. MR **80g**:41026
9. ______: Application of fixed point theorems in approximation theory. In: V. Lakshmikantham (Ed.), *Applied nonlinear analysis* (pp. 389–394). Academic Press, New York-London, 1979. MR **80j**:41043
10. W. Takahashi: A convexity in metric space and nonexpansive mappings, I. *Kodai Math. Sem. Rep.* **22** (1970), 142–149. MR **42**#2467

(M. SHARMA) DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, AMRITSAR 143005, INDIA
Email address: `msharma62@yahoo.co.in`

(T. D. NARANG) DEPARTMENT OF MATHEMATICS, GURU NANAK DEV UNIVERSITY, AMRITSAR 143005, INDIA
Email address: `address:tdnarang1948@yahoo.co.in`