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COMMON FIXED POINT RESULTS ON FUZZY METRIC SPACES AND MODULAR METRIC SPACES VIA SIMULATION FUNCTION

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Abstract. In this paper, we prove common fixed point theorems for two mappings by using simulation function on fuzzy metric spaces. We also deduce some consequences in modular metric spaces.

1. Introduction and Preliminaries

George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8] and defined the Hausdorff topology on fuzzy metric spaces which has important application [14] in quantum particle physics.

Recently, the notion of simulation function was given by Khojasteh et al. [7]. In [9] and [11] authors revised the definition of simulation function introduced by Khojasteh et al. [7].

Definition 1.1 ([9, 11]). A mapping $\zeta\colon [0,\infty)\times[0,\infty) \to R$ is a simulation function if it satisfies the following conditions:
$(\zeta_1)\zeta(t,s) < s-t$ for all $t,s > 0$.
(ζ_2) If $\{t_n\},\{s_n\}$are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in N$ then $\limsup_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.
The set of all simulation functions is denoted by Z.

Several examples of simulation function are given in [4], [7], [9]. [10], [13], [15].

It is clear from (ζ_1) that $\zeta(t,t) < 0$ when $t > 0$.

Definition 1.2 (Schweizer and Sklar [12]). A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if it satisfies the following conditions:

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(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,\ 1]$,
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,\ b,\ c,\ d \in [0,\ 1]$.

A few examples of continuous t-norm are

$$a * b = ab,\ a * b = \min\{a,\ b\},\ a * b = \max\{a + b - 1,\ 0\}.$$

Definition 1.3 (George and Veeramani [5]). A fuzzy metric space is an ordered triple $(X,\ M,\ *)$ such that X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\ \infty)$ satisfying the following conditions, for all $x,\ y,$ $z \in X$ and $s,\ t > 0:$

$(FM - 1)$ $M(x,\ y,\ 0) > 0,$
$(FM - 2)$ $M(x,\ y,\ t) = 1$ iff $x = y,$
$(FM - 3)$ $M(x,\ y,\ t) = M(y,\ x,\ t),$
$(FM - 4) M(x,\ y,\ t) * M(y,\ z,\ s) \le M(x,\ z,\ t + s),$
$(FM - 5)$ $M(x,\ y,\ \cdot) : [0,\ \infty) \to [0,\ 1]$ is continuous.

then the triple $(X,\ M,\ *)$ is called a fuzzy metric space. If we replace $(FM - 4)$ by

$$(FM - 6) M(x, y, t) * M(y, z, t) \le M(x, z, t)$$

then the triple $(X, M*)$ is called a non-Archimedean fuzzy metric space. We note that if $(X, M, .)$ is nondecreasing for all $x,\ y,\ z \in X$ then $(FM - 6)$ is equivalent to

$$(FM - 6) M(x, y, t) * M(y, z, s) \le M(x, z, \max\{t, s\})$$

that implies $(FM - 4)$.Thus each non-Archimedean fuzzy metric space is a fuzzy metric space, if $(X, M, .)$ is nondecreasing for all $x,\ y,\ z \in X$.

Definition 1.4. Let $(X,\ M,\ *)$ be a fuzzy metric space. Then

(i) a sequence $\{x_n\}$ converges to $x_0 \in X$ iff for all $t > 0 \lim_{n\to\infty} M(x_n, x_0, t) = 1$

(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence [5] if and only if for all $\varepsilon \in (0, 1)$ and $t > 0$ there exists n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \ge n_0$

(iii) $(X,\ M,\ *)$ is complete [6] if every Cauchy sequence converges to some $x \in X$.

Definition 1.5. ([1]). Let $(X,\ M,\ *)$ be a fuzzy metric space. The fuzzy metric M is called *triangular* when

$$\frac{1}{M(x, y, t)} - 1 \le \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1 \text{ for all } x, y, z \in X \text{ all } t > 0.$$

2. Common Fixed Point via Simulation Function on Fuzzy Metric Spaces

Theorem 2.1. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with M triangular and let $A, B : X \to X$ be two given mappings. Let there exists $\zeta \in Z$ such that*

$$\zeta\Big(\frac{1}{M(Ax, Ay, t)} - 1, \frac{1}{M(Bx, By, t)} - 1\Big) \geq 0 \text{ for all } x, y \in X \tag{2.1}$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Proof. First of all we will prove that if coincidence point of A and B exist then it is unique.

Suppose if possible v_1 and v_2 are two distinct coincidence points of A and B then there exists two points u_1, $u_2 \in$ X such that

$$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_2$$

then by (2.1) we have

$$\begin{aligned} 0 &\leq \zeta\Big(\frac{1}{M(Au_1, Au_2, t)} - 1, \frac{1}{M(Bu_1, Bu_2, t)} - 1\Big) \\ &= \zeta\Big(\frac{1}{M(v_1, v_2, t)} - 1, \frac{1}{M(v_1, v_2, t)} - 1\Big) \\ &< 0, \end{aligned}$$

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. Since $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we obtain $Ax_n = Bx_{n+1}$ for all $n \in N$

Let $Ax_n = Bx_{n+1} = y_n$.If $y_n = y_{n+1}$ for some $n \in N$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus x_{n+1} is the unique coincidence point of A and B. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in N$. Hence we have

$$\begin{aligned} 0 &\leq \zeta\Big(\frac{1}{M(Ax_n, Ax_{n+1}, t)} - 1, \frac{1}{M(Bx_n, Bx_{n+1}, t)} - 1\Big) \\ &= \zeta\Big(\frac{1}{M(y_n, y_{n+1}, t)} - 1, \frac{1}{M(y_{n-1}, y_n, t)} - 1\Big) \qquad (2.2) \\ &< S(y_{n-1}, y_n, t) - S(y_n, y_{n+1}, t), \end{aligned}$$

where $S(y_{n-1}, y_n, t) = \frac{1}{M(y_{n-1}, y_n, t)} - 1$.

Therefore $\{S(y_{n-1}, y_n, t)\}$ is a decreasing sequence of + ive real numbers. Thus there exists $z \geq 0$ such that

$$\lim_{n\to\infty} S(y_{n-1}, y_n, t) = z \tag{2.3}$$

Suppose $z > 0$ then by (2.2) and (ζ_2) it follows that

$$0 \leq \lim_{n\to\infty} \sup \zeta(S(y_n, y_{n+1}, t), S(y_{n-1}, y_n, t)) < 0$$

where $t_n = S(y_n, y_{n+1}, t) < S(y_{n-1}, y_n, t) = s_n$ and $t_n, s_n \to z > 0$.

Clearly this is a contradiction and so $z = 0$.By (2.3) we obtain

$$\lim_{n\to\infty} M(y_{n-1} y_n, t) = 1 \tag{2.4}$$

Now we prove that the sequence $\{y_n\}$ is Cauchy. Suppose if possible $\{y_n\}$ is not a Cauchy sequence in X, therefore $\lim_{m,n\to\infty} \inf M(y_m, y_n, t_0) < 1$for some $t_0 > 0$.

Suppose there exists $0 < \epsilon < 1$and two sub sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$such that n_k is the smallest index for which $n_k > m_k \geq k$ and

$$M(y_{m_k}, y_{n_k}, t_o) \leq 1 - \epsilon \tag{2.5}$$

and

$$M(y_{m_k}, y_{n_{k-1}}, t_o) > 1 - \epsilon \tag{2.6}$$

Now we have

$$\begin{aligned} 1 - \epsilon &\geq M(y_{m_k}, y_{n_k}, t_o) \\ &\geq M(y_{m_k}, y_{n_{k-1}}, t_o) * M(y_{n_{k-1}}, y_{n_k}, t_0) \\ &> 1 - \epsilon * M(y_{n_{k-1}}, y_{n_k}, t_0) \end{aligned}$$

Letting $k \to \infty$ and using (2.4), we get

$$\lim_{k\to\infty} M(y_{m_k}, y_{n_k}, t_o) = 1 - \epsilon \tag{2.7}$$

By the same reasoning as above, we obtain

$$\begin{aligned} 1 - \epsilon &\geq M(y_{m_k}, y_{n_k}, t_o) \\ &\geq M(y_{m_k}, y_{m_{k-1}}, t_o) * M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) * M(y_{n_{k-1}}, y_{n_k}, t_0) \end{aligned}$$

and

$$M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) \geq M(y_{m_{k-1}}, y_{m_k}, t_0) * M(y_{m_k}, y_{n_k}, t_o) * M(y_{n_k}, y_{n_{k-1}}, t_0)$$

By letting $k \to \infty$ and using (2.4) and (2.7), we obtain

$$\lim_{k\to\infty} M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = 1- \in \tag{2.8}$$

Using (2.7) and (2.8), we obtain

$$\begin{aligned}\lim_{k\to\infty} S(y_{m_k}, y_{n_k}, t_o) &= \lim_{k\to\infty} \frac{1}{M(y_{m_k}, y_{n_k}, t_o)} - 1 \\ &= \lim_{k\to\infty} \frac{1 - M(y_{m_k}, y_{n_k}, t_o)}{M(y_{m_k}, y_{n_k}, t_o)} \\ &= \frac{1-(1-\epsilon)}{1-\epsilon} \\ &= \frac{\epsilon}{1-\epsilon}\end{aligned}$$

and

$$\lim_{k\to\infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_o) = \frac{\epsilon}{1-\epsilon}$$

Let

$$\begin{aligned}t_k &= S(y_{m_k}, y_{n_k}, t_o) \\ s_k &= S(y_{m_{k-1}}, y_{n_{k-1}}, t_o).\end{aligned}$$

Thus by using (2.1) and (ζ_2) we have

$$0 \le \lim_{k\to\infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_o), S(y_{m_{k-1}}, y_{n_{k-1}}, t_o)) < 0.$$

Above inequality is not true and hence $\{y_n\}$ is a Cauchy sequence in X. Now since AX or BX is a complete subset of $(X, M, *)$ therefore there exists $u \in X$ such that $y_n \to Bu$ as $n \to \infty$. If there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Au$ then letting $k \to \infty$ we get $Au = Bu$ and hence the claim. So we suppose that $y_{n_k} \neq Au$ for all $n \in N$.

Since $y_{n-1} \neq y_n$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Bu$ for $k \in N$.Using (2.1) we have

$$\begin{aligned}0 &\le \zeta(\frac{1}{M(Ax_{n_{k+1}}, Au, t)} - 1, \frac{1}{M(Bx_{n_{k+1}}, Bu, t)} - 1) \\ &= \zeta(S(y_{n_{k+1}}, Au, t), S(y_{n_k}, Bu, t)) \\ &< S(y_{n_k}, Bu, t) - S(y_{n_{k+1},} Au, t).\end{aligned}$$

This shows that $y_{n_{k+1}} \to Au$ and hence $Au = Bu$ is a unique coincidence point of A and B. If A and B are weakly compatible then by using well known result due to Jungck, we can prove the existence of unique common fixed point of A and B.

Theorem 2.2. *Let $(X, M, *)$ be a non- Archimedean fuzzy metric space with M triangular and $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in Z$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$\zeta(\frac{1}{M(Ax, Ay, t)} - 1, \phi(\frac{1}{M(Bx, By, t)} - 1) \geq 0 \text{ for all } x, y \in X \tag{2.9}$$

$$0 < \phi(t) \leq t \text{ for all } t \in (0, +\infty) \text{ and } \phi(0) = 0 \tag{2.10}$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Proof. First of all we will prove that if coincidence point of A and B exist then it is unique.

suppose if possible v_1 and v_2 are two distinct coincidence points of A and B then there exists two points u_1, $u_2 \in X$ such that

$$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_2$$

then by (2.9) we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{M(Au_1, Au_2, t)} - 1, \phi(\frac{1}{M(Bu_1, Bu_2, t)} - 1)) \\ &< \phi(\frac{1}{M(v_1, v_2, t)} - 1) - \frac{1}{M(v_1, v_2, t)} - 1) \\ &\leq 0, \end{aligned}$$

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. Since $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we obtain $Ax_n = Bx_{n+1}$ for all $n \in N$

Let $Ax_n = Bx_{n+1} = y_n$.If $y_n = y_{n+1}$ for some $n \in N$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus x_{n+1} is the unique coincidence point of A and B. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in N$.Hence we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{M(Ax_n, Ax_{n+1}, t)} - 1, \phi(\frac{1}{M(Bx_n, Bx_{n+1}, t)} - 1) \\ &= \zeta(\frac{1}{M(y_n, y_{n+1}, t)} - 1, \phi(\frac{1}{M(y_{n-1}, y_n, t)} - 1) \end{aligned}$$

$$(2.11) \qquad < \phi(\frac{1}{M(y_{n-1}, y_n, t)} - 1), (\frac{1}{M(y_n, y_{n+1}, t)} - 1)$$
$$= S(y_{n-1}, y_{n,}t) - S(y_n, y_{n+1,}t) \text{ for all } n \in N$$

where $S(y_{n-1}, y_n, t) = \frac{1}{M(y_{n-1}, y_n, t)} - 1$.

Therefore $\{ S(y_{n-1}, y_n, t)\}$ is a decreasing sequence of + ive real numbers. Thus there exists $z \geq 0$ such that

$$(2.12) \qquad \lim_{n\to\infty} S(y_{n-1}, y_n, t) = z$$

Suppose $z > 0$ then

$$0 \leq \lim_{n\to\infty} \sup \zeta(S(y_n, y_{n+1}, t), \phi(S(y_{n-1}, y_n, t)) < 0$$

where $t_n = S(y_n, y_{n+1}, t), s_n = \phi(S(y_{n-1}, y_n, t)) < S(y_{n-1}, y_n, t)$, and $t_n < s_n, t_n, s_n \to z > 0$.

This is a contradiction. Thus we have

$$\lim_{n\to\infty} S(y_{n-1} y_n, t) = 0$$

By (2.12)we obtain

$$(2.13) \qquad \lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$$

Now we claim that the sequence $\{y_n\}$ is Cauchy sequence in (X, d).Suppose if possible $\{y_n\}$ is not a Cauchy sequence in X, therefore $\lim_{m,n\to\infty} \inf M(y_m, y_n, t_0) < 1$ for some $t_0 > 0$.

Suppose there exists $0 < \epsilon < 1$and two sub sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$such that n_k is the smallest index for which $n_k > m_k \geq k$ and

$$(2.14) \qquad M(y_{m_k}, y_{n_k}, t_o) \leq 1 - \epsilon$$

and

$$(2.15) \qquad M(y_{m_k}, y_{n_{k-1}}, \dot{t_o}) > 1 - \epsilon$$

Now we have

$$1 - \epsilon \geq M(y_{m_k}, y_{n_k}, t_o)$$
$$\geq M(y_{m_k}, y_{n_{k-1}}, \dot{t_o}) * M(y_{n_{k-1}}, y_{n_k}, t_0)$$
$$\geq 1 - \epsilon * M(y_{n_{k-1}}, y_{n_k}, t_0)$$

Letting $k \to \infty$ and using (2.13), we get

$$(2.16) \qquad \lim_{k\to\infty} M(y_{m_k}, y_{n_k}, t_o) = 1 - \epsilon$$

By the same reasoning as above, we obtain

$$\begin{aligned}1-\epsilon &\geq M(y_{m_k}, y_{n_k}, t_o)\\ &\geq M(y_{m_k}, y_{m_{k-1}}, t_o) * M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) * M(y_{n_{k-1}}, y_{n_k}, t_0)\end{aligned}$$

and

$$M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) \geq M(y_{m_{k-1}}, y_{m_k}, t_0) * M(y_{m_k}, y_{n_k}, t_o) * M(y_{n_k}, y_{n_{k-1}}, t_0)$$

From the last inequality, by letting $k \to \infty$ and using (2.13), (2.16) we get

$$\lim_{k\to\infty} M(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = 1- \in \tag{2.17}$$

By letting $k \to \infty$ and using (2.16) and (2.17) we obtain

$$\begin{aligned}\lim_{k\to\infty} S(y_{m_k}, y_{n_k}, t_o) &= \lim_{k\to\infty} \frac{1}{M(y_{m_k}, y_{n_k}, t_o)} - 1\\ &= \lim_{k\to\infty} \frac{1 - M(y_{m_k}, y_{n_k}, t_o)}{M(y_{m_k}, y_{n_k}, t_o)}\\ &= \frac{1-(1-\epsilon)}{1-\epsilon}\\ &= \frac{\epsilon}{1-\epsilon}\end{aligned}$$

and

$$\lim_{k\to\infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_o) = \frac{\epsilon}{1-\epsilon}$$

Let

$$t_k = S(y_{m_k}, y_{n_k}, t_o)$$

$$s_k = \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_o)) < S(y_{m_{k-1}}, y_{n_{k-1}} t_0)$$

By (2.9), we have

$$\begin{aligned}0 &\leq \zeta(\frac{1}{M(Ax_{m_k}, Ay_{n_k}, t_0)} - 1, \phi(\frac{1}{M(Bx_{m_k}, Bx_{n_k}, t_0)} - 1)\\ &= \zeta(\frac{1}{M(y_{m_k}, y_{n_k}, t_0)} - 1, \phi(\frac{1}{M(y_{m_{k-1}}, y_{n_{k-1},} t_0)} - 1)\\ &= \zeta(S(y_{m_k}, y_{n_k}, t_0), \phi(S((y_{m_{k-1}}, y_{n_{k-1},} t_0))\\ &< \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) - S(y_{m_k}, y_{n_k}, t_0)\\ &\to 0 \text{ as } k \to \infty\end{aligned} \tag{2.18}$$

From (2.18) we deduce that

$$\lim_{k\to\infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_o), \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_o)) = 0.$$

Clearly this is a contradiction to (ζ_2) and hence we conclude that $\{y_n\}$ is a Cauchy sequence in X. Now since AX or BX is a complete subset of $(X, M, *)$ therefore there exists $u \in X$ such that $y_n \to Bu$ as $n \to \infty$. If there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Au$ then letting $k \to \infty$ we get $Au = Bu$ and hence the claim. So we suppose that $y_{n_k} \neq Au$ for all $n \in N$.

Since $y_{n-1} \neq y_n$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Bu$ for $k \in N$.Using (2.9) we have

$$\begin{aligned}
0 &\leq \zeta(\frac{1}{M(Ax_{n_{k+1}}, Au, t)} - 1, \phi(\frac{1}{M(Bx_{n_{k+1}}, Bu, t)} - 1) \\
&= \zeta(S(y_{n_{k+1}}, Au, t), \phi(S(y_{n_k}, Bu, t))) \\
&< \phi(S(y_{n_k}, Bu, t) - S(y_{n_{k+1},} Au, t)). \\
&< S(y_{n_k}, Bu, t) - S(y_{n_{k+1}}, Au, t) \text{ for all } n \in N
\end{aligned}$$

This shows that $y_{n_{k+1}} \to Au$ and hence $Au = Bu$ is a unique coincidence point of A and B. If A and B are weakly compatible then by using well known result due to Jungck, we can prove the existence of unique common fixed point of A and B.

Theorem 2.3. *Let $(X, M, *)$ be a non- Archimedean fuzzy metric space and $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in Z$ and a function $k \in (0, \frac{1}{2})$ such that for all $x, y \in X$*

$$\begin{aligned}
&\zeta(\frac{1}{M(Ax, Ay, t)}-1, k \max\{\frac{1}{M(Bx, By, t)}-1, \\
&\frac{1}{M(Bx, Ax, t)}-1, \frac{1}{M(By, Ay, t)} - 1, \frac{1}{M(Bx, Ay, t)}-1\}) \geq 0
\end{aligned} \tag{2.19}$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Corollary 2.4. *If in (2.19) we put $Bx = x$ for all $x \in X$, then $A : X \to X$ has a unique fixed point in (X, d).*

3. Extended Approach to a Modular Metric

Definition 3.1 ([2, 3])**.** Let $\omega : (0, \infty) \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $\lambda, \mu > 0$ and $x, , y, z \in X$

(i) $x = y$ iff $\omega(\lambda, x, y) = 0$ for all $\lambda > 0$.

(ii) $\omega(\lambda, x, y) = \omega(\lambda, y, x)$

(iii)$\omega(\lambda+\mu, x, y) \leq \omega(\lambda, x, y)+\omega(\mu, z, y)$.

Then ω is called a modular metric on X. If we replace (i) by

(iv) $\omega(\lambda, x, x)=0$ for all $\lambda>0$,

then ω is called pseudo modular metric on X. If we replace (iii) by

(v) $\omega(\lambda, x, y) \leq \omega(\lambda, x, z)+\omega(\lambda, z, y)$ for all $\lambda>0$ and $x, y, z \in X$.

Then ω is called non-Archimedean. Moreover ω is called convex if the following inequality is satisfied for all $\lambda, \mu>0$ and $x, , y, z \in X$

(vi) $\omega(\lambda+\mu, x, z) \leq \frac{\lambda}{\lambda+\mu} \omega(\lambda, x, z)+\frac{\mu}{\lambda+\mu} \omega(\mu, z, y)$.

Remark 3.2. (i) A metric on a set X is a finite distance between any two points of X while a modular on a same set X is a way to consider a nonnegative "field of velocities" precisely an average velocity $\omega(\lambda, x, y)$ is associated to each $\lambda>0, \omega(\lambda, x, y)$ that is one takes time λ to move from x to y.

(ii)([6]). Let $(X, M, *)$ be a triangular fuzzy metric space. Define a function $\omega:(0, \infty) \times X \times X \rightarrow[0, \infty)$ as

$$\omega(\lambda, x, y)=\frac{1}{M(x, y, \lambda)}-1 \tag{3.1}$$

for all $x, y \in X$ and $\lambda>0$.Then ω_{λ} is a modular metric on X.

Definition 3.3. Let X_{ω} be a modular metric space. Then

(i) $\{x_{n}\}$ in X_{ω} is called $\omega-$convergent to $x \in X_{\omega}$, if $\omega(\lambda, x_{n}, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda>0$. In this case we say x is the $\omega-$limit of $\{x_{n}\}$.

(ii) $\{x_{n}\}$ in X_{ω} is called $\omega-$Cauchy if $\omega(\lambda, x_{n}, x_{m}) \rightarrow 0$ as $m, m \rightarrow \infty$ for all $\lambda>0$.

(iii) A subset Y of X_{ω} is called $\omega-$complete if any $\omega-$Cauchy sequence in Y is a $\omega-$convergent sequence and it's $\omega-$limit is in Y.

Now we state two existence results for unique fixed point in the setting of modular space. Clearly these results are modular counterparts of Theorem 3.1 and Theorem 3.2.

Theorem 3.4. *Let X_{ω} be a non-Archimedean modular metric space and let A, B : $X \rightarrow X$ be two given mappings. Let there exists $\zeta \in Z$ such that*

$$\zeta(\omega(\lambda, A x, A y), \omega(\lambda, B x, B y)) \geq 0 \tag{3.2}$$

for all $x, y \in X$ and for all $\lambda>0$.

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Theorem 3.5. *Let X_ω be a non-Archimedean modular metric space and let $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in Z$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$\zeta(\omega(\lambda, Ax, Ay), \phi(\omega(\lambda, Bx, By)) \geq 0 \tag{3.3}$$

for all $x, y \in X$ and for all $\lambda > 0$.

$$0 < \phi(t) \leq t \text{ for all } t \in (0, \infty) \text{ and } \phi(0) = 0$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

The proof of Theorem 3.4 and Theorem 3.5 are established by applying Theorem 2.1 and Theorem 2.2. We give outline of the proof of Theorem 3.4.

Proof. Let M be a fuzzy metric induced by ω and defined by (3.1). It follows that the triple $(X, M, *)$ is non-Archimedean fuzzy metric space. Then by (3.2) we have

$$\zeta(\frac{1}{M(Ax, Ay, \lambda)} - 1, \frac{1}{M(Bx, By, \lambda)} - 1) \geq 0$$

for all $x, y \in X_\omega$ and for all $\lambda > 0$.Therefore, we apply Theorem 3.1 to conclude that A and B have a unique common fixed point in X.

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