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SOME SPECIAL CURVES IN THREE DIMENSIONAL f-KENMOTSU MANIFOLDS

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Abstract. In this paper we study Biharmonic curves, Legendre curves and Magnetic curves in three dimensional f-Kenmotsu manifolds. We also study 1-type curves in a three dimensional f-Kenmotsu manifold by using the mean curvature vector field of the curve. As a consequence we obtain for a biharmonic helix in a three dimensional f-Kenmotsu manifold with the curvature κ and the torsion τ, $\kappa^2+\tau^2=-(f^2+f')$. Also we prove that if a 1-type non-geodesic biharmonic curve γ is helix, then $\lambda=-(f^2+f')$.

1. Introduction

In the study of f-Kenmotsu manifolds, Legendre curves on contact manifolds have been studied by Baikoussis and Blair in the paper [2]. Belkhelfa et al. [3] have investigated Legendre curves in Riemannian and Lorentzian manifolds.
In [7], Cabrerizo et al. have introduced a geometric approach to the study of magnetic fields on three dimensional Sasakian manifolds. A curve γ is called a magnetic curve in three dimensional f-Kenmotsu manifolds if $\nabla_{\dot\gamma}\dot\gamma=\phi\dot\gamma$ [2]. A magnetic curve is the trajectory of magnetic fields. Geodesics on a manifold are curves which do not experience any kind of forces where the magnetic curves experience due to magnetic fields. If the magnetic field disappears, its magnetic curve become a geodesic. In this way a magnetic curve is a generalization of a geodesic.

Let M be a 3-dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot\gamma}$ denote

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the covariant derivative along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases hold:

- γ is of osculating order 1, i.e,$\nabla_t t = 0$ (geodesic), $t = \dot{\gamma}$. Here, $\cdot$ denotes differentiation with respect to the arc length parameter.
- γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot{\gamma}), n$ and a non-negative function κ (curvature) along γ such that $\nabla_t t = \kappa n$, $\nabla_t n = -\kappa t$.
- γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(= \dot{\gamma})$, n, b and two non-negative functions κ(curvature) and τ(torsion) along γ such that

$$\nabla_t t = \kappa n, \tag{1.1}$$

$$\nabla_t n = -\kappa t + \tau b, \tag{1.2}$$

$$\nabla_t b = -\tau n. \tag{1.3}$$

With respect to the Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is said to be a helix in M if κ and τ both are positive constants and the curve is called a generalized helix if $\frac{\kappa}{\tau}$ is a constant.

2. PRELIMINARIES

Let M be an $(2n + 1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) [4]. As usually denote by Φ the fundamental 2-form of M, $\Phi(X, Y) = g(X, \phi Y)$, for X, $Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on M. For further use, we recall the following definitions ([4], [5]). The manifold M and its structure (ϕ, ξ, η, g) is said to be:

- normal if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently, $[\phi, \phi] + 2d\eta \otimes \xi = 0$),
- almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$,
- cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla \phi = 0$, ∇ being covariant differentiation with respect to the Levi-Civita connection).

The manifold M is said to be locally conformal cosymplectic (respectively, almost cosymplectic) if M has an open covering U_t endowed with differentiable functions

$\sigma_t : U_t \to \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ is defined by

$$\phi_t = \phi, \xi_t = e^{\sigma_t}\xi, \eta_t = e^{-\sigma_t}\eta, g_t = e^{-2\sigma_t}g \tag{2.1}$$

is cosymplectic (respectively, almost cosymplectic).
Osaka and Rosa [19] studied normal locally conformal almost cosymplectic manifold. They gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature properties. Among others Calin and Crasmareanu [10] proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.
By an f-Kenmotsu manifold we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic.
Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \tag{2.2}$$

$$\phi\xi = 0, \ \ \eta \circ \phi = 0, \ \ \eta(X) = g(X, \xi), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, $Y \in \chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1, 1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [20]:

$$(\nabla_X \phi)(Y) = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.5}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = constant \neq 0$, then the manifold is a α-Kenmotsu manifold. 1-Kenmotsu manifold is a Kenmotsu manifold ([16], [21]). If $f = 0$, then the manifold is cosymplectic [20]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$, f' denotes covariant derivation of f with respect to ξ.
For an f-Kenmotsu manifold from (2.2) it follows that

$$\nabla_X \xi = f\{X - \eta(X)\xi\}. \tag{2.6}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. In general this does not hold if $dimM = 3$ [21].

In a three dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}g(Y,Z)X - g(X,Z)Y, \end{aligned} \tag{2.7}$$

In a three dimensional f-Kenmotsu manifold, we have ([18], [21])

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')(g(Y,Z)X - g(X,Z)Y) \\ &\quad -(\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(g(Y,Z)\xi - g(\xi,Z)Y) \\ &\quad +\eta(Y)(g(\xi,Z)X - g(X,Z)\xi)\}. \end{aligned} \tag{2.8}$$

$$S(X,Y) = (\frac{r}{2} + 2f^2 + 2f')g(Y,Z)X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.9}$$

where r is a scalar curvature of M and $f' = \xi f$.
From (2.5), we obtain

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

and (2.6) yields

$$S(X,\xi) = -(f^2 + f')\eta(X). \tag{2.11}$$

Proposition 2.1. *Let γ be a unit speed curve on a three dimensional f-Kenmotsu manifold and T, N and B be the tangent, principal normal and binormal of the curve γ respectively. Then*

$$\eta(T)' = \kappa\eta(N) + f(1 - \eta(T)^2),$$

$$\eta(N)' = -\kappa\eta(T) + \tau\eta(B) - f\eta(T)\eta(N),$$

and

$$\eta(B)' = -\tau\eta(N) - f\eta(T)\eta(B).$$

Proof. Let γ be a unit speed curve on a three dimensional f-Kenmotsu manifold. Differentiating $\eta(T)$, $\eta(N)$ and $\eta(B)$ along γ, we have

$$\begin{aligned} \eta(T)' &= g(\nabla_T T, \xi) + g(T, \nabla_T \xi) \\ &= \kappa\eta(N) + g(T, f(T - \eta(T)\xi)) \\ &= \kappa\eta(N) + f(1 - \eta(T)^2). \end{aligned} \tag{2.12}$$

$$\begin{aligned} \eta(N)' &= g(\nabla_T N, \xi) + g(N, \nabla_T \xi) \\ &= g(-\kappa T + \tau B, \xi) + g(N, f(T - \eta(T)\xi)) \\ &= -\kappa\eta(T) + \tau\eta(B) - f\eta(T)\eta(N). \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
\eta(B)' &= g(\nabla_T B,\xi) + g(B,\nabla_T \xi) \\
&= g(-\tau N,\xi) + g(B, f(T-\eta(T)\xi)) \\
&= -\tau\eta(N) - f\eta(T)\eta(B). \qquad (2.14)
\end{aligned}
$$

This completes the proof. □

A Frenet curve is called a slant curve if it makes a constant angle with the Reeb vector field ξ [9]. If a unit speed curve on an almost contact metric manifold is slant curve, then $\eta(\dot{\gamma}) = \cos\theta$, where θ is a constant and is called slant angle. In particular, if the angle is $\frac{\pi}{2}$, the curve becomes almost contact curve or Legendre curve. A slant curve is called proper if it is neither parallel nor perpendicular to the Reeb vector ξ.

Remark 2.2. For a curve γ in a three dimensional f-Kenmotsu manifold, the following conditions are equivalent
(i) the curve γ is slant curve,
(ii) $\eta(T)' = 0$,
(iii) $\eta(N) = -\frac{f}{\kappa}(1-\eta(T)^2)$.

Remark 2.3. If a curve γ is Legendre in a three dimensional f-Kenmotsu manifold, then from the (2.12), we have

$$\eta(N) = -\frac{f}{\kappa}. \qquad (2.15)$$

3. Biharmonic Curves in Three Dimensional f-Kenmotsu Manifolds

The theory of biharmonic functions is a rich subject. Biharmonic functions have been studied by Maxwell in 1862 and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on. There are a few results on biharmonic curves in arbitrary Riemannian manifolds. Biharmonic curves on a surface was studied by R. Caddeo, et al. in the paper [8]. Later, in [17] S. Montaldo and C. Oniciuc studied biharmonic maps between Riemannian manifolds. In the paper [12] D. Fetcu studied Biharmonic Legendre curves in Sasakian space forms. Certain biharmonic curves on different manifolds have been studied by several authors such as ([6], [13]).

Definition 3.1. A helix γ is said to be *biharmonic* with respect to the Levi-Civita connection ∇ if it satisfies [13]

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0,$$

where $\dot{\gamma} = T$, and R is the curvature tensor of type $(1, 3)$.

Theorem 3.2. *Let γ be a biharmonic helix in a three dimensional f-Kenmotsu manifold with the curvature κ and the torsion τ. Then $\kappa^2 + \tau^2 = -(f^2 + f')$.*

Proof. Let γ be a biharmonic helix in a three dimensional f-Kenmotsu manifold. Then

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0, \tag{3.1}$$

where $\dot{\gamma} = T$, tangent vector and the curvature κ and torsion τ are constant.
Let N and B be principal normal and binormal respectively. Then the Frenet-Serret equations are

$$\nabla_T T = \kappa N, \tag{3.2}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{3.3}$$

and

$$\nabla_T B = -\tau N. \tag{3.4}$$

Differentiating (3.2) with respect to T, we have

$$\begin{aligned} \nabla_T^2 T &= \nabla_T(\kappa N) \\ &= \kappa \nabla_T N \\ &= \kappa(-\kappa T + \tau B) \\ &= -\kappa^2 T + \kappa\tau B. \end{aligned} \tag{3.5}$$

Again differentiating the foregoing with respect to T, we get

$$\begin{aligned} \nabla_T^3 T &= \nabla_T(-\kappa^2 T + \kappa\tau B) \\ &= -\kappa^2(\kappa N) + \kappa\tau(-\tau N) \\ &= -\kappa^3 N - \kappa\tau^2 N. \end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} R(\nabla_T T, T)T &= (\frac{r}{2} + 2f^2 + 2f')\{g(T,T)\nabla_T T - g(\nabla_T T, T)T\} \\ &\quad -(\frac{r}{2} + 3f^2 + 3f')\{\eta(\nabla_T T)(g(T,T)\xi - g(\xi,T)T) \\ &\quad +\eta(T)(g(\xi,T)\nabla_T T - g(\nabla_T T, T)\xi)\} \\ &= (\frac{r}{2} + 2f^2 + 2f')\{\kappa N - 0\} - (\frac{r}{2} + 3f^2 + 3f') \\ &\quad \{\kappa\eta(N)(\xi - \eta(T)T) + \eta(T)(\eta(T)\kappa N - 0)\} \\ &= (\frac{r}{2} + 2f^2 + 2f')\kappa N - (\frac{r}{2} + 3f^2 + 3f')(\kappa\eta(N)\xi \\ &\quad -\kappa\eta(N)\eta(T)T + \eta(T)^2\kappa N). \end{aligned} \tag{3.7}$$

Since the curve is biharmonic helix. Then using (3.6) and (3.7) in (3.1), we obtain

$$\begin{aligned} &-\kappa^3 N - \kappa\tau^2 N + (\frac{r}{2} + 2f^2 + 2f')\kappa N \\ &-(\frac{r}{2} + 3f^2 + 3f')(\kappa\eta(N)\xi - \kappa\eta(N)\eta(T)T + \eta(T)^2\kappa N) = 0. \end{aligned} \tag{3.8}$$

Taking inner product in (3.8) with ξ, we get

$$\begin{aligned} &-\kappa(\kappa^2 + \tau^2)\eta(N) + (\frac{r}{2} + 2f^2 + 2f')\kappa\eta(N) \\ &-(\frac{r}{2} + 3f^2 + 3f')(\kappa\eta(N) - \kappa\eta(N)\eta(T)^2 + \kappa\eta(T)^2\eta(N)) = 0. \end{aligned} \tag{3.9}$$

This implies

$$-\kappa(\kappa^2 + \tau^2)\eta(N) - (f^2 + f')\kappa\eta(N) = 0. \tag{3.10}$$

Since κ and $\eta(N)$ are non-zero, we have

$$\kappa^2 + \tau^2 = -(f^2 + f'). \tag{3.11}$$

This completes the proof. □

Definition 3.3. A curve γ is called a *curve with proper mean curvature vector field* H if there exist $\lambda \in C^k(\gamma)$ such that

$$\Delta H = \lambda H.$$

The curve γ is also called 1- *type.*

In particular, if $\lambda = 0$ then γ is known as a curve with the harmonic mean curvature vector field [14]. Hence the Laplace operator Δ acts on the vector valued function H and it is given by

$$\Delta H = -\nabla_T \nabla_T \nabla_T T.$$

Making use of Frenet equations, we get

$$-3\kappa\dot{\kappa}T + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)N + (2\dot{\kappa}\tau + \kappa\dot{\tau})B = -\lambda\kappa N. \tag{3.12}$$

If both κ and τ are constants, then

$$\lambda = \kappa^2 + \tau^2. \tag{3.13}$$

For more details see ([1], [14] and [15]).

Theorem 3.4. *If a 1-type non-geodesic biharmonic curve γ is helix, then $\lambda = -(f^2 + f')$.*

Proof. Let γ be a biharmonic helix. Then κ and τ are constants. From (3.11), we have

$$\kappa^2 + \tau^2 = -(f^2 + f'). \tag{3.14}$$

Also for a 1-type non-geodesic curve, we have from (3.13)

$$\lambda = \kappa^2 + \tau^2. \tag{3.15}$$

Comparing the equations (3.14) and (3.15), we obtain

$$\lambda = -(f^2 + f'). \tag{3.16}$$

This completes the proof of the theorem. □

4. LEGENDRE CURVES IN THREE DIMENSIONAL f-KENMOTSU MANIFOLDS

A Frenet curve γ in a Riemannian manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $D = ker\eta$, i.e., if $\eta(\dot{\gamma}) = 0$. Legendre curves have been studied by ([22], [23]). For more details we refer ([2], [3]).

Proposition 4.1. *Let M be a three dimensional f-Kenmotsu manifold. If a Legendre curve $\gamma : I \to M$ is not geodesic, then it's curvature and torsion are given by*

$$\kappa = \sqrt{f^2 + \delta^2},$$

and

$$\tau = \frac{f\dot{\delta} - \delta\dot{f}}{\kappa^2},$$

where δ is a function on I.

Proof. Let γ be a Legendre curve on a 3-dimensional f-Kenmotsu manifold. Note that $\dot{\gamma}$, $\phi\dot{\gamma}$ and ξ are orthonormal vector fields along γ. Differentiating $g(\dot{\gamma},\xi)=0$ along γ, we get

$$g(\nabla_{\dot{\gamma}}\dot{\gamma},\xi)+g(f(\dot{\gamma}-\eta(\dot{\gamma})\xi),\dot{\gamma})=0. \tag{4.1}$$

It follows that

$$g(\nabla_{\dot{\gamma}}\dot{\gamma},\xi)+f=0, \tag{4.2}$$

and hence

$$g(\nabla_{\dot{\gamma}}\dot{\gamma},\xi)=-f. \tag{4.3}$$

Therefore

$$\nabla_{\dot{\gamma}}E_1=\nabla_{\dot{\gamma}}\dot{\gamma}=-f\xi+\delta\phi\dot{\gamma}, \tag{4.4}$$

where δ is certain function on I. Hence the curvature κ of the curve γ is given by

$$\kappa=\sqrt{f^2+\delta^2}. \tag{4.5}$$

Differentiating the following vector field E_2

$$E_2=\frac{1}{\kappa}\nabla_{\dot{\gamma}}E_1=-\frac{f}{\kappa}\xi+\frac{\delta}{\kappa}\phi\dot{\gamma} \tag{4.6}$$

along γ, we obtain

$$\begin{aligned}\nabla_{\dot{\gamma}}E_2 &= -\frac{\kappa\dot{f}-f\dot{\kappa}}{\kappa^2}\xi-\frac{f}{k}\nabla_{\dot{\gamma}}\xi+\frac{\kappa\dot{\delta}-\delta\dot{\kappa}}{\kappa^2}\phi\dot{\gamma}+\frac{\delta}{\kappa}\nabla_{\dot{\gamma}}(\phi\dot{\gamma})\\ &= -\frac{\kappa\dot{f}-f\dot{\kappa}}{\kappa^2}\xi-\frac{f^2}{k}\dot{\gamma}+\frac{\kappa\dot{\delta}-\delta\dot{\kappa}}{\kappa^2}\phi\dot{\gamma}+\frac{\delta}{\kappa}(-\delta\dot{\gamma})\\ &= -\frac{f^2+\delta^2}{\kappa}\dot{\gamma}-\frac{\kappa\dot{f}-f\dot{\kappa}}{\kappa^2}\xi+\frac{\kappa\dot{\delta}-\delta\dot{\kappa}}{\kappa^2}\phi\dot{\gamma}. \qquad (4.7)\end{aligned}$$

Again

$$\frac{\kappa\dot{f}-f\dot{\kappa}}{\kappa^2}=\frac{\delta}{\kappa}\frac{\delta\dot{f}-f\dot{\delta}}{\kappa^2} \tag{4.8}$$

and

$$\frac{\kappa\dot{\delta}-\delta\dot{\kappa}}{\kappa^2}=\frac{f}{\kappa}\frac{f\dot{\delta}-\delta\dot{f}}{\kappa^2}. \tag{4.9}$$

Thus using (4.8), (4.9) in (4.7), we have

$$\nabla_{\dot{\gamma}}E_2=-\kappa\dot{\gamma}+\frac{\delta a}{\kappa}\xi+\frac{fa}{\kappa}\phi\dot{\gamma}, \tag{4.10}$$

where $a = \frac{f\dot{\delta}-\delta\dot{f}}{\kappa^2}$. Therefore from (4.10), we get

$$\begin{aligned} \tau E_3 &= \nabla_{\dot{\gamma}} E_2 + \kappa E_1 \\ &= \frac{\delta a}{\kappa}\xi + \frac{fa}{\kappa}\phi\dot{\gamma}. \end{aligned} \tag{4.11}$$

Hence from the foregoing equation it follows that

$$\begin{aligned} \tau &= \sqrt{(\frac{\delta a}{\kappa})^2 + (\frac{fa}{\kappa})^2} \\ &= a = \frac{f\dot{\delta} - \delta\dot{f}}{\kappa^2}. \end{aligned} \tag{4.12}$$

This completes the proof. □

The curvature measures the extent to which a curve is not contained in a straight line so that straight lines have zero curvature, and the torsion measures the extent to which a curve is not contained in a plane so that plane curves have zero torsion [11]. Thus for a plane curve torsion $\tau = 0$.

Theorem 4.2. *Let γ be a Legendre curve on a three dimensional f-Kenmotsu manifold. If the unit vector ξ is parallel to principal normal vector N or binormal vector B. Then the manifold is cosymplectic and the curve is plane curve.*

Proof. Let γ be a Legendre curve on a three dimensional f-Kenmotsu manifold. If ξ is along binormal vector B. Then $\{\dot{\gamma}, \phi\dot{\gamma}, \xi\}$ are orthonormal vector field along γ and

$$T = \dot{\gamma}, \;\; N = \phi\dot{\gamma}, \;\; B = \xi.$$

Let κ and τ be the curvature and torsion of the curve γ. Then

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa\phi\dot{\gamma}, \tag{4.13}$$

$$\nabla_{\dot{\gamma}}\phi\dot{\gamma} = -\kappa\dot{\gamma} + \tau\xi. \tag{4.14}$$

and

$$\nabla_{\dot{\gamma}}\xi = -\tau\phi\dot{\gamma}. \tag{4.15}$$

Also

$$\begin{aligned} \nabla_{\dot{\gamma}}\phi\dot{\gamma} &= (\nabla_{\dot{\gamma}}\phi)\dot{\gamma} + \phi(\nabla_{\dot{\gamma}}\dot{\gamma}) \\ &= f(g(\phi\dot{\gamma}, \dot{\gamma}) - \eta(\dot{\gamma})\xi) + \kappa\phi^2\dot{\gamma} \\ &= 0 + \kappa(-\dot{\gamma} + \eta(\dot{\gamma})\xi) \\ &= -\kappa\dot{\gamma}. \end{aligned} \tag{4.16}$$

and

$$\nabla_{\dot\gamma}\xi = f(\dot\gamma - \eta(\dot\gamma)\xi)$$
$$= f\dot\gamma. \tag{4.17}$$

Then comparing the equations (4.14), (4.15) with (4.16), (4.17) respectively, we get

$$\tau = 0, \tag{4.18}$$

and

$$f = 0. \tag{4.19}$$

If ξ is along principal normal vector N, then the proof is same as above. This completes the proof Theorem. □

Theorem 4.3. *A Legendre curve in three dimensional f-Kenmotsu manifold is of 1-type with $\lambda = \frac{\kappa^2 \dot f - \ddot f}{f}$, where $\gamma f = \dot f$.*

Proof. Let γ be a Legendre curve in a three dimensional f-Kenmotsu manifold. Then $\eta(\dot\gamma) = 0$, where tangent $T = \dot\gamma$. Differentiating $\eta(\dot\gamma) = 0$ with respect to $\dot\gamma$, we get

$$g(\nabla_{\dot\gamma}\dot\gamma, \xi) + g(\dot\gamma, \nabla_{\dot\gamma}\xi) = 0. \tag{4.20}$$

From which it follows that

$$g(\nabla_{\dot\gamma}\dot\gamma, \xi) = -f. \tag{4.21}$$

Differentiating again with respect to $\dot\gamma$, we have

$$g(\nabla^2_{\dot\gamma}\dot\gamma, \xi) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\xi) = -\dot f. \tag{4.22}$$

This implies

$$g(\nabla^2_{\dot\gamma}\dot\gamma, \xi) = -\dot f. \tag{4.23}$$

Differentiating the foregoing equation along γ , we obtain

$$g(\nabla^3_{\dot\gamma}\dot\gamma, \xi) + g(\nabla^2_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\xi) = -\ddot f. \tag{4.24}$$

It follows that

$$g(\nabla^3_{\dot\gamma}\dot\gamma, \xi) + g(\kappa\nabla_{\dot\gamma}N + \dot\kappa N, f\dot\gamma) = -\ddot f, \tag{4.25}$$

and hence

$$g(\nabla^3_{\dot\gamma}\dot\gamma, \xi) = \kappa^2 f - \ddot f. \tag{4.26}$$

If γ is a 1-type curve with $\lambda \in C^k$. Then $\nabla^3_{\dot{\gamma}}\dot{\gamma} = -\lambda\kappa N$. Then from (4.26), we get

$$\lambda = -\frac{\kappa^2 \dot{f} - \ddot{f}}{\kappa\eta(N)}. \tag{4.27}$$

Using (2.15) in (4.27), we have

$$\lambda = \frac{\kappa^2 \dot{f} - \ddot{f}}{f}. \tag{4.28}$$

This completes the proof. □

Theorem 4.4. *If γ is a magnetic helix in three dimensional f-Kenmotsu manifolds, then $\eta(N) = 0$ and $\frac{\eta(T)}{\eta(B)} = \frac{\tau}{\kappa}$.*

Proof. Let γ be a magnetic helix curve in a three dimensional f-Kenmotsu manifold. Then

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}, \tag{4.29}$$

where $\dot{\gamma} = T$(tangent vector). Using Frenet formula, we have

$$\kappa N = \phi T. \tag{4.30}$$

Taking inner product of (4.30) with ξ, we get

$$\eta(N) = 0. \tag{4.31}$$

Differentiating (4.29) with respect to T, we have

$$\nabla^2_T T = \nabla_T(\phi T). \tag{4.32}$$

It follows that

$$\nabla_T(\kappa N) = \eta(T)(\xi - f\phi T) - T. \tag{4.33}$$

This implies

$$-\kappa^2 T + \kappa\tau B = \eta(T)(\xi - f\phi T) - T. \tag{4.34}$$

Taking inner product of (4.34) with ξ, we obtain

$$-\kappa^2\eta(T) + \kappa\tau\eta(B) = \eta(T) - \eta(T). \tag{4.35}$$

Therefore

$$\frac{\eta(T)}{\eta(B)} = \frac{\tau}{\kappa}. \tag{4.36}$$

This completes the proof. □

Theorem 4.5. *Any magnetic helix curve on three dimensional f-Kenmotsu manifolds is of* 1*-type and*

$$\lambda = \kappa^2 \frac{\eta(T)^2 + \eta(B)^2}{\eta(B)^2} = \tau^2 \frac{\eta(T)^2 + \eta(B)^2}{\eta(T)^2}. \tag{4.37}$$

Proof. Let γ be a magnetic helix on a three dimensional f-Kenmotsu manifold. From (4.36), we get

$$\frac{\kappa^2}{\eta(B)^2} = \frac{\tau^2}{\eta(T)^2} = \frac{\kappa^2 + \tau^2}{\eta(T)^2 + \eta(B)^2}. \tag{4.38}$$

If γ is a 1-type curve, then there exists $\lambda \in C^\infty(\gamma)$ such that $\kappa^2 + \tau^2 = \lambda$. Then from (4.38), we get

$$\lambda = \kappa^2 \frac{\eta(T)^2 + \eta(B)^2}{\eta(B)^2} = \tau^2 \frac{\eta(T)^2 + \eta(B)^2}{\eta(T)^2}. \tag{4.39}$$

This completes the proof of the theorem. □

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References

1. J. Arroya, O. Garay & J.J. Mencia: When is a periodic function the curvature of a closed plane curve? The American Mathematical Monthly **115** (2008), 405-414.
2. C. Baikoussis & D.E. Blair: On Legendre curves in contact 3-manifolds. Geom. Dedicata **49** (1994), 135-142.
3. M. Belkhalfa, I.E. Hirica, R. Rosaca & L. Verstraelen: On Legendre curves in Riemannian and Sasakian spaces. Soochow J. Math. **28** (2002), 81-91.
4. D.E. Blair: Contact manifolds in Riemannian geometry. Lecture Notes in Mathematics Vol. 509, Springer-Verlag, Berlin-New York, (1976).
5. D.E. Blair: Riemannian Geometry of contact and symplectic manifolds. Birkhäuser, Boston, 2010.
6. B.Y. Chen, S. Ishikawa: Biharmonic surfaces in pseudo Euclidean spaces. Mem. Fac. Sci. Kyushu Univ. Ser. A. **45** (1991), 323-347.
7. J.L. Cabrerizo, M. Fernandez & J.S. Gomez: The contact magnetic flow in 3D Sasakian manifolds. J. Phys. A.Math., Theor. **42** (2009), 195201(10pp).
8. R. Caddeo, S. Montaldo & P. Piu: Biharmonic curves on a surface. Rend, Mat. Appl. **21** (2001), 143-157.

9. J.T. Cho, J.I. Inoguchi & J.E. Lee: On slant curves in Sasakian 3-manifolds. Bull. Austral. Math. Soc. **74** (2006), 359-367.
10. C. Calin & M. Crasmareanu: From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds. Bull. Malays. Math. Soc. **33** (2010), 361-368.
11. P.Do Carmo: Riemannian Geometry. Birkhauser Boston, 1992.
12. D. Fetcu: Biharmonic Legendre curves in Sasakian space forms. J. Korean Math. Soc. **45** (2008), 393-404.
13. J.I. Inoguchi & J.E. Lee: Biminimal curves in two dimensional space forms. Commun Korean Math Soc. **27** (2012), 771-780.
14. J.I. Inoguchi: Submanifolds with harmonic mean curvature vector field in contact 3-manifold. Cooloq. Math. **100** (2004), 163-167.
15. H. Kocayigit & H.H. Hacisalihoglu: 1-Type curves and biharmonic curves in euclidean 3-space. Int. Elec. J. Geo. **4** (2011), 97-101.
16. K. Kenmotsu: A class of almost contact Riemannian manifolds. Tohoku Math. J. **24** (1972), 93-103.
17. S. Montaldo & C. Oniciuc: A short survey on biharmonic maps between Riemannian manifolds. Rev. Un. Mat. Argentina **47** (2006), 1-22.
18. P. Majhi: Almost ricci solution and gradient almost ricci solution on 3-dimensional f-Kenmotsu manifolds. Kyungpook Math. J. **57** (2017), 309-318.
19. Z. Olszak & R. Rosca: Normal locally conformal almost cosymplectic manifolds. Publ. Math. Debrecen **39** (1991), 315-323.
20. Z. Olszak: Locally conformal almost cosymplectic manifolds. Colloq. Math. **57** (1989), 73-87.
21. G. Pitis: A remark on Kenmotsu manifolds. Bul. Univ. Brasov Ser. C, **30** (1988), 31-32.
22. A. Sarkar & D. Biswas: Legendre curves on three dimensional Heisenberg Groups. Facta Universitatis **28** (2013), 241-248.
23. J. Welyczko: On Legendre curves in 3-dimensional normal almost contact metric manifolds. Soochow J. Math. **33** (2007), 929-937.

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