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ALEXANDROV TOPOLOGIES AND NON-SYMMETRIC PSEUDO-METRICS

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Abstract. In this paper, we investigate the properties of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators on $[0, \infty]$. Moreover, we investigate the relations among Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators. We give their examples.

1. Introduction

Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Pawlak [12,13] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1-11,14,15]. Kim [6-10] investigated the properties of Alexandrov topologies, fuzzy preorders and join-preserving maps in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators on $[0, \infty]$. We give their examples. In fact, categories of Alexandrov topologies, non-symmetric pseudo-metrics and lower approximation operators are isomorphic.

2. Preliminaries

Let $([0, \infty], \leq, \vee, +, \wedge, \rightarrow, \infty, 0)$ be a structure where

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$$x \to y = \bigwedge\{z \in [0,\infty] \mid z + x \geq y\} = (y - x) \vee 0,$$

$$\infty + a = a + \infty = \infty, \forall a \in [0,\infty], \infty \to \infty = 0.$$

Definition 2.1. Let X be a set. A function $d_X : X \times X \to [0,\infty]$ is called a *non-symmetric pseudo-metric* if it satisfies the following conditions:

(M1) $d_X(x,x) = 0$ for all $x \in X$,

(M2) $d_X(x,y) + d_X(y,z) \geq d_X(x,z)$, for all $x, y, z \in X$.

The pair (X, d_X) is called a *non-symmetric pseudo-metric space.*

Remark 2.2. (1) We define a function $d_{[0,\infty]^X} : [0,\infty]^X \times [0,\infty]^X \to [0,\infty]$ as $d_{[0,\infty]^X}(A,B) = \bigvee_{x\in X}(A(x) \to B(x)) = \bigvee_{x\in X}((B(x)-A(x))\vee 0)$. Then $([0,\infty]^X, d_{[0,\infty]^X})$ is a non-symmetric pseudo-metric space.

(2) If (X, d_X) is a non-symmetric pseudo-metric space and we define a function $d_X^{-1}(x,y) = d_X(y,x)$, then (X, d_X^{-1}) is a non-symmetric pseudo-metric space.

(3) Let (X, d_X) be a non-symmetric pseudo-metric space and define $(d_X \oplus d_X)(x,z) = \bigwedge_{y\in X}(d_X(x,y) + d_X(y,z))$ for each $x, z \in X$. By (M2), $(d_X \oplus d_X)(x,z) \geq d_X(x,z)$ and $(d_X \oplus d_X)(x,z) \leq d_X(x,x) + d_X(x,z) = d(x,z)$. Hence $(d_X \oplus d_X) = d_X$.

(4) If d_X is a non-symmetric pseudo-metric and $d_X(x,y) = d_X(y,x)$ for each $x, y \in X$, then d_X is a pseudo-metric

Example 2.3. (1) Let $X = \{a, b, c\}$ be a set and define maps $d_X^i : X \times X \to [0,\infty]$ for $i = 1, 2, 3$ as follows:

$$d_X^1 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 1 \\ 15 & 7 & 0 \end{pmatrix}, d_X^2 = \begin{pmatrix} 0 & 6 & 3 \\ 7 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}, d_X^3 = \begin{pmatrix} 0 & 3 & 7 \\ 6 & 0 & 9 \\ 5 & 4 & 0 \end{pmatrix}.$$

Since $d_X^1(c,b) + d_X^1(b,a) = 13 < d_X^1(c,a) = 15$, d_X^1 is not a non-symmetric pseudo-metric. Since d_X^2 and d_X^3 are non-symmetric pseudo-metrics, $d_X^k \oplus d_X^k = d_X^k$ for $k = 2, 3$.

3. Alexandrov Topologies and Non-symmetric Pseudo-metrics

We define the following two definitions as a sense in [2, 5-10].

Definition 3.1. A subset $\tau_X \subset [0,\infty]^X$ is called an *Alexandrov topology* on X iff it satisfies the following conditions:

(AT1) $\alpha_X \in \tau_X$ where $\alpha_X(x) = \alpha$ for each $x \in X$ and $\alpha \in [0,\infty]$.

(AT2) If $A_i \in \tau_X$ for all $i \in I$, then $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau_X$.

(AT3) If $A \in \tau_X$ and $\alpha \in [0,\infty]$, then $\alpha + A, \alpha \to A \in \tau_X$ where $(\alpha \to A)(x) = (A(x) - \alpha) \vee 0$.

The pair (X, τ_X) is called an *Alexandrov topological space.*

Definition 3.2. A map $\mathcal{H} : [0,\infty]^X \to [0,\infty]^X$ is called a *lower approximation operator* if it satisfies the following conditions, for all $A, A_i \in [0,\infty]^X$, and $\alpha \in [0,\infty]$,

(H1) $\mathcal{H}(\alpha + A) = \alpha + \mathcal{H}(A)$ where $(\alpha + A)(x) = \alpha + A(x)$,

(H2) $\mathcal{H}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{H}(A_i)$,

(H3) $\mathcal{H}(A) \le A$,

(H4) $\mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A)$.

Theorem 3.3. *Let $d_X \in [0,\infty]^{X\times X}$ be a non-symmetric pseudo-metric. Define $\mathcal{H}_{d_X}(A) : [0,\infty]^X \to [0,\infty]^X$ as follows*

$$\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X} (A(x) + d_X(x,y)).$$

Then $\mathcal{H}_{d_X}$ is a lower approximation operator.

Proof. Since $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) + d_X(x,y))$,

(H1) $\mathcal{H}_{d_X}(\alpha + A) = \alpha + \mathcal{H}_{d_X}(A)$.

(H2) $\mathcal{H}_{d_X}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{H}_{d_X}(A_i)$.

(H3) $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) + d_X(x,y)) \le A(y) + d_X(y,y) = A(y)$.

(H4) For all $A \in [0,\infty]^X, z \in X$,

$$\begin{aligned}
&\mathcal{H}_{d_X}(\mathcal{H}_{d_X}(A))(z) = \textstyle\bigwedge_{y\in X}(\mathcal{H}_{d_X}(A)(y) + d_X(y,z)) \\
&= \textstyle\bigwedge_{y\in X}((\bigwedge_{x\in X}(A(x) + d_X(x,y))) + d_X(y,z)) \\
&= \textstyle\bigwedge_{x\in X}(A(x) + \bigwedge_{y\in X}(d_X(x,y) + d_X(y,z))) \\
&= \textstyle\bigwedge_{x\in X}(A(x) + d_X(x,z)) = \mathcal{H}_{d_X}(A)(z).
\end{aligned}$$

Hence $\mathcal{H}_{d_X}$ is a lower approximation operator. □

Theorem 3.4. *A map $\mathcal{H} : [0,\infty]^X \to [0,\infty]^X$ is a lower approximation operator iff there exist a non-symmetric pseudo-metric $d_{\mathcal{H}}$ on X such that*

$$\mathcal{H}(A)(y) = \bigwedge_{x\in X} (A(x) + d_{\mathcal{H}}(x,y)).$$

Proof. (⇒) Put $d_{\mathcal{H}} : X \times X \to [0,\infty]$ as $d_{\mathcal{H}}(x,y) = \mathcal{H}(0_x)(y)$ where $0_x(x) = 0$ and $0_x(y) = \infty$ for $x \ne y \in X$. (M1) $d_{\mathcal{H}}(x,x) = \mathcal{H}(0_x)(x) \le 0_x(x) = 0$.

(M2) Since $A = \bigwedge_{y\in X}(A(y) + 0_y)$ and $\mathcal{H}(0_x) = \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + 0_y)$,

$$\begin{aligned}
&\bigwedge_{y\in X}(d_{\mathcal{H}}(x,y) + d_{\mathcal{H}}(y,z)) \\
&= \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) \text{ (by (H2))} \\
&= \mathcal{H}(\bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + 0_y)(z)) = \mathcal{H}(\mathcal{H}(0_x))(z) \\
&= \mathcal{H}(0_x)(z) = d_{\mathcal{H}}(x,z).
\end{aligned}$$

Hence $d_{\mathcal{H}}$ is a non-symmetric pseudo-metric. Moreover,

$$\begin{aligned}
&\mathcal{H}(A)(y) = \mathcal{H}(\bigwedge_{x\in X}(A(x) + 0_x))(y) \\
&= \bigwedge_{x\in X}(A(x) + \mathcal{H}(0_x)(y)) \\
&= \bigwedge_{x\in X}(A(x) + d_{\mathcal{H}}(x,y))).
\end{aligned}$$

($\Leftarrow$) It follow from Theorem 3.3. □

Theorem 3.5. *Let d_X be a non-symmetric pseudo-metric on X. Define $\tau_{d_X} = \{A \in [0,\infty]^X \mid A(x) + d_X(x,y) \geq A(y)\}$. Then the following properties hold.*

(1) τ_{d_X} is an Alexandrov topology on X.

(2) $d_X(x,-) \in \tau_{d_X}$. Moreover, $A \in \tau_{d_X}$ iff $A = \bigwedge_{x\in X}(A(x) + d_X(x,-)) = \mathcal{H}_{d_X}(A)$.

Proof. (1) (AT1) Since $\alpha_X(x) + d_X(x,y) \geq \alpha_X(y)$, we have $\alpha_X \in \tau_{d_X}$.

(AT2) If $A_i \in \tau_{d_X}$ for all $i \in I$, then

$$\begin{aligned}
&(\bigwedge_{i\in I} A_i)(x) + d_X(x,y) = \bigwedge_{i\in I}(A_i(x) + d_X(x,y)) \\
&\geq \bigwedge_{i\in I} A_i(y), \\
&(\bigvee_{i\in I} A_i)(x) + d_X(x,y) = \bigvee_{i\in I}(A_i(x) + d_X(x,y)) \\
&\geq \bigvee_{i\in I} A_i(y).
\end{aligned}$$

Hence $\bigwedge_{i\in I} A_i, \bigvee_{i\in I} A_i \in \tau_{d_X}$.

(AT3) If $A \in \tau_{e_X}$ and $\alpha \in [0,\infty]$, then

$$\begin{aligned}
&(\alpha + A)(x) + d_X(x,y) \geq (\alpha + A)(y), \\
&(\alpha \to A)(x) + d_X(x,y) = ((A(x) - \alpha) \vee 0) + d(x,y) \\
&= ((A(x) - \alpha) + d(x,y)) \vee d(x,y) \\
&\geq (A(y) - \alpha) \vee 0 = (\alpha \to A)(y).
\end{aligned}$$

So, $\alpha + A, \alpha \to A \in \tau_{d_X}$. Hence τ_{d_X} is an Alexandrov topology on X.

(2) Since $d_X(x,y) + d_X(y,z) \geq d_X(x,z)$, $d_X(x,-) \in \tau_{d_X}$. Let $A \in \tau_{d_X}$. Then $\bigwedge_{x\in X}(A(x)+d_X(x,y)) \geq A(y)$ and $\bigwedge_{x\in X}(A(x)+d_X(x,y)) \leq A(y)+d_X(y,y) = A(y)$. Hence $A = \bigwedge_{x\in X}(A(x) + d_X(x,-)) = \mathcal{H}_{d_X}(A)$.

Conversely, since $\mathcal{H}_{d_X}(A)(y) + d_X(y,z) = \bigwedge_{x\in X}(A(x) + d_X(x,y)) + d_X(y,z) \geq \bigwedge_{x\in X}(A(x) + d_X(x,z)) = \mathcal{H}_{d_X}(A)(z)$. So, $A = \mathcal{H}_{d_X}(A) \in \tau_{d_X}$. □

Theorem 3.6. *Let $\mathcal{H} : [0,\infty]^X \to [0,\infty]^X$ be a lower approximation operator. Then the following properties hold.*

(1) $\tau_{\mathcal{H}} = \{A \in [0,\infty]^X \mid \mathcal{H}(A) = A\}$ *is an Alexandrov topology on* X *such that* $\tau_{\mathcal{H}} = \{\mathcal{H}(A) \mid A \in [0,\infty]^X\}$.

(2) *Define* $d_{\mathcal{H}} : X \times X \to [0,\infty]$ *as* $d_{\mathcal{H}}(x,y) = \mathcal{H}(0_x)(y)$ *where* $0_x(x) = 0$ *and* $0_x(y) = \infty$ *for* $x \neq y \in X$. *Then* $d_{\mathcal{H}}$ *is a non-symmetric pseudo-metric on* X *such that* $\mathcal{H}_{d_{\mathcal{H}}} = \mathcal{H}$ *and* $\tau_{d_{\mathcal{H}}} = \tau_{\mathcal{H}}$.

Proof. (1) (AT1) Since $\mathcal{H}(\alpha_X) = \mathcal{H}(\alpha + 0_X) = \alpha + 0_X = \alpha_X$, then $\alpha_X \in \tau_{\mathcal{H}}$.

(AT2) For $A_i \in \tau_{\mathcal{H}}$ for each $i \in \Gamma$, by (H2), $\bigwedge_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$. Since $\bigvee_{i\in\Gamma} A_i = \bigvee_{i\in\Gamma} \mathcal{H}(A_i) \leq \mathcal{H}(\bigvee_{i\in\Gamma} A_i) \leq \bigvee_{i\in\Gamma} A_i$, Thus, $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{H}}$.

(AT3) For $A \in \tau_{\mathcal{H}}$, by (H1), $\alpha + A \in \tau_{\mathcal{H}}$.

Since $\alpha + \mathcal{H}(\alpha \to A) = \mathcal{H}(\alpha + (\alpha \to A)) \geq \mathcal{H}(A)$, $\mathcal{H}(\alpha \to A) \geq (\mathcal{H}(A) - \alpha) \vee 0 = \alpha \to \mathcal{H}(A) = \alpha \to A$. Then $\alpha \to A \in \tau_{\mathcal{H}}$. Hence $\tau_{\mathcal{H}}$ is an Alexandrov topology on X. Let $A \in \tau_{\mathcal{H}}$. Then $A = \mathcal{H}(A) \in \{\mathcal{H}(A) \mid A \in [0,\infty]^X\}$. Let $\mathcal{H}(A) \in \{\mathcal{H}(A) \mid A \in [0,\infty]^X\}$. Since $\mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A)$, $\mathcal{H}(A) \in \tau_{\mathcal{H}}$.

(2) By Theorem 3.4, $d_{\mathcal{H}}$ is a non-symmetric pseudo-metric on X. Moreover,

$$\begin{aligned}
\mathcal{H}_{d_{\mathcal{H}}}(A)(y) &= \textstyle\bigwedge_{x\in X}(A(x) + d_{\mathcal{H}}(x,y))) \\
&= \textstyle\bigwedge_{x\in X}(A(x) + \mathcal{H}(0_x)(y)) \\
&= \mathcal{H}(\textstyle\bigwedge_{x\in X}(A(x) + 0_x))(y) = \mathcal{H}(A)(y),
\end{aligned}$$

$\tau_{d_{\mathcal{H}}} = \tau_{\mathcal{H}}$ from:

$$A \in \tau_{d_{\mathcal{H}}} \text{ iff } \mathcal{H}_{d_{\mathcal{H}}}(A) = A \text{ iff } \mathcal{H}(A) = A \text{ iff } A \in \tau_{\mathcal{H}}.$$

□

Example 3.7. (1) Define maps $d^i : [0,\infty] \times [0,\infty] \to [0,\infty]$ for $i = 0,1,2,3$ as follows:

$$d^0(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \infty, & \text{if } x \neq y, \end{cases} \quad d^1(x,y) = \begin{cases} 0, & \text{if } x \geq y, \\ \infty, & \text{if } x < y, \end{cases}$$

$$d^2(x,y) = \begin{cases} 0, & \text{if } x \leq y, \\ \infty, & \text{if } x > y, \end{cases} \quad d^3(x,y) = 0.$$

Since $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) + d_X(x,y))$, we can obtain

$$\begin{aligned}
&\mathcal{H}_{d^0}(A)(y) = \textstyle\bigwedge_{x\in X}(A(x) + d^0_X(x,y)) = A(y), \\
&\mathcal{H}_{d^1}(A) = \textstyle\bigwedge_{x\geq y} A(x), \\
&\mathcal{H}_{d^2}(A) = \textstyle\bigwedge_{x\leq y} A(x), \\
&\mathcal{H}_{d^3}(A) = \textstyle\bigwedge_{x\in X} A(x).
\end{aligned}$$

$$\begin{aligned}
\tau_{d^0} &= [0,\infty]^{[0,\infty]},\\
\tau_{d^1} &= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) \le A(y) \text{ if } x \le y\},\\
\tau_{d^2} &= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) \ge A(y) \text{ if } x \le y\},\\
\tau_{d^3} &= \{\alpha_X \in [0,\infty]^{[0,\infty]} \mid \alpha \in [0,\infty]\}.
\end{aligned}$$

Theorem 3.8. *Let τ be Alexandrov topology on X. Then the following properties hold.*

(1) *Define $\mathcal{H}_\tau : [0,\infty]^X \to [0,\infty]^X$ as follows:*

$$\mathcal{H}_\tau(A) = \bigvee\{B \mid B \le A,\ B \in \tau\}.$$

Then $\mathcal{H}_\tau$ is a lower approximation operator such that $\tau_{\mathcal{H}_\tau} = \tau$, $\mathcal{H}_{\tau_{\mathcal{H}}} = \mathcal{H}$.

(2) *Define $d_\tau : X \times X \to [0,\infty]$ as*

$$d_\tau(x,y) = \bigvee_{A\in\tau} (A(x) \to A(y)) = \bigvee_{A\in\tau} ((A(y) - A(x)) \vee 0)$$

Then d_τ is a non-symmetric pseudo-metric such that $\tau = \tau_{\mathcal{H}_{d_\tau}} = \tau_{d_\tau}$. Moreover, $\mathcal{H}_\tau = \mathcal{H}_{d_\tau}$ and $d_\tau = d_{\mathcal{H}_\tau}$.

(3) *If $\mathcal{H} : [0,\infty]^X \to [0,\infty]^X$ is a lower approximation operator, then*

$$\bigwedge_{y\in X} (\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) = \mathcal{H}(0_x)(z)$$

for all $x,y,z \in X$ and $d_{\tau_{\mathcal{H}}} = d_{\mathcal{H}}$.

Proof. (1) We show $\mathcal{H}_\tau(A) = \bigvee\{B \mid B \le A,\ B \in \tau\}$ is a lower approximation operator.

(H1) For $\alpha \in [0,\infty], A \in [0,\infty]^X$,

$$\begin{aligned}
&\alpha + \mathcal{H}_\tau(A)\\
&= \alpha + \bigvee\{B \mid B \le A,\ B \in \tau\}\\
&= \bigvee\{\alpha + B \mid \alpha + B \le \alpha + A,\ \alpha + B \in \tau\}\\
&= \mathcal{H}_\tau(\alpha + A).
\end{aligned}$$

(H2) Since $\mathcal{H}_\tau(A) \le \mathcal{H}_\tau(B)$ for $A \le B$, we have $\bigwedge_{i\in\Gamma} \mathcal{H}_\tau(A_i) \ge \mathcal{H}_\tau(\bigwedge_{i\in\Gamma} A_i)$. Since $\bigwedge_{i\in\Gamma} A_i \ge \bigwedge_{i\in\Gamma} \mathcal{H}_\tau(A_i) \in \tau$, then $\mathcal{H}_\tau(\bigwedge_{i\in\Gamma} A_i) \ge \bigwedge_{i\in\Gamma} \mathcal{H}_\tau(A_i)$.

(H3) It follows from the definition.

(H4) Since $\mathcal{H}_\tau(A) \in \tau$, we have $\mathcal{H}_\tau(\mathcal{H}_\tau(A)) = \mathcal{H}_\tau(A)$.

Let $A \in \tau_{\mathcal{H}_\tau}$. Then $A = \mathcal{H}_\tau(A) \in \tau$. Hence $\tau_{\mathcal{H}_\tau} \subset \tau$.

Let $A \in \tau$. Then $\mathcal{H}_\tau(A) = A$. So, $A \in \tau_{\mathcal{H}_\tau}$. Hence $\tau \subset \tau_{\mathcal{H}_\tau}$.

Since $\mathcal{H}_{\tau_{\mathcal{H}}}(A) = \bigvee\{B \mid B \le A,\ B \in \tau_{\mathcal{H}}\}$ and $A \ge \mathcal{H}(\mathcal{H}(A)) = \mathcal{H}(A)$, we have $\mathcal{H}(A) \le \mathcal{H}_{\tau_{\mathcal{H}}}(A)$. For $B_i \in \tau_{\mathcal{H}}$, since $\mathcal{H}(\bigvee_{i\in\Gamma} B_i) \ge \bigvee_{i\in\Gamma} \mathcal{H}(B_i) = \bigvee_{i\in\Gamma} B_i$, then $\mathcal{H}(\mathcal{H}_{\tau_{\mathcal{H}}}(A)) = \mathcal{H}_{\tau_{\mathcal{H}}}(A)$. So, $\mathcal{H}(A) \ge \mathcal{H}_{\tau_{\mathcal{H}}}(A)$.

(2) We easily show that d_τ is a non-symmetric pseudo-metric.

Let $A \in \tau$. Then $\mathcal{H}_{d_\tau}(A) = A$ because

$$\begin{aligned}&\mathcal{H}_{d_\tau}(A)(y) = \bigwedge_{x\in X}(A(x) + d_\tau(x,y)) = \bigwedge_{x\in X}(A(x) + \bigvee_{B\in\tau}(B(x) \to B(y))\\ &\geq \bigwedge_{x\in X}(A(x) + (A(x) \to A(y))) = \bigwedge_{x\in X}(A(x) + ((A(y) - A(x)) \vee 0))) \geq A(y),\end{aligned}$$

So, $\tau \subset \tau_{\mathcal{H}_{d_\tau}}$

Let $A = \mathcal{H}_{d_\tau}(A)$. Then

$$A = \mathcal{H}_{d_\tau}(A) = \bigwedge_{x\in X}(A(x) + \bigvee_{B\in\tau}(B(x) \to B)) \in \tau.$$

So, $\tau_{\mathcal{H}_{d_\tau}} \subset \tau$.

Let $A \in \tau$. Then $A \in \tau_{d_\tau}$ because

$$\begin{aligned}&A(x) + d_\tau(x,y) = A(x) + \bigvee_{B\in\tau}(B(x) \to B(y))\\ &\geq A(x) + (A(x) \to A(y)) = A(x) + ((A(y) - A(x)) \vee 0))) \geq A(y).\end{aligned}$$

So, $\tau \subset \tau_{d_\tau}$

Let $A \in \tau_{d_\tau}$. Then $A \in \tau$ because

$$A = \mathcal{H}_{d_\tau}(A) = \bigwedge_{x\in X}(A(x) + \bigvee_{B\in\tau}(B(x) \to B)) \in \tau.$$

Since $A \geq \mathcal{H}_{d_\tau}(A) \in \tau$, then $\mathcal{H}_\tau(A) \geq \mathcal{H}_{d_\tau}(A)$. Since

$$\begin{aligned}&\mathcal{H}_{d_\tau}(A)(y) = \bigwedge_{x\in X}(A(x) + \bigvee_{B\in\tau}(B(x) \to B(y))\\ &\geq \bigwedge_{x\in X}(\mathcal{H}_\tau(A)(x) + (\mathcal{H}_\tau(A)(x) \to \mathcal{H}_\tau(A)(y)) \geq \mathcal{H}_\tau(A)(y),\end{aligned}$$

Hence $\mathcal{H}_{d_\tau}(A) = \mathcal{H}_\tau(A)$.

(3) Since $\mathcal{H}(0_x) = \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + 0_y(-))$, $\bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)) = \mathcal{H}(0_x)(z)$ because

$$\begin{aligned}&\mathcal{H}(0_x)(z) = \mathcal{H}(\mathcal{H}(0_x)(z) = \mathcal{H}(\bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + 0_y(-)))(z)\\ &= \bigwedge_{y\in X}(\mathcal{H}(0_x)(y) + \mathcal{H}(0_y)(z)).\end{aligned}$$

$$\begin{aligned}&d_{\tau_\mathcal{H}}(x,y) = \bigvee_{A\in\tau_\mathcal{H}}((A(y) - A(x)) \vee 0)\\ &\geq \bigvee_{z\in X}((\mathcal{H}(0_z)(y) - \mathcal{H}(0_z)(x)) \vee 0) \geq (\mathcal{H}(0_x)(y) - \mathcal{H}(0_x)(x)) \vee 0\\ &\geq (\mathcal{H}(0_x)(y) - (0_x)(x)) \vee 0 = \mathcal{H}(0_x)(y),\end{aligned}$$

$$\begin{aligned}&d_{\tau_\mathcal{H}}(x,y) = \bigvee_{A\in\tau_\mathcal{H}}((A(y) - A(x)) \vee 0) = \bigvee_{A\in[0,\infty]^X}((\mathcal{H}(A)(y) - \mathcal{H}(A)(x)) \vee 0)\\ &= \bigvee_{A\in[0,\infty]^X}((\bigwedge_{z\in X}(A(z) + \mathcal{H}(0_z)(y)) - \bigwedge_{z\in X}(A(z) + \mathcal{H}(0_z)(x))) \vee 0)\\ &\leq \bigvee_{z\in X}((\mathcal{H}(0_z)(y) - \mathcal{H}(0_z)(x)) \vee 0) \leq \mathcal{H}(0_x)(y).\end{aligned}$$

Hence $d_{\tau_\mathcal{H}}(x,y) = \mathcal{H}(0_x)(y) = d_\mathcal{H}(x,y)$. □

Theorem 3.9. *Let $d_X \in [0,\infty]^{X\times X}$ be a non-symmetric pseudo-metric. Then $d_{\mathcal{H}_{d_X}}(x,y) = \mathcal{H}_{d_X}(0_X)(y) = d_X(x,y) = d_{\tau_{d_X}}(x,y)$ for each $x,y \in X$.*

Proof. Since $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) + d_X(x,y))$, $\mathcal{H}_{d_X}(0_x)(y) = \bigwedge_{x\in X}(0_x(x) + d_X(x,y)) = d_X(x,y)$. Since $d_X(x,-) \in \tau_{d_X}$ and $A = \mathcal{H}_{d_X}(A)$ for $A \in \tau_{d_X}$,

$$\begin{aligned}
&d_{\tau_{d_X}}(x,y) = \bigvee_{A\in\tau_{d_X}}((A(y) - A(x)) \vee 0)\\
&\geq (d_X(x,y) - d_X(x,x)) \vee 0 = d_X(x,y),\\
&d_{\tau_{d_X}}(x,y) = \bigvee_{A\in\tau_{d_X}}((A(y) - A(x)) \vee 0)\\
&= \bigvee_{A\in\tau_{d_X}}((\mathcal{H}_{d_X}(A)(y) - \mathcal{H}_{d_X}(A)(x)) \vee 0)\\
&= \bigvee_{A\in\tau_{d_X}}((\bigwedge_{z\in X}(A(z) + d_X(z,y)) - (\bigwedge_{z\in X}(A(z) + d_X(z,x))) \vee 0)\\
&\leq \bigvee_{z\in X}((d_X(z,y) - d_X(z,x)) \vee 0) \leq d_X(x,y).
\end{aligned}$$

Example 3.10. Let $X = \{a,b,c\}$ be a set and $A \in [0,\infty]^X$ as

$$A(a) = 7, A(b) = 5, A(c) = 10$$

(1) Define $d_A(x,y) = (A(y) - A(x)) \vee 0$ as

$$d_A = \begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{H}_{d_A}(B) = \begin{pmatrix} B(a) \wedge (2 + B(b)) \wedge B(c) \\ B(a) \wedge B(b) \wedge B(c) \\ (3 + B(a)) \wedge (5 + B(b)) \wedge B(c) \end{pmatrix}.$$

Then $d_A(a,) = (0,0,3), d_A(b,) = (2,0,5), d_A(c,) = (0,0,0), A \in \tau_{d_A}$. Moreover, $\tau_{d_A} = \{\mathcal{H}_{d_A}(B) \mid B \in [0,\infty]^X\}$.

(2) Define $d^1_A(x,y) = |A(x) - A(y)|$ as

$$d^1_A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 0 \end{pmatrix}, \mathcal{H}_{d^1_A}(B) = \begin{pmatrix} B(a) \wedge (2 + B(b)) \wedge (3 + B(c)) \\ (2 + B(a)) \wedge B(b) \wedge (5 + B(c)) \\ (3 + B(a)) \wedge (5 + B(b)) \wedge B(c) \end{pmatrix}.$$

Then $d^1_A(a,) = (0,2,3), d^1_A(b,) = (2,0,5), d_A(c,) = (3,5,0), A \in \tau_{d^1_A}$. Moreover, $\tau_{d^1_A} = \{\mathcal{H}_{d^1_A}(B) \mid B \in [0,\infty]^X\}$.

(3) Define d^2_X and $d^2_X \oplus d^2_X(x,z) = \bigwedge_{y\in X}(d^2_X(x,y) + d^2_X(y,z))$ as

$$d^2_X = \begin{pmatrix} 0 & 4 & 1 \\ 7 & 0 & 3 \\ 2 & 9 & 0 \end{pmatrix}, d^2_X \oplus d^2_X = \begin{pmatrix} 0 & 4 & 1 \\ 5 & 0 & 3 \\ 2 & 6 & 0 \end{pmatrix}.$$

Since $d^2_X(b,c) + d^2_X(c,a) = 5 < d^1_X(b,a) = 7$ and $d^2_X(c,a) + d^2_X(a,b) = 6 < d^1_X(b,a) = 9$, d^2_X is not a non-symmetric pseudo-metric.

(4) Define $d^3_X = d^2_X \oplus d^2_X$.

$$d^3_X = \begin{pmatrix} 0 & 4 & 1 \\ 5 & 0 & 3 \\ 2 & 6 & 0 \end{pmatrix}, \mathcal{H}_{d^3_X}(B) = \begin{pmatrix} B(a) \wedge (5 + B(b)) \wedge (2 + B(c)) \\ (4 + B(a)) \wedge B(b) \wedge (6 + B(c)) \\ (1 + B(a)) \wedge (3 + B(b)) \wedge B(c) \end{pmatrix}.$$

Since d^3_X is a non-symmetric pseudo-metric, $d^3_X \oplus d^3_X = d^3_X$. Then $d^3_X(a,) = (0,4,1), d^3_X(b,) = (5,0,3), d^3_X(c,) = (2,6,0) \in \tau_{d^3_X}$. Moreover, $\tau_{d^3_X} = \{\mathcal{H}_{d^3_X}(B) \mid B \in [0,\infty]^X\}$.

4. Categories of Non-symmetric Pseudo-metrics, Lower Approximation Operators and Alexandrov Topologies

Let **LA** be a category with object $(X, \mathcal{H}_X)$ where $\mathcal{H}_X$ is a lower approximation operator with a morphism $f : (X, \mathcal{H}_X) \to (Y, \mathcal{H}_Y)$ such that $f^{\leftarrow}(\mathcal{H}_Y(B)) \leq \mathcal{H}_Y(f^{\leftarrow}(B))$ for all $B \in [0,\infty]^Y$.

Let **NPM** be a category with object (X, d_X) where d_X is a non-symmetric pseudo-metric with a morphism $f : (X, d_X) \to (Y, d_Y)$ such that $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$.

Theorem 4.1. *Two categories* **LA** *and* **NPM** *are isomorphic.*

Proof. Define H : **NPM** $\to$ **LA** as $H(X, d_X) = (X, \mathcal{H}_{d_X})$ where $\mathcal{H}_{d_X}(A)(y) = \bigwedge_{x\in X}(A(x) + d_X(x, y))$ from Theorem 3.3. Let $d_Y(f(x), f(z)) \leq d_X(x, z)$. Then

$$\begin{aligned} f^{\leftarrow}(\mathcal{H}_Y(B))(x) &= \textstyle\bigwedge_{w\in Y}(B(w) + d_Y(w, f(x))) \\ &\leq \textstyle\bigwedge_{z\in X}(B(f(z)) + d_Y(f(z), f(x))) \\ &\leq \textstyle\bigwedge_{z\in X}(f^{\leftarrow}(B)(z) + d_X(z, x)) = \mathcal{H}_Y(f^{\leftarrow}(B))(x). \end{aligned}$$

Hence H is a functor.

Define a functor G : **LA** $\to$ **NPM** as $G(X, \mathcal{H}_X) = (X, d_{\mathcal{H}_X})$ where $d_{\mathcal{H}_X}(x, y) = \mathcal{H}_X(0_x)(y)$ from Theorem 3.6(2). Let $f^{\leftarrow}(\mathcal{H}_Y(B)) \leq \mathcal{H}_Y(f^{\rightarrow}(B))$. Since

$$\begin{aligned} d_{\mathcal{H}_Y}(f(x), f(z)) &= \mathcal{H}_Y(0_{f(x)})(f(y)) = f^{\leftarrow}(\mathcal{H}_Y(0_{f(x)}))(y) \\ &\leq \mathcal{H}_X(f^{\leftarrow}(0_{f(x)}))(y) \leq \mathcal{H}_Y(0_x)(y) = d_X(x, y). \end{aligned}$$

Hence G is a functor. Moreover, by Theorem 3.9, $G(H(X, d_X)) = G(X, \mathcal{H}_{d_X}) = (X, d_{\mathcal{H}_{d_X}}) = (X, d_X)$ and, by Theorem 3.6(2), $H(G(X, \mathcal{H}_X)) = H(X, d_{\mathcal{H}_X}) = (X, \mathcal{H}_{d_{\mathcal{H}_X}}) = (X, \mathcal{H}_X)$. Thus, **LA** and **NPM** are isomorphic. □

Let **ATOP** be a category with object (X, τ_X) where τ_X is an Alexandrov topology with a morphism $f : (X, \tau_X) \to (Y, \tau_Y)$ such that $f^{\leftarrow}(B) \in \tau_X$ for all $B \in \tau_Y$.

Theorem 4.2. *Two categories* **ATOP** *and* **LA** *are isomorphic.*

Proof. Define U : **ATOP** $\to$ **LA** as $U(X,\tau_X)=(X,\mathcal{H}_{\tau_X})$ where $\mathcal{H}_{\tau_X}(A)=\bigvee\{B\in[0,\infty]^X \mid B\leq A, B\in\tau_X\}$ from Theorem 3.8(1). For $B\in\tau_{d_X}$,

$$\begin{aligned}&f^{\leftarrow}(\mathcal{H}_{\tau_Y}(B))=\bigvee\{f^{\leftarrow}(C)\mid C\leq B, C\in\tau_Y\}\\&\leq\bigvee\{f^{\leftarrow}(C)\mid f^{\leftarrow}(C)\leq f^{\leftarrow}(B), f^{\leftarrow}(C)\in\tau_X\}\\&\leq\mathcal{H}_{\tau_X}(f^{\leftarrow}(B)).\end{aligned}$$

Hence U is a functor.

Define W : **LA** $\to$ **ATP** as $W(X,\mathcal{H}_X)=(X,\tau_{\mathcal{H}_X})$ where $\tau_{\mathcal{H}_X}=\{A\in[0,\infty]^X\mid A=\mathcal{H}_X(A)\}$ from Theorem 3.6(1). For $B=\mathcal{H}_Y(B)$, $f^{\leftarrow}(B)\in\tau_{\mathcal{H}_X}$ because

$$f^{\leftarrow}(B)=f^{\leftarrow}(\mathcal{H}_{\tau_Y}(B))\leq\mathcal{H}_{\tau_X}(f^{\leftarrow}(B))\leq f^{\leftarrow}(B).$$

Hence W is a functor. Moreover, by Theorem 3.8(1), $U(W(X,\mathcal{H}_X))=U(X,\tau_{\mathcal{H}_X})=(X,\mathcal{H}_{\tau_{\mathcal{H}_X}})=(X,\mathcal{H}_X)$ and, by Theorem 3.8(1), $W(U(X,\tau_X))=W(X,\mathcal{H}_{\tau_X})=(X,\tau_{\mathcal{H}_{\tau_X}})=(X,\tau_X)$. Thus, **LA** and **NPM** are isomorphic. □

Theorem 4.3. *Two categories* **ATOP** *and* **NPM** *are isomorphic.*

Proof. Define T : **NPM** $\to$ **ATOP** as $T(X,d_X)=(X,\tau_{d_X})$ where $\tau_{d_X}=\{A\in[0,\infty]^X\mid A(x)+d_X(x,y)\geq A(y)\}$ from Theorem 3.5. Let $d_Y(f(x),f(z))\leq d_X(x,z)$. For $B\in\tau_{d_Y}$, we have

$$\begin{aligned}&f^{\leftarrow}(B)(x)+d_X(x,z)\geq B(f(x))+d_Y(f(x),f(z))\\&\geq B(f(z))=f^{\leftarrow}(B)(z).\end{aligned}$$

Hence T is a functor.

Define P : **ATOP** $\to$ **NPM** as $P(X,\tau_X)=(X,d_{\tau_X})$ where $d_{\tau_X}(x,y)=\bigvee_{A\in\tau_X}(A(x)\to A(y))$ from Theorem 3.8(2). Let $f^{\leftarrow}(B)\in\tau_X$ for $B\in\tau_Y$. We have

$$\begin{aligned}&d_{\tau_Y}(f(x),f(y))=\bigvee_{B\in\tau_Y}(B(f(x))\to B(f(x)))\\&=\bigvee_{B\in\tau_Y}(f^{\leftarrow}(B)(x)\to f^{\leftarrow}(B)(x))\\&\leq\bigvee_{A\in\tau_X}(A(x)\to A(y))=d_{\tau_X}(x,y)\end{aligned}$$

Hence P is a functor. Moreover, by Theorem 3.8(2),$T(P(X,\tau_X))=T(X,d_{\tau_X})=(X,\tau_{d_{\tau_X}})=(X,\tau_X)$ and, by Theorem 3.9, $P(T(X,d_X))=P(X,\tau_{d_X})=(X,d_{\tau_{d_X}})=(X,d_X)$ □

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