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EXISTENCE OF COINCIDENCE POINT UNDER GENERALIZED GERAGHTY-TYPE CONTRACTION WITH APPLICATION

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Abstract. We establish coincidence point theorem for S-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. With the help of obtain result, we derive two dimensional results for generalized compatible pair of mappings F, $G : X^2 \to X$. As an application, we obtain the solution of integral equation and also give an example to show the usefulness of our results. Our results improve, sharpen, enrich and generalize various known results.

1. Introduction

The Banach contraction principle is a classical, useful and powerful tool in nonlinear analysis. Weak contractions are generalizations of the Banach contraction, which have been studied by various authors. Several authors studied the existence of fixed point for weak contractions and generalized contractions in partially ordered metric spaces. In particular, Geraghty proved in [8] an interesting generalization of Banach contraction principle which had a lot of applications. For more details one can consult $[1-7]$.

Hussain et al. [9] obtained some coupled coincidence point results with the help of newly defined concept of generalized compatibility. Erhan et al. [7], declared that the results established in Hussain et al. [9] can be deduce from the coincidence point results in the existing literature.

In this paper, we establish coincidence point theorem for S-non-decreasing mappings under Geraghty-type contraction on partially ordered metric spaces. With the help of the obtain result, we indicate the formation of a coupled coincidence point

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theorem of generalized compatible pair of mappings F, $G : X^2 \to X$. As an application, we obtain the solution of integral equation and also give an example to show the fruitfulness of our results. Our results improve, sharpen, enrich and generalize the results of Kadelburg et al. [10] and various known results.

2. Preliminaries

In the sequel, X is a non-empty set. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the n^{th} Cartesian product $X \times X \times ... \times X$ (n times). Let $S : X \to X$ be a mapping. For simplicity, we denote $S(x)$ by Sx where $x \in X$.

Definition 2.1 ([9]). Suppose that F, $G : X^2 \to X$ are two mappings. F is said to be *G-increasing* with respect to $\preceq$ if for all x, y, u, $v \in X$, with $G(x, y) \preceq G(u, v)$ we have $F(x, y) \preceq F(u, v)$.

Definition 2.2 ([9]). Let F, $G : X^2 \to X$ be two mappings. We say that the pair $\{F, G\}$ is *commuting* if $F(G(x, y), G(y, x)) = G(F(x, y), F(y, x))$, for all $x, y \in X$.

Definition 2.3 ([9]). Suppose that F, $G : X^2 \to X$ are two mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of mappings F and G if $F(x, y) = G(x, y)$ and $F(y, x) = G(y, x)$.

Definition 2.4 ([9]). Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is *g-increasing* with respect to $\preceq$ if for any x, $y \in X$,

$$\begin{aligned} gx_1 &\preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \\ gy_1 &\preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.5 ([9]). Let $(X, \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is *increasing* with respect to $\preceq$ if for any x, $y \in X$,

$$\begin{aligned} x_1 &\preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \\ y_1 &\preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.6 ([9]). Let F, $G : X^2 \to X$ are two mappings. We say that the pair $\{F, G\}$ is *generalized compatible* if

$$\begin{aligned}\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0,\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x \in X,\\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y \in X.\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Definition 2.7 ([7])**.** Let $(X,\ \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v,\ \text{ for all } (u,\ v),\ (x,\ y) \in X^2. \tag{2.1}$$

Definition 2.8 ([1])**.** Let $(X,\ d,\ \preceq)$ be a partially ordered metric space. Two mappings $T,\ S : X \to X$ are said to be *O-compatible* if

$$\lim_{n\to\infty} d(STx_n,\ TSx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that $\{Sx_n\}$ is $\preceq$-monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n \in X.$$

Lemma 2.1 ([12])**.** Let $(X,\ d)$ be a metric space. Define $\delta : X^2 \times X^2 \to [0,\ +\infty)$ by

$$\delta((x,\ y),\ (u,\ v)) = \max\{d(x,\ u),\ d(y,\ v)\},\ \text{ for all } (x,\ y),\ (u,\ v) \in X^2.$$

Then δ is metric on X^2 and $(X,\ d)$ is complete if and only if $(X^2,\ \delta)$ is complete.

3. Main Results

In [10], Kadelburg et al. introduced the class Θ of all functions $\theta : [0,\ +\infty) \to [0,\ 1)$ satisfying that for any sequence $\{t_n\}$ of non-negative real numbers $\theta(t_n) \to 1$ implies that $t_n \to 0$.

Now, we will prove our main result.

Theorem 3.1. *Let $(X,\, d,\, \preceq)$ be a partially ordered metric space and let $T,\, S : X \to X$ be two mappings such that T is $(S,\, \preceq)$-non-decreasing, $T(X) \subseteq S(X)$ and there exists $\theta \in \Theta$ such that*

$$d(Tx,\ Ty) \leq \theta(d(Sx,\ Sy))d(Sx,\ Sy), \tag{3.1}$$

for all $x,\, y \in X$ where $Sx \preceq Sy$. There exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$. Also assume that one of the following conditions holds.

(a) $(X,\, d)$ is complete, T and S are continuous and the pair $(T,\, S)$ is O-compatible,

(b) $(S(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular,

(c) $(X,\, d)$ is complete, S is continuous and monotone non-decreasing, the pair $(T,\, S)$ is O-compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular.

Then S and T have a coincidence point.

Proof. Let $x_0 \in X$ is arbitrary. Since $T(X) \subseteq S(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = Sx_1$. Then $Sx_0 \preceq Tx_0 = Sx_1$. As T is $(S,\, \preceq)$-non-decreasing and so $Tx_0 \preceq Tx_1$. Repeating this procedure, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{Sx_n\}$ is $\preceq$-non-decreasing, $Sx_{n+1} = Tx_n \preceq Tx_{n+1} = Sx_{n+2}$ and

$$Sx_{n+1} = Tx_n, \text{ for all } n \geq 0. \tag{3.2}$$

Let $\zeta_n = d(Sx_n,\, Sx_{n+1})$, for all $n \geq 0$. By using contractive condition (3.1), we have

$$d(Sx_{n+1},\ Sx_{n+2}) = d(Tx_n,\ Tx_{n+1}) \leq \theta(d(Sx_n,\ Sx_{n+1}))d(Sx_n,\ Sx_{n+1}), \tag{3.3}$$

which, by the fact that $\theta < 1$, implies

$$d(Sx_{n+1},\ Sx_{n+2}) < d(Sx_n,\ Sx_{n+1}), \text{ that is, } \zeta_{n+1} < \zeta_n \text{ for all } n \geq 0.$$

Thus the sequence $\{\zeta_n\}_{n\geq 0}$ is decreasing. Hence there exists an $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(Sx_n,\ Sx_{n+1}) = \zeta. \tag{3.4}$$

We claim that $\zeta = 0$. If possible, suppose $\zeta > 0$. Then from (3.3), we obtain

$$\frac{\zeta_{n+1}}{\zeta_n} \leq \theta(\zeta_n) < 1.$$

On taking limit as $n \to \infty$, we get

$$\theta(\zeta_n) \to 1 \text{ as } n \to \infty.$$

Using the properties of function θ, we have

$$\zeta_n = d(Sx_n,\ Sx_{n+1}) \to 0 \text{ as } n \to \infty,$$

which contradicts the assumption that $\zeta > 0$. Hence, by (3.4), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(Sx_n,\ Sx_{n+1}) = 0. \tag{3.5}$$

We now claim that $\{Sx_n\}_{n\geq 0}$ is a Cauchy sequence in $(X,\ d)$. Suppose, to the contrary, that the sequence $\{Sx_n\}_{n\geq 0}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}_{n\geq 0}$ with $n(k) > m(k) \geq k$ such that

$$d(Sx_{n(k)},\ Sx_{m(k)}) \geq \varepsilon. \tag{3.6}$$

Let $n(k)$ be the smallest positive integer satisfying (3.6). Then

$$d(Sx_{n(k)-1},\ Sx_{m(k)}) < \varepsilon. \tag{3.7}$$

By using (3.6), (3.7) and triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq r_k = d(Sx_{n(k)},\ Sx_{m(k)}) \\ &\leq d(Sx_{n(k)},\ Sx_{n(k)-1}) + d(Sx_{n(k)-1},\ Sx_{m(k)}) \\ &< d(Sx_{n(k)},\ Sx_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.5), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(Sx_{n(k)},\ Sx_{m(k)}) = \varepsilon. \tag{3.8}$$

By the triangle inequality, we have

$$\begin{aligned} r_k &= d(Sx_{n(k)},\ Sx_{m(k)}) \\ &\leq d(Sx_{n(k)},\ Sx_{n(k)+1}) + d(Sx_{n(k)+1},\ Sx_{m(k)+1}) + d(Sx_{m(k)+1},\ Sx_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + d(Tx_{n(k)},\ Tx_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + \theta(d(Sx_{n(k)},\ Sx_{m(k)}))d(Sx_{n(k)},\ Sx_{m(k)}) \\ &\leq \zeta_{n(k)} + \zeta_{m(k)} + r_k. \end{aligned}$$

This shows that

$$r_k \leq \zeta_{n(k)} + \zeta_{m(k)} + \theta(r_k)r_k \leq \zeta_{n(k)} + \zeta_{m(k)} + r_k.$$

On taking limit as $n \to \infty$ in the above inequality, by using (3.5) and (3.8), we get

$$\theta(r_k) \to 1 \text{ as } n \to \infty.$$

Using the properties of function θ, we obtain

$$r_k = d(Sx_{n(k)},\ Sx_{m(k)}) \to 0 \text{ as } k \to \infty,$$

which implies that

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(Sx_{n(k)},\ Sx_{m(k)}) = 0,$$

which contradicts the fact that $\varepsilon > 0$. Consequently $\{Sx_n\}_{n\geq 0}$ is a Cauchy sequence in X. We claim that T and S have a coincidence point between cases $(a) - (c)$.

Suppose (a) holds, that is, (X, d) is complete, T and S are continuous and the pair (T, S) is O-compatible. Since (X, d) is complete, therefore there exists $z \in X$ such that $\{Sx_n\} \to z$. It follows, from (3.2), that $\{Tx_n\} \to z$. As T and S are continuous, so $\{TSx_n\} \to Tz$ and $\{SSx_n\} \to Sz$. Since the pair (T, S) is O-compatible, we conclude that

$$d(Sz,\ Tz) = \lim_{n\to\infty} d(SSx_{n+1},\ TSx_n) = \lim_{n\to\infty} d(STx_n,\ TSx_n) = 0,$$

that is, z is a coincidence point of S and T.

Suppose now (b) holds, that is, $(S(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. As $\{Sx_n\}_{n\geq 0}$ is a Cauchy sequence in the complete space $(S(X), d)$, so there exists $y \in S(X)$ such that $\{Sx_n\} \to y$. Let $z \in X$ be any point such that $y = Sz$, then $\{Sx_n\} \to Sz$. Also, since $(X, d, \preceq)$ is non-decreasing-regular and $\{Sx_n\}$ is $\preceq$-non-decreasing converging to Sz, therefore we get $Sx_n \preceq Sz$ for all $n \geq 0$. Applying the contractive condition (3.1), we have

$$d(Sx_{n+1},\ Tz) = d(Tx_n,\ Tz) \leq \theta(d(Sx_n,\ Sz))d(Sx_n,\ Sz),$$

which, by the fact $\theta < 1$, implies

$$d(Sx_{n+1},\ Tz) \leq d(Sx_n,\ Sz).$$

Letting $n \to \infty$ in the above inequality and using $\lim_{n\to\infty} Sx_n = Sz$, we get $d(Sz, Tz) = 0$, that is, z is a coincidence point of S and T.

Suppose now that (c) holds, that is, (X, d) is complete, S is continuous and monotone non-decreasing, the pair (T, S) is O-compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete and so there exists $z \in X$ such that $\{Sx_n\} \to z$. It follows, from (3.2), that $\{Tx_n\} \to z$. As S is continuous, then $\{SSx_n\} \to Sz$. Furthermore, since the pair (T, S) is O-compatible, it means that $\{TSx_n\} \to Sz$.

As $(X, d, \preceq)$ is non-decreasing-regular and $\{Sx_n\}$ is $\preceq$-non-decreasing converging to z, we obtain that $Sx_n \preceq z$, which, by the monotonicity of S, implies $SSx_n \preceq Sz$. Thus, by using contractive condition (3.1), we get

$$d(TSx_n,\ Tz) \leq \theta(d(SSx_n,\ Sz))d(SSx_n,\ Sz),$$

which, by the fact $\theta < 1$, implies

$$d(TSx_n,\ Tz) \leq d(SSx_n,\ Sz).$$

On taking $n \to \infty$ and by using $SSx_n \to Sz$ and $TSx_n \to Sz$ as $n \to \infty$, we get $d(Sz,\, Tz) = 0$, that is, z is a coincidence point of S and T. □

Taking $\theta(s) = k$ with $k \in [0,\, 1)$ for all $s \geq 0$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let $(X,\, d,\, \preceq)$ be a partially ordered metric space and let $T,\, S : X \to X$ be two mappings T is $(S,\, \preceq)$-non-decreasing, $T(X) \subseteq S(X)$ and there exists $k \in [0,\, 1)$ such that*

$$d(Tx,\ Ty) \leq kd(Sx,\ Sy),$$

for all $x,\, y \in X$ where $Sx \preceq Sy$. There exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$. Also assume that one of the conditions $(a) - (c)$ of Theorem 3.1 holds. Then S and T have a coincidence point.

4. Coupled Coincidence Point Results

Now, we find the two dimensional version of Theorem 3.1. For this, we shall consider the partially ordered metric space $(X^2,\, \delta,\, \sqsubseteq)$, where δ was defined in Lemma 2.1 and $\sqsubseteq$ was introduced in (2.1). We define the mappings $T_F,\, T_G : X^2 \to X^2$, for all $(x,\, y) \in X^2$, by

$$T_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } T_G(x,\ y) = (G(x,\ y),\ G(y,\ x)).$$

Lemma 4.1 ([2]). *Let $(X,\, d,\, \preceq)$ be a partially ordered metric space and let $F,\, G : X^2 \to X$ be two mappings. Then*

(1) $(X,\, d)$ is complete if and only if $(X^2,\, \delta)$ is complete.

(2) If $(X,\, d,\, \preceq)$ is regular, then $(X^2,\, \delta,\, \sqsubseteq)$ is also regular.

(3) If F is d-continuous, then T_F is δ-continuous.

(4) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G,\, \sqsubseteq)$-non-decreasing.

(5) If there exist two elements $x_0,\, y_0 \in X$ with $G(x_0,\, y_0) \preceq F(x_0,\, y_0)$ and $G(y_0,\, x_0) \succeq F(y_0,\, x_0)$, then there exists a point $(x_0,\, y_0) \in X^2$ such that $T_G(x_0,\, y_0) \sqsubseteq T_F(x_0,\, y_0)$.

(6) For any $x,\, y \in X$, there exist $u,\, v \in X$ such that $F(x,\, y) = G(u,\, v)$ and $F(y,\, x) = G(v,\, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) If the pair $\{F, G\}$ is generalized compatible, then the mappings T_F and T_G are O-compatible in $(X^2, \delta, \sqsubseteq)$.

(8) A point $(x, y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ are two generalized compatible mappings for which there exists $\theta \in \Theta$ such that*

$$
\begin{aligned}
&d(F(x, y), F(u, v)) \\
\leq\ &\theta(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}) \\
&\times \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\},
\end{aligned} \tag{4.1}
$$

for all $x, y, u, v \in X$, with $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Suppose F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements x_0, y_0 $\in X$ with

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).$$

Suppose that for any $x, y \in X$, there exist $u, v \in X$ such that

$$F(x, y) = G(u, v) \text{ and } F(y, x) = G(v, u). \tag{4.2}$$

Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Proof. Let $x, y, u, v \in X$ be such that $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Then by using (4.1), we have

$$
\begin{aligned}
&d(F(x, y), F(u, v)) \\
\leq\ &\theta(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}) \\
&\times \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}.
\end{aligned}
$$

Furthermore $G(y, x) \succeq G(v, u)$ and $G(x, y) \preceq G(u, v)$, the contractive condition (4.1) also assure that

$$
\begin{aligned}
&d(F(y, x), F(v, u)) \\
\leq\ &\theta(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}) \\
&\times \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}.
\end{aligned}
$$

Combining them, we get

$$
\begin{aligned}
(4.3)\qquad & \max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\} \\
\leq\ & \theta(\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}) \\
& \times \max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}).
\end{aligned}
$$

Thus, by using (4.3), we get

$$
\begin{aligned}
& \delta(T_F(x,\ y),\ T_F(u,\ v))) \\
=\ & \delta((F(x,\ y),\ F(y,\ x)),\ (F(u,\ v),\ F(v,\ u))) \\
=\ & \max\{d(F(x,\ y),\ F(u,\ v)),\ d(F(y,\ x),\ F(v,\ u))\} \\
\leq\ & \theta(\max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\}) \\
& \times \max\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\} \\
\leq\ & \theta(\delta(T_G(x,\ y),\ T_G(u,\ v)))\delta(T_G(x,\ y),\ T_G(u,\ v)).
\end{aligned}
$$

It is only require to apply Theorem 3.1 to the mappings $T = T_F$ and $S = T_G$ in the partially ordered metric space $(X^2,\ \delta,\ \sqsubseteq)$ with the help of all items of Lemma 4.1. □

Corollary 4.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X^2 \to X$ are two commuting mappings for which there exists $\theta \in \Theta$ satisfying* (4.1), *for all $x,\ y,\ u,\ v \in X$ with $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Suppose F is G-increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ satisfying (4.2). *Also suppose that either*

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Corollary 4.3. *Let $(X,\ \preceq)$ are a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F is g-increasing with respect to $\preceq$ and there exists $\theta \in \Theta$ such that*

$$
\begin{aligned}
(4.4)\qquad & d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \theta(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \max\{d(gx,\ gu),\ d(gy,\ gv)\},
\end{aligned}
$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F, g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Corollary 4.4. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F is g-increasing with respect to $\preceq$ for which there exists $\theta \in \Theta$ satisfying* (4.4), *for all $x, y, u, v \in X$, with $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F, g\}$ is commuting. Also suppose that either*

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

Corollary 4.5. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ is an increasing mapping with respect to $\preceq$ and there exists $\theta \in \Theta$ such that*

$$d(F(x, y), F(u, v)) \leq \theta(\max\{d(x, u), d(y, v)\}) \max\{d(x, u), d(y, v)\},$$

for all $x, y, u, v \in X$, with $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Then F has a coupled fixed point.

In a similar way, we may state the results analog of Corollary 3.2.

Example 4.1. Let $X=\mathbb{R}$ furnished with the usual metric $d: X^2 \to [0, +\infty)$ with the natural ordering of real numbers $\leq$. Let $F, G: X^2 \to X$ be defined as

$$F(x, y) = \begin{cases} \ln(1+x^2-y^2), & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

and

$$G(x, y) = \begin{cases} x^2-y^2, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Define $\theta: [0, +\infty) \to [0, 1)$ as follows

$$\theta(s) = \begin{cases} \frac{\ln(1+s)}{s}, & s > 0, \\ 0, & s = 0. \end{cases}$$

Firstly, we shall show that the contractive condition of Theorem 4.1 should satisfy by the mappings F and G. Let $x, y, u, v \in X$ such that $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, we have

$$\begin{aligned}
& d(F(x, y), F(u, v)) \\
= \; & |F(x, y) - F(u, v)| \\
= \; & \left|\ln\left(1+x^2-y^2\right) - \ln\left(1+u^2-v^2\right)\right| \\
= \; & \left|\ln \frac{1+x^2-y^2}{1+u^2-v^2}\right| \\
= \; & \left|\ln\left(1+\frac{(x^2-y^2)-(u^2-v^2)}{1+u^2-v^2}\right)\right| \\
\leq \; & \ln\left(1+\left|(x^2-y^2)-(u^2-v^2)\right|\right) \\
\leq \; & \ln(1+|G(x, y)-G(u, v)|) \\
\leq \; & \ln(1+d(G(x, y), G(u, v))) \\
\leq \; & \frac{\ln(1+\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\})}{\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}} \\
& \times \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\} \\
\leq \; & \theta(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}) \\
& \times \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}.
\end{aligned}$$

Thus the contractive condition of Theorem 4.1 is satisfied for all $x, y, u, v \in X$. Furthermore, like in [9], all the other conditions of Theorem 4.1 are satisfied and $z=(0, 0)$ is a coupled coincidence point of F and G.

5. Application to Integral Equations

In this fragment, we study the existence of the solution to a Fredholm nonlinear integral equation. Consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q) + K_2(p,\ q))[f(q,\ x(q)) + g(q,\ x(q))]dq + h(p), \tag{5.1}$$

for all $p \in I = [a,\ b]$.

Let Θ denote the set of all functions $\theta : [0,\ +\infty) \to [0,\ +\infty)$ having the following properties:

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq \ln(p+1)$.

Definition 5.1 ([11])**.** A pair $(\alpha,\ \beta) \in X^2$ with $X = C(I,\ \mathbb{R})$, where $C(I,\ \mathbb{R})$ denote the set of all continuous functions from I to $\mathbb{R}$, is called a *coupled lower-upper solution* of (5.1) if, for all $p \in I$,

$$\begin{aligned}\alpha(p) &\leq \int_a^b K_1(p,\ q)[f(q,\ \alpha(q)) + g(q,\ \beta(q))]dq \\ &\quad + \int_a^b K_2(p,\ q)[f(q,\ \beta(q)) + g(q,\ \alpha(q))]dq + h(p), \\ \beta(p) &\geq \int_a^b K_1(p,\ q)[f(q,\ \beta(q)) + g(q,\ \alpha(q))]dq \\ &\quad + \int_a^b K_2(p,\ q)[f(q,\ \alpha(q)) + g(q,\ \beta(q))]dq + h(p).\end{aligned}$$

Theorem 5.1. *Consider the integral equation* (5.1) *with* $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, f, $g \in C(I \times \mathbb{R},\ \mathbb{R})$ *and* $h \in C(I,\ \mathbb{R})$ *satisfying the following conditions:*

(i) $K_1(p,\ q) \geq 0$ *and* $K_2(p,\ q) \geq 0$ *for all* $p,\ q \in I$.

(ii) *There exist positive numbers* λ, μ *and* $\theta \in \Theta$ *such that for all* $x,\ y \in \mathbb{R}$ *with* $x \succeq y$, *the following conditions hold:*

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\theta(x - y), \tag{5.2}$$

$$0 \leq g(q,\ x) - g(q,\ y) \leq \mu\theta(x - y). \tag{5.3}$$

(iii)

$$\max\{\lambda,\ \mu\} \sup_{p \in I} \int_a^b [K_1(p,\ q) + K_2(p,\ q)]dq \leq \frac{1}{2}. \tag{5.4}$$

Suppose that there exists a coupled lower-upper solution $(\alpha,\ \beta)$ *of* (5.1). *Then the integral equation* (5.1) *has a solution in* $C(I,\ \mathbb{R})$.

Proof. Consider $X = C(I,\ \mathbb{R})$ with the following partial order

$$x \preceq y \Longleftrightarrow x(p) \leq y(p),\ \forall p \in I,$$

for all $x,\ y \in C(I,\ \mathbb{R})$. It is noticeable that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p \in I} |x(p) - y(p)| \,.$$

Define the following partial order on X^2 : for $(x,\ y),\ (u,\ v) \in X^2$,

$$(x,\ y) \preceq (u,\ v) \Longleftrightarrow x(p) \leq u(p) \text{ and } y(p) \geq v(p),\ \forall p \in I.$$

Define $\theta : [0,\ +\infty) \to [0,\ 1)$ as follows

$$\theta(s) = \begin{cases} \frac{\ln(1+s)}{s},\ s > 0, \\ 0,\ s = 0. \end{cases}$$

and the mapping $F : X^2 \to X$ by

$$\begin{aligned} F(x,\ y)(p) &= \int_a^b K_1(p,\ q)[f(q,\ x(q)) + g(q,\ y(q))]dq \\ &\quad + \int_a^b K_2(p,\ q)[f(q,\ y(q)) + g(q,\ x(q))]dq + h(p), \end{aligned}$$

for all $p \in I$. One can easily prove, like in [9], that F is increasing. Then, for all x, y, u, $v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = & \int_a^b K_1(p,\ q)[(f(q,\ x(q)) - f(q,\ u(q))) + (g(q,\ y(q)) - g(q,\ v(q)))]dq \\ & + \int_a^b K_2(p,\ q)[(f(q,\ y(q)) - f(q,\ v(q))) + (g(q,\ x(q)) - g(q,\ u(q)))]dq. \end{aligned}$$

Thus, by using (5.2) and (5.3), we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ & \leq \int_a^b K_1(p,\ q)\left[\lambda\theta\left(x(q) - u(q)\right) + \mu\theta\left(y(q) - v(q)\right)\right]dq \\ & \quad + \int_a^b K_2(p,\ q)\left[\lambda\theta\left(y(q) - v(q)\right) + \mu\theta\left(x(q) - u(q)\right)\right]dq. \end{aligned} \tag{5.5}$$

Since the function θ is non-decreasing and $x \succeq u$, $y \preceq v$, we have

$$\begin{aligned}\theta\,(x(q)-u(q)) &\leq \theta(\sup_{q\in I}|x(q)-u(q)|)=\theta(d(x,\ u)),\\ \theta\,(y(q)-v(q)) &\leq \theta(\sup_{q\in I}|y(q)-v(q)|)=\theta(d(y,\ v)).\end{aligned}$$

Hence by (5.5), we obtain

$$\begin{aligned}&|F(x,\ y)(p)-F(u,\ v)(p)|\\ \leq\ &\int_a^b K_1(p,\ q)[\lambda\theta(d(x,\ u))+\mu\theta(d(y,\ v))]dq\\ &+\int_a^b K_2(p,\ q)[\lambda\theta(d(y,\ v))+\mu\theta(d(x,\ u))]dq.\end{aligned}$$

Now, taking the supremum with respect to p, by using (5.4), we get

$$\begin{aligned}&d(F(x,\ y),\ F(u,\ v))\\ \leq\ &\max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b(K_1(p,\ q)+K_2(p,\ q))dq.[\theta(d(x,\ u))+\theta(d(y,\ v))]\\ \leq\ &\frac{\theta(d(x,\ u))+\theta(d(y,\ v))}{2}.\end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v))\leq\frac{\theta(d(x,\ u))+\theta(d(y,\ v))}{2}. \tag{5.6}$$

Now, since θ is non-decreasing, we have

$$\begin{aligned}\theta(d(x,\ u)) &\leq \theta(\max\{d(x,\ u),\ d(y,\ v)\}),\\ \theta(d(y,\ v)) &\leq \theta(\max\{d(x,\ u),\ d(y,\ v)\}),\end{aligned}$$

which implies, by (ii_θ), that

$$\begin{aligned}\frac{\theta(d(x,\ u))+\theta(d(y,\ v))}{2} &\leq \theta(\max\{d(x,\ u),\ d(y,\ v)\})\\ &\leq \ln(1+\max\{d(x,\ u),\ d(y,\ v)\}).\end{aligned}$$

Thus, by (5.6), we have

$$d(F(x,\ y),\ F(u,\ v))\leq\ln(1+\max\{d(x,\ u),\ d(y,\ v)\}). \tag{5.7}$$

Now, by (5.7), we have

$$\begin{aligned}
& d(F(x,\ y),\ F(u,\ v)) \\
\leq\ & \ln(1+\max\{d(x,\ u),\ d(y,\ v)\}) \\
\leq\ & \frac{\ln(1+\max\{d(x,\ u),\ d(y,\ v)\})}{\max\{d(x,\ u),\ d(y,\ v)\}} \times \max\{d(x,\ u),\ d(y,\ v)\} \\
\leq\ & \theta(\max\{d(x,\ u),\ d(y,\ v)\}) \max\{d(x,\ u),\ d(y,\ v)\}),
\end{aligned}$$

which is the contractive condition of Corollary 4.5. Let $(\alpha,\ \beta) \in X^2$ be a coupled upper-lower solution of (5.1), then we have $\alpha(p) \leq F(\alpha,\ \beta)(p)$ and $\beta(p) \geq F(\ \beta,\ \alpha)(p)$, for all $p \in I$. Thus all the hypothesis of Corollary 4.5 are satisfied. Consequently, F has a coupled fixed point $(x,\ y) \in X^2$ which is the solution of integral equation (5.1) in $X = C(I,\ \mathbb{R})$.

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