J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 13, Number 1 (February 2006), Pages 1–10

UNSTEADY HARTMANN FLOW WITH HEAT TRANSFER IN THE PRESENCE OF UNIFORM SUCTION AND INJECTION

HAZEM A. ATTIA

ABSTRACT. The unsteady Hartmann flow of an electrically conducting, viscous, incompressible fluid bounded by two parallel non-conducting porous plates is studied with heat transfer. An external uniform magnetic field and a uniform suction and injection are applied perpendicular to the plates while the fluid motion is subjected to a constant pressure gradient. The two plates are kept at different but constant temperatures while the Joule and viscous dissipations are included in the energy equation. The effect of the magnetic field and the uniform suction and injection on both the velocity and temperature distributions is examined.

1. INTRODUCTION

Magnetohydrodynamic flow between two parallel plates, known as Hartmann flow, is a classical problem that has many applications in magnetohydrodynamic (MHD) power generators, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, the petroleum industry, purification of crude oil and fluid droplets and sprays. Hartmann and Lazarus [1] studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary, and insulated plates. Since then, considerable research has been done that examined the effect of various physical processes on Hartmann flow [2-12].

In the present study, we study the unsteady flow and heat transfer of an incompressible, viscous, electrically conducting fluid between two infinite non-conducting horizontal porous plates. The fluid is acted upon by a constant pressure gradient, a uniform suction and injection and a uniform magnetic field perpendicular to the plates. The induced magnetic field is neglected by assuming a very small magnetic

Received by the editors April 18, 2005. Revised February 28, 2006.
2000 *Mathematics Subject Classification.* 78XX .
Key words and phrases. Hartmann flow, heat transfer, hydromagnetic, numerical solution .

$\vec{v}(y,t) = u(y,t)\vec{i} + v_o\vec{j}$

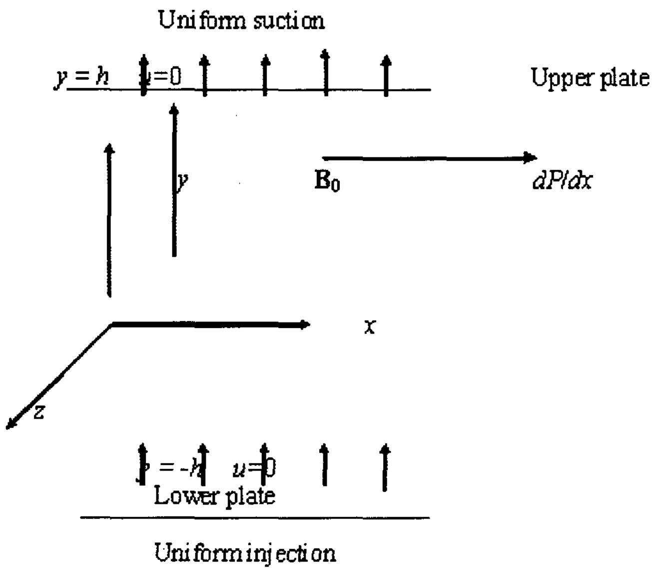


Fig. 1. The geometry of the problem.

Reynolds number [4, 5]. The two plates are maintained at two different but constant temperatures. This configuration approximates well several practical situations such as heat exchangers, flow meters, and pipes that connect system components. The cooling of these devices can be achieved by utilizing a porous surface through which a coolant, either a liquid or gas, is forced. Therefore, the results obtained here are important for the design of the wall and the cooling arrangements of these devices. The equations of motion are solved analytically using Laplace transforms method while the energy equation is solved numerically and includes the Joule and the viscous dissipations. The effects of the magnetic field and the suction and injection on both the velocity and temperature distributions are studied. In the following section, a detailed description is given to the problem and the governing equations for the velocity and temperature fields are derived. Then, the analytical solution for the velocity problem and the numerical solution for the temperature problem are obtained. Discussion for some selected results for the velocity and temperature distributions is given.

2. Description of the Problem

The two non-conducting plates of infinite extent are located at the $y = h$ planes, as shown in Fig. 1. The lower and upper plates are kept at the two constant temperatures T_1 and T_2, respectively, where $T_2 > T_1$. The fluid flows between the two plates under the influence of a constant pressure gradient dP/dx in the x-direction, and a uniform suction from above and injection from below starting at $t = 0$. The whole system is subjected to a uniform magnetic field B_o in the positive y-direction. This is the total magnetic field acting on the fluid since the induced magnetic field is neglected. From the geometry of the problem, it is evident that $\partial/\partial x = \partial/\partial z = 0$ for all quantities apart from the pressure gradient dP/dx, which is assumed constant. The velocity vector of the fluid is Fig. 1 with the initial and boundary conditions $u(y.0) = 0$, and $u(h,t) = 0$ for $t > 0$. The temperature $T(y,t)$ at any point in the fluid satisfies both the initial and boundary conditions $T(y,0) = T_1$, $T(+h,t) = T_2$ and $T(-h,t) = T_1$ for $t > 0$. The fluid flow is governed by the momentum equation

$$\rho\frac{\partial u}{\partial t} + \rho\nu_o\frac{\partial u}{\partial y} = -\frac{dP}{dx} + \mu\frac{\partial^2 u}{\partial y^2} - \sigma B_o^2 u \tag{1}$$

where ρ, μ and σ are the constant density, the constant coefficient of viscosity and the constant electrical conductivity of the fluid, respectively. To find the temperature distribution inside the fluid we use the energy equation [13]

$$\rho c\frac{\partial T}{\partial t} + \rho c\nu_o\frac{\partial T}{\partial y} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \sigma B_o^2 u^2 \tag{2}$$

where c and k are the specific heat capacity and the thermal conductivity of the fluid, respectively. The second and third terms on the right side represent the viscous and Joule dissipations, respectively. The problem is simplified by writing the equations in non-dimensional form. The characteristic length is taken to be h, and the characteristic time is $\rho h^2/\mu^2$ while the characteristic velocity is $\mu/\rho h$. We define the following non-dimensional quantities:

$$\hat{x} = \frac{x}{h},\ \hat{y} = \frac{y}{h},\ \hat{z} = \frac{z}{h},\ \hat{u} = \frac{\rho h u}{\mu},\ \hat{P} = \frac{P\rho h^2}{\mu^2},\ \hat{t} = \frac{tu}{\rho h^2},\ \hat{T} = \frac{T - T_1}{T_2 - T_1},$$

$S = \frac{\rho\nu_o h}{\mu}$ is the suction parameter, $Pr = \frac{\mu c}{k}$ is the Prandtl number, $Ec = \frac{\mu^2}{\rho^2 c h^2 (T_2 - T_1)}$ is the Eckert number, $Ha^2 = \frac{\sigma B_o^2 h^2}{\mu}$ where Ha is the Hartmann number, In terms of these non-dimensional variables and parameters, the basic Eqs. (1)-(2) are written as (the "hats" will be dropped for convenience)

$$\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{\partial^2 u}{\partial y^2} - Ha^2 u \tag{3}$$

and

$$\frac{\partial T}{\partial t} + S\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2} + Ec\left(\frac{\partial u}{\partial y}\right)^2 + EcHa^2u^2. \tag{4}$$

The initial and boundary conditions for the velocity become

$$u(y,0) = 0,\ u(\pm 1, t) = 0, \tag{5}$$

and the initial and boundary conditions for the temperature are given by

$$T(y,0) = 0,\ T(+1,t) = 1,\ T(-1,t) = 0. \tag{6}$$

3. Analytical Solution of the Equations of Motion

Equation (3) is a linear nonhomogeneous partial differential equation which can be solved analytically using Laplace transforms (LT) subject to the initial and boundary conditions given by Eq. (5). This solution gives the velocity field as functions of space and time. Taking the LT of Eq. (3), we find

$$\frac{d^2U(y,s)}{dy^2} - S\frac{dU(y,s)}{dy} - K(s)U(y,s) = -\frac{C}{s} \tag{7}$$

where $U(y,s) = L(u(y,t))$, C is the constant value of $-dP/dx$, and $K(s) = Ha^2 + s$. The solution of Eq. (7) with y as an independent variable is

$$U(y,s) = \frac{C}{Ks}\left(1 + \exp(Sy/2)\left[\frac{\sinh(S/2)\sinh(\alpha y)}{\sinh(\alpha)} - \frac{\cosh(S/2)\cosh(\alpha y)}{\cosh(\alpha)}\right]\right)$$

where $\alpha^2 = S^2/4 + K$. Using the complex inversion formula and the residue theorem [14], the inverse transform of $U(y,s)$ is determined as

$$\begin{aligned} U(y,t) &= \frac{C}{Ha^2}\left(1 + \exp(Sy/2)\left[\frac{\sinh(S/2)\sinh(\alpha_1 y)}{\sinh(\alpha_1)} - \frac{\cosh(S/2)\cosh(\alpha_1 y)}{\cosh(\alpha_1)}\right]\right) \\ &+ 2\pi C\exp(Sy/2)\sum_{n=1}^{\infty}(-1)^n\left[(n-1)\frac{\exp(Dn_1 t)}{(Dn_1 + Ha^2)Dn_1}\sinh(S/2)\sin(\pi(n-1)y)\right. \\ &\left.+ (n-0.5)\frac{\exp(Dn_2 t)}{(Dn_2 + Ha^2)Dn_2}\cosh(S/2)\cos(\pi(n-0.5)y)\right] \end{aligned} \tag{8}$$

where,

$$Dn_1 = -[\pi^2(n-1)^2 + S^2/4 + Ha^2],$$
$$Dn_2 = -[\pi^2(n-0.5)^2 + S^2/4 + Ha^2],\ \alpha_1 = S^2/4 + Ha^2.$$

The location and nature of the poles are listed as follows:

(a) Simple pole at $s = 0$, $s = -S^2/4 - Ha^2 - \pi^2 n^2$, $s = -S^2/4 - Ha^2 - \pi^2(2n-1)^2/4$,

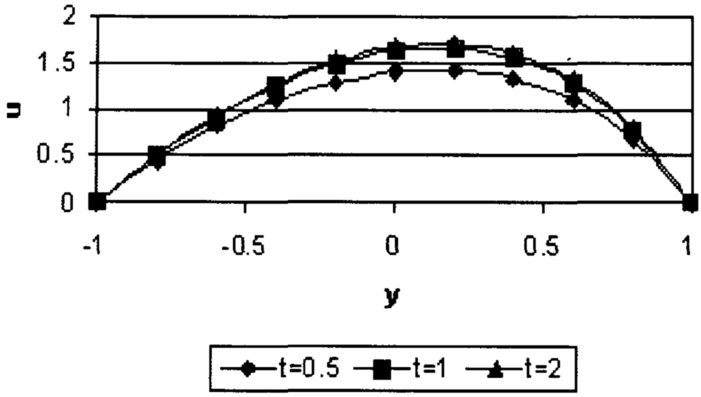
(b) Removal singularity at $s = -Ha^2$.

Equation (8) shows that u is directly proportional to the pressure gradient, that is u/C is independent of C. The expression for the velocity u is to be evaluated for different values of the parameters Ha and S.

4. Numerical Solution of the Energy Equation

The exact solution of the equation of motion, given by Eq. (8), determines the velocity field for different values of the parameters Ha and S. The values of the velocity components, when substituted in the right side of the inhomogeneous energy equation (4), make it too difficult to solve analytically. The energy equation is to be solved numerically with the initial and boundary conditions given by Eq. (6) using finite differences [15]. The Crank-Nicolson implicit method is applied. The finite difference equations are written at the mid-point of the computational cell and the different terms are replaced by their second-order central difference approximations in the y-direction. The diffusion term is replaced by the average of the central differences at two successive time levels. The viscous and Joule dissipation terms are evaluated using the velocity components and their derivatives in the y-direction which are obtained from the exact solution. Finally, the block tri-diagonal system is solved using Thomas' algorithm.

The computational domain is divided into meshes each of dimension Δt and Δy in time and space, respectively. We define the variables $\nu = \partial u/\partial y$ and $H = \partial T/\partial y$ to reduce the second order differential Eq. (4) to first order differential equation



(a)

which has the following finite difference representation:

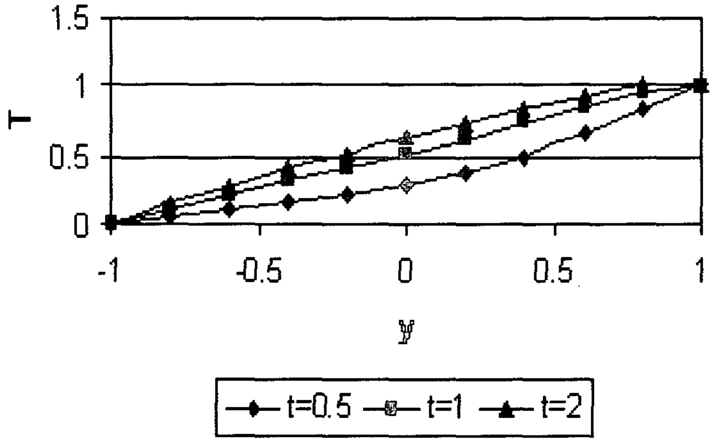
$$\left(\frac{T_{i+1,j+1}-T_{i,j+1}+T_{i+1,j}-T_{i,j}}{2\Delta t}\right)+S\left(\frac{H_{i+1,j+1}-H_{i,j+1}+H_{i+1,j}-H_{i,j}}{4Pr}\right)$$
$$=\left(\frac{(H_{i+1,j+1}+H_{i,j+1})-(H_{i+1,j}+H_{i,j})}{2\Delta y Pr}\right)$$
$$+Ec\left(\frac{\bar{v}_{i+1,j+1}+\bar{v}_{i,j+1}+\bar{v}_{i+1,j}+\bar{v}_{i,j}}{2}\right)^2$$
$$+EcHa^2\left(\frac{\bar{u}_{i+1,j+1}+\bar{u}_{i,j+1}+\bar{u}_{i+1,j}+\bar{u}_{i,j}}{2}\right)^2$$

Unlike the velocity u, the temperature distribution depends on C. All calculations have been carried out for $C=5$, $Pr=1$ and $Ec=0.2$.

5. RESULTS AND DISCUSSION

Figure 2 presents the velocity and temperature distributions as functions of y for different values of the time starting from $t=0$ to the steady state. Figures 2a and 2b are evaluated for $Ha=1$ and $S=1$. The velocity curves are asymmetric about the $y=0$ plane because of the suction as shown in Fig. 2a. It is observed that the velocity component u reaches the steady state faster than T. This is expected, because u acts as the source of temperature.

Figure 3 shows the effect of the Hartmann number Ha on the time development of the velocity u and temperature T at the center of the channel ($y=0$). In this

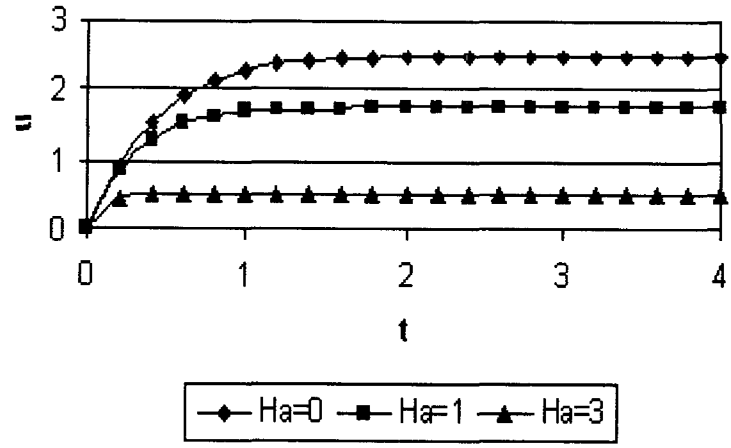


(b)

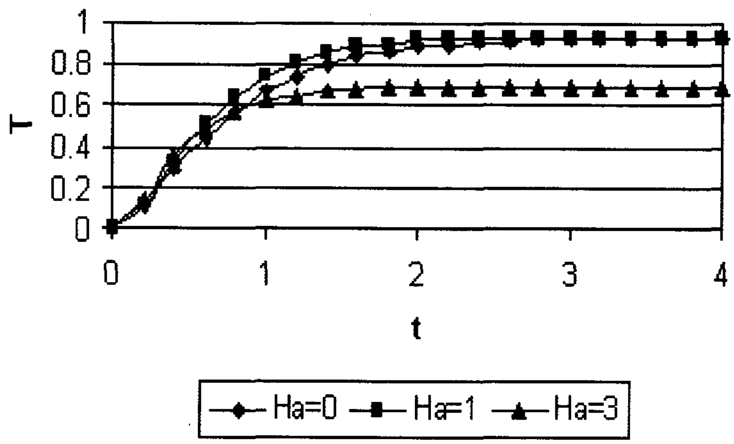
Fig. 2. Temporal evolution of the profile of (a) u and (b) T when $Ha = 1$ and $S = 1$.

figure, $S = 0$ (suction suppressed). It is clear from Fig. 3a that increasing the parameter Ha decreases $u(0,t)$ and its steady state time. This is due to increasing the magnetic damping force on u. Figure 3b indicates that increasing Ha increases also $T(0,t)$ at small times but decreases it at larger times. This is due to the fact that, for small times, u is small and an increase in Ha increases the Joule dissipation which is proportional to Ha. Therefore, the temperature increases. For larger times, increasing Ha decreases u and, in turn, decreases the Joule and viscous dissipations which, in turn, decreases T. This accounts for the crossing of the T curves with time for various values of Ha.

Figure 4 shows the effect of the suction parameter on the time development of the velocity u and temperature T at the centre of the channel ($y = 0$). In this figure, $Ha = 0$ (hydrodynamic case). In Fig. 4a, it is observed that increasing the suction decreases the velocity u at the center and its steady state time due to the convection of fluid from regions in the lower half to the center, which has higher fluid speed. In Fig. 4b, the temperature at the center is affected more by the convection term, which pumps the fluid from the cold lower half towards the centre.



(a)

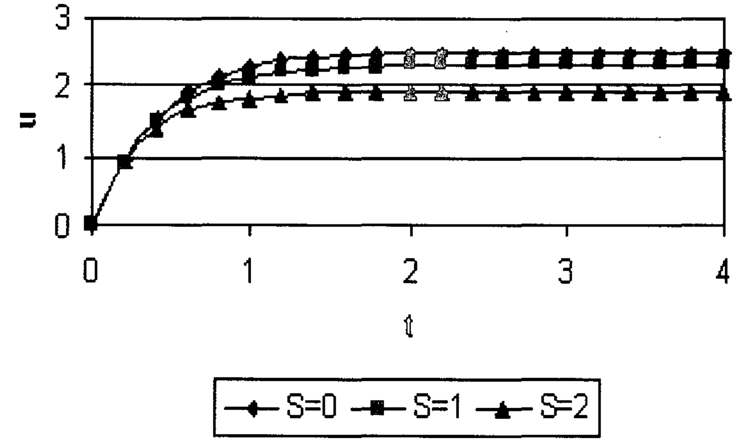


(b)

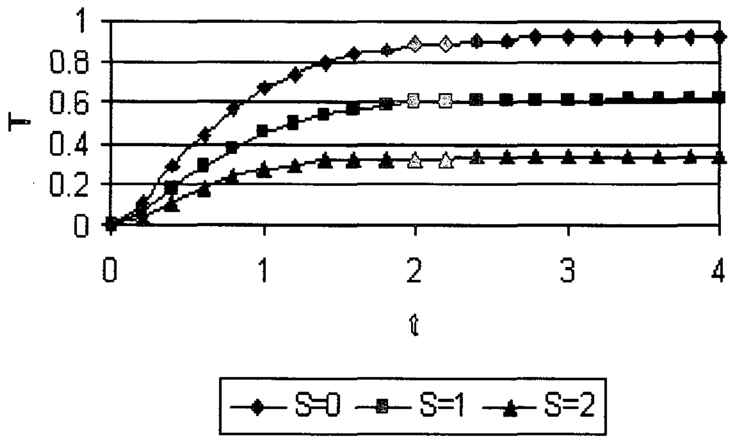
Fig. 3. Effect of Ha on the time variation of: (a) u at $y = 0$; (b) T at $y = 0$. ($S = 0$)

6. CONCLUSION

The unsteady Hartmann flow of a conducting fluid under the influence of an applied uniform magnetic field has been studied in the presence of uniform suction and injection. The effect of the magnetic field and the suction and injection velocity on the velocity and temperature distributions has been investigated. It is found that



(a)



(b)

Fig. 4. Effect of S on the time variation of: (a) u at $y = 0$; (b) T at $y = 0$. $(Ha = 0)$

both the magnetic field and suction or injection velocity has a marked effect on both the velocity and temperature distributions. It is of interest to see that the effect of the magnetic field on the temperature at the center of the channel depends on time. For small time, increasing the magnetic field increases the temperature, however, for large time, increasing the magnetic field decreases the temperature.

REFERENCES

1. J. Hartmann & F. Lazarus: Kgl. Danske Videnskab. Selskab. *Mat.-Fys. Medd* **15** (1937), no.6-7.
2. I. N. Tao: *J. of Aerospace Sci* **27, 334** (1960).
3. R. A. Alpher: *Int. J. Heat and Mass Transfer.* **3** (1961), 108.
4. G. W. Sutton & A. Sherman: *Quart. J. Mech. Appl. Math.*, **13, 85** 1960.
5. K. Cramer & S.-I. Pai: *Nihonkai Math. J.* **6** (1995), no. 2, 201–206.
6. S. D. Nigam & S. N. Singh: *Integral Equations Operator Theory* **43** (2002), no. 4, 379–384.
7. I. Tani: *J. of Aerospace Sci.* **29, 287** 1962.
8. V. M. Soundalgekar, N. V. Vighnesam, & H. S. Takhar, *IEEE Trans. Plasma Sci.* PS-7(3), 178 (1979).
9. V. M. Soundalgekar & A. G. Uplekar: *IEEE Trans. Plasma Sci.* PS-14(5), 579 (1986).
10. H. A. Attia: *Res. Comm.* **26**, no. 1, 115 (1999).
11. Singh, G. S.: MHD fluctuating flow of a viscous conducting fluid over a porous plate plat with time-dependent suction. *Indian J., Pure Appl. Math.* **1**, 37 (1970).
12. Srivastava, L. M. & Agarwal, R. P.: On unsteady viscous MHD flow through a porous channel with constant suction. *Acta Phys. Acad. Scien. Hungar.* **45**, 163 (1978).
13. H. Schlichting: Theory and problems of Laplace transform *McGraw-Hill Book Co.* 1968.
14. M. R. Spiegel: *McGraw-Hill Book Co.* 1986.
15. W. F. Ames: Numerical solutions of partial differential equations. 2nd ed., *Academic Press, New York* 1977.

HAZEM A. ATTIA DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, AL-QASSEEM UNIVERSITY, P.O. BOX 237, BURAIDAH 81999, KSA. PRESENT RESIDENCE : 38 MOHAMED SHAHINE AT. #5, AGGOZA, GIZA, EGYPT
Email address: ah1113@yahoo.com