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NUMERICAL METHODS FOR FUZZY SYSTEM OF LINEAR EQUATIONS WITH CRISP COEFFICIENTS

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Abstract. In this paper, numerical algorithms for solving a fuzzy system of linear equations with crisp coefficients are presented. We illustrate the efficiency and accuracy of the proposed methods by solving some numerical examples. We also provide a graphical representation of the fuzzy solutions in three-dimension as a visual reference of the solution of the fuzzy system.

1. Introduction

The system of equations plays a vital role in various areas such as mathematics, statistics, social sciences, economics, finance, and engineering. Since many real world problems requiring a system of equations are too complicated to be defined in precise terms, uncertainty is often needed. So, the variables or parameters may be expressed in terms of an interval or a fuzzy number, which was initially introduced and investigated by Zadeh [11].

The fuzzy system of equations were investigated by various authors using different approaches. Friedman *et al.* [5] studied a general fuzzy linear system using the embedding approach. Wang *et al.* [10] suggested iteration algorithms for a system of fuzzy linear equations of the form $X = AX + U$. Asady *et al.* [2] developed a method for solving $m \times n$ fuzzy linear system for $m \leq n$. Salahuddin [9] used the random resolvent operator techniques for a fuzzy system of nonlinear equations. Behera and Chakraverty [3] investigated on fuzzy complex system of linear equations. Rivaz and Abad [7] presented a domain decomposition method for system of fuzzy sylvester equations. Allahviranloo *et al.* [1] proposed a signed decomposition method for fully fuzzy linear systems.

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Many of those works were done on fuzzy numbers by α-cut approach or an ordered pair of functions. On the other hand, a slightly different form of fuzzy number so called a linear fuzzy real number was discussed in [6, 8]. Very few researchers have developed methods fuzzy system of linear equations on linear fuzzy real numbers. In this paper, we present numerical algorithms for the fuzzy system of linear equations with crisp coefficients on linear fuzzy real numbers.

The paper is organized as follows. In Section 2, we provide some preliminary definitions of linear fuzzy real numbers. In Section 3, numerical algorithms and experiments are presented to solve a fuzzy system of linear equations. Lastly, we will make concluding remarks in Section 4.

2. Preliminaries

As preliminaries, we introduce some definitions and properties of linear fuzzy real numbers which are used in this research. We first define a fuzzy number with an associated triple of real numbers as follows.

Definition 2.1 ([6, Linear fuzzy real number])**.** Let R be the set of all real numbers. For some real numbers a, b, c, let $\mu : R \to [0, 1]$ be a function defined by

$$\mu(x) = \begin{cases} 0, & \text{if } x < a \text{ or } x > c, \\ \frac{x-a}{b-a}, & \text{if } a \leq x < b, \\ 1, & \text{if } x = b, \\ \frac{c-x}{c-b}, & \text{if } b < x \leq c. \end{cases}$$

Then a notation $\mu(a, b, c)$ is called a *linear fuzzy real number* with an associated triple of real numbers (a, b, c), where $a \leq b \leq c$, shown in Figure 1.

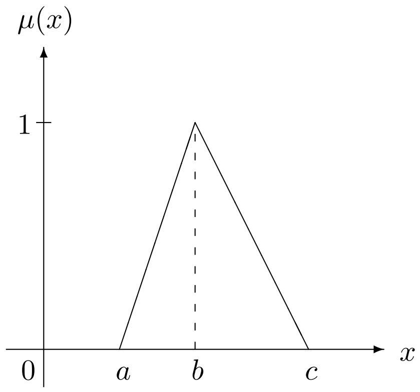


Figure 1. Linear fuzzy real number $\mu(a, b, c)$

Let LFR be the set of all linear fuzzy real numbers. Since any real number $t \in R$ can be written as a linear fuzzy real number $r(t) \in LFR$, where $r(t) = \mu(t, t, t)$, we can see $R \subseteq LFR$. As a linear fuzzy real number, we consider $r(t)$ to represent the real number t itself.

Definition 2.2 ([6, Fuzzy arithmetic]). Let $\mu_1 = \mu(a_1, b_1, c_1)$ and $\mu_2 = \mu(a_2, b_2, c_2)$ be two linear fuzzy real numbers. Then addition, subtraction, multiplication, and division of μ_1 and $\mu_2 \in LFR$ are defined by

(1) $\mu_1 + \mu_2 = \mu(a_1 + a_2, b_1 + b_2, c_1 + c_2)$
(2) $\mu_1 - \mu_2 = \mu(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(3) $\mu_1 \cdot \mu_2 = \mu(\min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\}, b_1b_2, \max\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\})$
(4) $\frac{\mu_1}{\mu_2} = \mu_1 \cdot \frac{1}{\mu_2}$ where $\frac{1}{\mu_2} = \mu(\min\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \text{median}\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, \max\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\})$.

We can easily see that $t \cdot \mu(a, b, c) = \mu(t \cdot a, t \cdot b, t \cdot c)$ for $t > 0$, because a real number t can be considered as a linear fuzzy real number $r(t) = \mu(t, t, t)$.

Definition 2.3 ([6, Fuzzy sequence]). Let $\{\mu^{(k)}\}_{k=0}^{\infty}$ be a sequence of LFR where $\mu^{(k)} = \mu(a^{(k)}, b^{(k)}, c^{(k)})$. The LFR sequence $\{\mu^{(k)}\}$ has the limit $\mu^* = \mu(a^*, b^*, c^*)$ and we write $\lim_{k\to\infty} \mu^{(k)} = \mu^*$, if the sequences $\{a^{(k)}\}$, $\{b^{(k)}\}$, and $\{c^{(k)}\}$ have the limit a^*, b^*, and c^*, respectively. If $\lim_{k\to\infty} \mu^{(k)}$ exists, we say the LFR sequence $\{\mu^{(k)}\}$ is *convergent*. Otherwise, we say the sequence is *divergent*.

3. Solving Fuzzy System of Linear Equations

In this section, we will construct an iterative algorithm for solving a fuzzy system of linear equations. Consider the following $n \times n$ fuzzy system of linear equations of the form:

$$\left\{\begin{array}{ccccccc} a_{11}\mu_{x_1} & + & a_{12}\mu_{x_2} & + \cdots + & a_{1n}\mu_{x_n} & = & b_1 \\ a_{21}\mu_{x_1} & + & a_{22}\mu_{x_2} & + \cdots + & a_{2n}\mu_{x_n} & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}\mu_{x_1} & + & a_{n2}\mu_{x_2} & + \cdots + & a_{nn}\mu_{x_n} & = & b_n. \end{array}\right. \tag{3.1}$$

In matrix notation, the above system may be written as $[A]\{X\} = \{b\}$, where the coefficient matrix $[A] = (a_{ij})$, $1 \le i, j \le n$, is a crisp real $n \times n$ matrix, $\{b\} = \{b_i\}$, $1 \le i \le n$, is a crisp real column vector and $\{X\} = \{\mu_{x_i}\}$, $1 \le i \le n$, is the unknown vector of linear fuzzy real numbers.

Many researchers have developed methods to solve a fuzzy system of linear equations. But the existing methods sometimes are lengthy and not computationally efficient for the large systems. So, it is important to develop an efficient numerical procedures to solve a large fuzzy system of linear equations. Solving the fuzzy system (3.1) over LFR is possible with a modification of the classical Jacobi iterative method of a crisp system over real numbers. The intention of this research is to see if the classical Jacobi method over real numbers R can be extended to linear fuzzy real numbers LFR in terms of the efficiency and the accuracy.

In order to solve the $n \times n$ fuzzy system (3.1) by Jacobi iterative technique, we start with an initial approximation $\left\{X^{(0)}\right\} = \left\{\mu_{x_i}^{(0)}\right\}$, $1 \le i \le n$, to the solution $\{X\} = \{\mu_{x_i}\}$ and generate a sequence of vectors $\left\{X^{(k)}\right\}_{k=0}^{\infty} = \left\{\mu_{x_i}^{(k)}\right\}_{k=0}^{\infty}$, $1 \le i \le n$, where

$$\mu_{x_i}^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} \mu_{x_j}^{(k-1)} \right], \text{ for } i = 1, \cdots, n. \tag{3.2}$$

Convergence of the sequence $\left\{X^{(k)}\right\}_{k=0}^{\infty}$ can be seen in the next theorem.

Theorem 3.1. *Suppose that* $\left\{X^{(k)}\right\}_{k=0}^{\infty}$ *is a sequence generated by (3.2). Then it converges to the solution* $\{X\}$ *of the system (3.1) provided that the diagonal entries of the crisp matrix* $[A]$ *are all non-zero.*

Proof. See in [4]. □

Now we provide the algorithm of the modified iterative scheme using (3.2), referred to as LFR Jacobi's algorithm, to solve the fuzzy system of linear equations (3.1) over LFR.

Algorithm 3.2. (LFR Jacobi's algorithm)

INPUT: fuzzy system of n equations, initial value $\mu_{x_i}^{(0)}$ for all i, integer N
OUTPUT: approximate sol. μ_{x_i} for all i
Step 1: For $k = 1, 2, \cdots, N$ do Step 2.
Step 2: For $i = 1, 2, \cdots, n$ do Step 3.
Step 3: $\mu_{x_i}^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} \mu_{x_j}^{(k-1)} \right]$
Step 4: OUTPUT(all $\mu_{x_i}^{(N)}$) and STOP.

Example 3.3. Consider the following 3×3 fuzzy system of linear equations:

$$\begin{cases} 10\mu_{x_1} - \mu_{x_2} + 2\mu_{x_3} = 10 \\ -\mu_{x_1} + 11\mu_{x_2} - \mu_{x_3} = 20 \\ 2\mu_{x_1} - \mu_{x_2} + 10\mu_{x_3} = 10 \end{cases} \tag{3.3}$$

Let $\mu_{x_1}^{(0)} = \mu(-1,0,1)$, $\mu_{x_2}^{(0)} = \mu(0,1,2)$, and $\mu_{x_3}^{(0)} = \mu(1,2,3) \in LFR$ be initial values of the fuzzy system (3.3). We note that any other values may be chosen as an initial approximation. Then we can generate an approximate solution sequence $\{X^{(k)}\}_{k=0}^{\infty}$ using LFR Jacobi's algorithm. The first seven terms of the sequence of the fuzzy system (3.3) are listed in Table 1, which is compared with the solution of the crisp Jacobi method. We can see that the approximate solutions converge to the exact solution within seven iterations and they are exact up to four decimal places. In other words, the sequence $\{X^{(k)}\} = \left\{\mu_{x_i}^{(k)}\right\}$, where $i = 1, 2, 3$, is convergent to the solution $\{X\} = \{\mu_{x_i}\}$.

Table 1. Approximate solutions by LFR Jacobi and crisp Jacobi

k	Sol. $\mu_{x_i}^{(k)}$ by LFR Jacobi	Sol. $x_i^{(k)}$ by crisp Jacobi
0	$\mu_{x_1}^{(0)} = \mu(-1.0000, 0.0000, 1.0000)$ $\mu_{x_2}^{(0)} = \mu(0.0000, 1.0000, 2.0000)$ $\mu_{x_3}^{(0)} = \mu(1.0000, 2.0000, 3.0000)$	$x_1^{(0)} = 0.0000$ $x_2^{(0)} = 1.0000$ $x_3^{(0)} = 2.0000$
1	$\mu_{x_1}^{(1)} = \mu(0.6000, 0.7000, 0.8000)$ $\mu_{x_2}^{(1)} = \mu(1.8182, 2.0000, 2.1818)$ $\mu_{x_3}^{(1)} = \mu(1.0000, 1.1000, 1.2000)$	$x_1^{(1)} = 0.7000$ $x_2^{(1)} = 2.0000$ $x_3^{(1)} = 1.1000$
2	$\mu_{x_1}^{(2)} = \mu(0.9782, 0.9800, 0.9818)$ $\mu_{x_2}^{(2)} = \mu(1.9636, 1.9818, 2.0000)$ $\mu_{x_3}^{(2)} = \mu(1.0582, 1.0600, 1.0618)$	$x_1^{(2)} = 0.9800$ $x_2^{(2)} = 1.9818$ $x_3^{(2)} = 1.0600$
3	$\mu_{x_1}^{(3)} = \mu(0.9847, 0.9862, 0.9876)$ $\mu_{x_2}^{(3)} = \mu(2.0033, 2.0036, 2.0040)$ $\mu_{x_3}^{(3)} = \mu(1.0007, 1.0022, 1.0036)$	$x_1^{(3)} = 0.9862$ $x_2^{(3)} = 2.0036$ $x_3^{(3)} = 1.0022$
4	$\mu_{x_1}^{(4)} = \mu(0.9997, 0.9999, 1.0002)$ $\mu_{x_2}^{(4)} = \mu(1.9987, 1.9989, 1.9992)$ $\mu_{x_3}^{(4)} = \mu(1.0029, 1.0031, 1.0034)$	$x_1^{(4)} = 0.9999$ $x_2^{(4)} = 1.9989$ $x_3^{(4)} = 1.0031$
5	$\mu_{x_1}^{(5)} = \mu(0.9992, 0.9993, 0.9993)$ $\mu_{x_2}^{(5)} = \mu(2.0002, 2.0003, 2.0003)$ $\mu_{x_3}^{(5)} = \mu(0.9999, 0.9999, 0.9999)$	$x_1^{(5)} = 0.9993$ $x_2^{(5)} = 2.0003$ $x_3^{(5)} = 0.9999$
6	$\mu_{x_1}^{(6)} = \mu(1.0000, 1.0000, 1.0000)$ $\mu_{x_2}^{(6)} = \mu(1.9999, 1.9999, 1.9999)$ $\mu_{x_3}^{(6)} = \mu(1.0002, 1.0002, 1.0002)$	$x_1^{(6)} = 1.0000$ $x_2^{(6)} = 1.9999$ $x_3^{(6)} = 1.0002$
7	$\mu_{x_1}^{(7)} = \mu(1.0000, 1.0000, 1.0000)$ $\mu_{x_2}^{(7)} = \mu(2.0000, 2.0000, 2.0000)$ $\mu_{x_3}^{(7)} = \mu(1.0000, 1.0000, 1.0000)$	$x_1^{(7)} = 1.0000$ $x_2^{(7)} = 2.0000$ $x_3^{(7)} = 1.0000$

It should be pointed that, for the sake of simplicity, Jacobi method is applied to the fuzzy system (3.1). However, other iterative methods may be applied to the system, too. For example, if Gauss-Seidel (GS) iterative method is used, then Step 3 in the algorithm mentioned above will be changed to the following:

$$\mu_{x_i}^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\mu_{x_j}^{(k)} - \sum_{j=i+1}^{n} a_{ij}\mu_{x_j}^{(k-1)}\right], \text{ for } i = 1, \cdots, n. \tag{3.4}$$

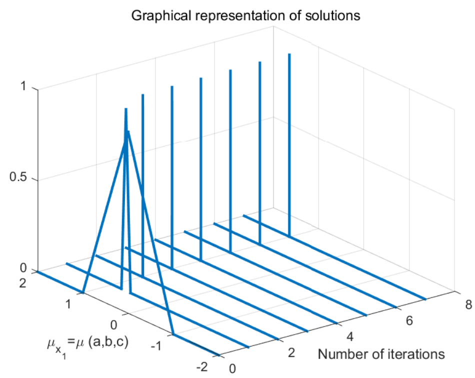
The first four terms of the sequence of the fuzzy system (3.3) based on Eq. (3.4) are listed in Table 2. As expected, LFR GS method is faster than LFR Jacobi method for the same problem of fuzzy system.

Table 2. Approximate solutions by LFR GS and crisp GS

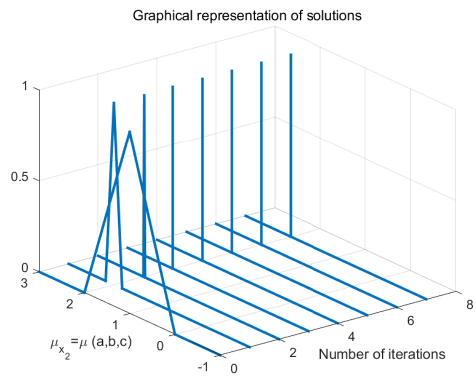
k	Sol. $\mu_{x_i}^{(k)}$ by LFR GS	Sol. $x_i^{(k)}$ by crisp GS
0	$\mu_{x_1}^{(0)} = \mu(-1.0000, 0.0000, 1.0000)$ $\mu_{x_2}^{(0)} = \mu(0.0000, 1.0000, 2.0000)$ $\mu_{x_3}^{(0)} = \mu(1.0000, 2.0000, 3.0000)$	$x_1^{(0)} = 0.0000$ $x_2^{(0)} = 1.0000$ $x_3^{(0)} = 2.0000$
1	$\mu_{x_1}^{(1)} = \mu(0.6000, 0.7000, 0.8000)$ $\mu_{x_2}^{(1)} = \mu(1.9818, 2.0636, 2.1455)$ $\mu_{x_3}^{(1)} = \mu(1.0382, 1.0664, 1.0945)$	$x_1^{(1)} = 0.7000$ $x_2^{(1)} = 2.0636$ $x_3^{(1)} = 1.0664$
2	$\mu_{x_1}^{(2)} = \mu(0.9905, 0.9931, 0.9956)$ $\mu_{x_2}^{(2)} = \mu(2.0026, 2.0054, 2.0082)$ $\mu_{x_3}^{(2)} = \mu(1.0017, 1.0019, 1.0022)$	$x_1^{(2)} = 0.9931$ $x_2^{(2)} = 2.0054$ $x_3^{(2)} = 1.0019$
3	$\mu_{x_1}^{(3)} = \mu(0.9999, 1.0002, 1.0004)$ $\mu_{x_2}^{(3)} = \mu(2.0001, 2.0002, 2.0002)$ $\mu_{x_3}^{(3)} = \mu(0.9999, 1.0000, 1.0000)$	$x_1^{(3)} = 1.0002$ $x_2^{(3)} = 2.0002$ $x_3^{(3)} = 1.0000$
4	$\mu_{x_1}^{(4)} = \mu(1.0000, 1.0000, 1.0000)$ $\mu_{x_2}^{(4)} = \mu(2.0000, 2.0000, 2.0000)$ $\mu_{x_3}^{(4)} = \mu(1.0000, 1.0000, 1.0000)$	$x_1^{(4)} = 1.0000$ $x_2^{(4)} = 2.0000$ $x_3^{(4)} = 1.0000$

In this research, the coefficient matrix is considered as a real crisp, whereas an unknown variable vector is considered as linear fuzzy real numbers. In the future, we plan to extend our research to a fuzzy system whose coefficient matrix is fuzzy.

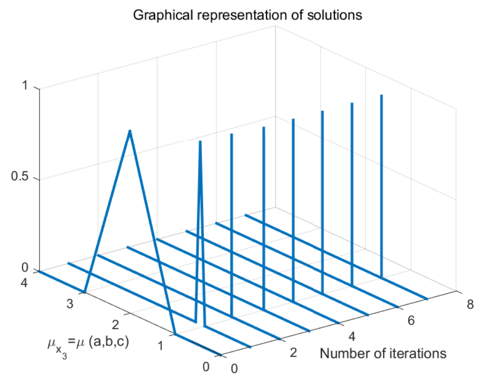
Finally, in Figure 2, we provide graphical representation of the fuzzy solutions of Table 1 in three-dimension as a visual reference of the solution of the fuzzy system (3.3). We can see the convergence of approximate solution sequence using LFR Jacobi method.



(a) Approximations of μ_{x1}



(b) Approximations of μ_{x2}



(c) Approximations of μ_{x3}

Figure 2. Graphical representation of fuzzy solutions by LFR Jacobi method

4. Conclusion

In this paper, we present numerical methods for solving a fuzzy system of linear equations over linear fuzzy real numbers with a modification of the crisp Jacobi method over real numbers. The numerical experiments show that the LFR Jacobi method is very efficient and accurate for solving a fuzzy system of linear equations over linear fuzzy real numbers. Graphical representation of the fuzzy solutions is also provided as a visual reference in three-dimension.

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