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APPLICATIONS OF SUBORDINATION PRINCIPLE FOR ANALYTIC FUNCTIONS CONCERNED WITH ROGOSINSKI'S LEMMA

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Abstract. In this paper, we improve a new boundary Schwarz lemma, for analytic functions in the unit disk. For new inequalities, the results of Rogosinski's lemma, Subordinate principle and Jack's lemma were used. Moreover, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

1. Introduction

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $\mathtt{E} = \{z : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows:

Let $\mathtt{E}$ be the unit disc in the complex plane $\mathbb{C}$. Let $f : \mathtt{E} \to \mathtt{E}$ be an analytic function with $f(0) = 0$. Under these conditions, $|f(z)| \leq |z|$ for all $z \in \mathtt{E}$ and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation; that is $f(z) = ze^{i\theta}$, θ real ([5], p.329). A sharpened version of this is Rogosinski's Lemma [11], which say that for all $z \in \mathtt{E}$

$$|f(z) - a_1| \leq r_1,$$

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where

$$a_1 = \frac{zf'(0)\left(1-|z|^2\right)}{1-|z|^2|f'(0)|^2} \text{ and } r_1 = \frac{|z|^2\left(1-|f'(0)|^2\right)}{1-|z|^2|f'(0)|^2}.$$

We will use the following definition and lemma to prove our result [5, 6].

Definition 1.1 (Subordination Principle)**.** Let f and g be analytic functions in $\mathtt{E}$. A function is said to be *subordinate* to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathtt{E}$ with $\omega(0)=0$, $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$.

Lemma 1.2 (Jack's Lemma)**.** *Let $f(z)$ be a non-constant analytic function in $\mathtt{E}$ with $f(0)=0$. If*

$$|f(z_0)| = \max\{|f(z)| : |z| \le |z_0|\},$$

then there exists a real number $k \ge 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = 1 + c_1 z + c_2 z^2 + ...$ that are analytic in $\mathtt{E}$. Also, let $\mathcal{H}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\frac{zf'(z)}{(f(z))^2} \prec \frac{\left(1+e^{-2i\beta}\right)z}{(1+ze^{-2i\beta})^2}, z \in \mathtt{E}, \ |\beta| < \frac{\pi}{2}.$$

The certain anaytic functions which are in the class of $\mathcal{H}$ on the unit disc $\mathtt{E}$ are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{H}$ by applying Jack's Lemma.

Let $f(z) \in \mathcal{H}$ and consider the following function

$$\phi(z) = \frac{f(z)-1}{f(z)+e^{-2i\beta}}.$$

It is an analytic function in $\mathtt{E}$ and $\phi(0)=0$. Now, let us show that $|\phi(z)|<1$ in $\mathtt{E}$. From the function $\phi(z)$, we have

$$\frac{zf'(z)}{(f(z))^2} = \frac{\left(1+e^{-2i\beta}\right)z\phi'(z)}{(1+\phi(z)e^{-2i\beta})^2}.$$

We suppose that there exists a $z_0 \in \mathtt{E}$ such that

$$\max_{|z|\le|z_0|} |\phi(z)| = |\phi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\phi(z_0) = e^{i\theta} \text{ and } \frac{z_0\phi'(z_0)}{\phi(z_0)} = k.$$

Therefore, we have that

$$\frac{z_0 f'(z_0)}{(f(z_0))^2} = \frac{\left(1+e^{-2i\beta}\right) z_0\phi'(z_0)}{\left(1+\phi(z_0)e^{-2i\beta}\right)^2} = \frac{\left(1+e^{-2i\beta}\right) k\phi(z_0)}{\left(1+\phi(z_0)e^{-2i\beta}\right)^2} = \frac{\left(1+e^{-2i\beta}\right) ke^{i\theta}}{\left(1+e^{i\theta}e^{-2i\beta}\right)^2}$$

which is not contained in $\frac{\left(1+e^{-2i\beta}\right)\mathtt{E}}{\left(1+\mathtt{E}e^{-2i\beta}\right)^2}$ since $|\phi(z_0)| = 1$ and $k \geq 1$. This contradicts the $f(z) \in \mathcal{H}$. This means that there is no point $z_0 \in \mathtt{E}$ such that $\max_{|z|\leq|z_0|} |\phi(z)| = |\phi(z_0)| = 1$. Hence, we take $|\phi(z)| < 1$ in $\mathtt{E}$. From the Schwarz lemma, we obtain

$$\phi(z) = \frac{f(z)-1}{f(z)+e^{-2i\beta}} = \frac{c_1 z + c_2 z^2 + \ldots}{1+e^{-2i\beta}+c_1 z + c_2 z^2 + \ldots}$$

$$\frac{\phi(z)}{z} = \frac{c_1 + c_2 z + \ldots}{1+e^{-2i\beta}+c_1 z + c_2 z^2 + \ldots},$$

$$\left|\phi'(0)\right| = \frac{|c_1|}{\left|1+e^{-2i\beta}\right|} \leq 1$$

and

$$|c_1| = \left|f'(0)\right| \leq \left|1+e^{-2i\beta}\right| = 2\cos\beta.$$

Lemma 1.3. *If $f(z) \in \mathcal{H}$, then we have the inequality*

$$\left|f'(0)\right| \leq 2\cos\beta. \tag{1.1}$$

This result is sharp and the extremal function is

$$f(z) = \frac{1+ze^{-2i\beta}}{1-z}.$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Schwarz lemma has several applications in the field of electrical and electronics engineering. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and nulti-notch filter design in signal processing [14, 15]. Others of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows:

If f extends continuously to some boundary point c with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then $|f'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. In addition to conditions of the boundary Schwarz Lemma, if f fixes the point zero, that is $f(0) = 0$, then the inequality

$$|f'(c)| \geq \frac{2}{1+|f'(0)|} \tag{1.2}$$

is obtained [13]. Inequality (1.2) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 7, 8, 9, 11, 12, 13, 14, 15, 16]. Mercer [10] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [11]. In addition, he obtain a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [12].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [17])

Lemma 1.4 (Julia-Wolff lemma)**.** *Let f be an analytic function in $\mathtt{E}$, $f(0) = 0$ and $f(\mathtt{E}) \subset \mathtt{E}$. If, in addition, the function f has an angular limit $f(c)$ at $c \in \partial\mathtt{E}$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

In this study, the modulus of the angular derivative of the $f(z)$ function has been considered from below at the boundary point in the unit disc. In this evaluation, $f(c) = \frac{1-e^{-2i\beta}}{2}$ condition has been taken into account.

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{H}$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. We also show that these estimations are sharp.

Theorem 2.1. *Let $f(z) \in \mathcal{H}$. Assume that, for some $c \in \partial\mathtt{E}$, f has an angular limit $f(c)$ at c, $f(c) = \frac{1-e^{-2i\beta}}{2}$. Then we have the inequality*

$$|f'(c)| \geq \frac{\cos\beta}{2}. \tag{2.1}$$

Moreover, the equality in (2.1) occurs for the function

$$f(z) = \frac{1+ze^{-2i\beta}}{1-z}.$$

Proof. Let

$$\phi(z)=\frac{f(z)-1}{f(z)+e^{-2i\beta}}.$$

$\phi(z)$ is an analytic function in $\mathtt{E}$, $\phi(0)=0$ and $|\phi(z)|<1$ for $z\in\mathtt{E}$. In addition, we take $|\phi(c)|=1$ for $c\in\partial\mathtt{E}$ and $f(c)=\frac{1-e^{-2i\beta}}{2}$.

From the definition of $\phi(z)$, with the simple calculations, we get

$$\phi'(z)=\frac{\left(1+e^{-2i\beta}\right)f'(z)}{\left(f(z)+e^{-2i\beta}\right)^2}.$$

Therefore, from on the boundary Schwarz lemma, we obtain

$$\begin{aligned}1 \quad &\leq \quad \left|\phi'(c)\right|=\left|\frac{\left(1+e^{-2i\beta}\right)f'(c)}{\left(f(c)+e^{-2i\beta}\right)^2}\right|=\left|\frac{\left(1+e^{-2i\beta}\right)f'(c)}{\left(\frac{1-e^{-2i\beta}}{2}+e^{-2i\beta}\right)^2}\right|\\ &= \quad \frac{4}{\left|1+e^{-2i\beta}\right|}\left|f'(c)\right|=\frac{2}{\cos\beta}\left|f'(c)\right|\end{aligned}$$

and

$$\left|f'(c)\right|\geq\frac{\cos\beta}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=\frac{1+ze^{-2i\beta}}{1-z}.$$

Then

$$f'(z)=\frac{1+e^{-2i\beta}}{(1-z)^2}$$

and

$$\left|f'(-1)\right|=\frac{\left|1+e^{-2i\beta}\right|}{4}=\frac{\cos\beta}{2}.$$

□

The inequality (2.1) can be strengthened as below by taking into account $c_1=f'(0)$ which is first coefficient in the expansion of the function $f(z)$.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f'(c)\right|\geq\frac{2\cos^2\beta}{2\cos\beta+\left|f'(0)\right|}. \tag{2.2}$$

The inequality (2.2) is sharp with equality for the function

$$f(z)=\frac{1+ze^{-2i\beta}}{1-z}.$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Therefore, from Rogosinski's Lemma, we obtain

$$|\phi(z) - a_1| \leq r_1,$$

where

$$a_1 = \frac{z\phi'(0)\left(1-|z|^2\right)}{1-|z|^2|\phi'(0)|^2} \quad \textit{and } r_1 = \frac{|z|^2\left(1-|\phi'(0)|^2\right)}{1-|z|^2|\phi'(0)|^2}.$$

Without loss of generality, we will assume that $c = 1$. Thus, we obtain

$$\begin{aligned}
\left|\frac{\phi(z)-1}{z-1}\right| &\geq \frac{1-|a_1|-r_1}{1-|z|} = \frac{1-\frac{|z||\phi'(0)|\left(1-|z|^2\right)}{1-|z|^2|\phi'(0)|^2} - \frac{|z|^2\left(1-|\phi'(0)|^2\right)}{1-|z|^2|\phi'(0)|^2}}{1-|z|} \\
&= \frac{1-|z|^2|\phi'(0)|^2 - |z||\phi'(0)|\left(1-|z|^2\right) - |z|^2\left(1-|\phi'(0)|^2\right)}{(1-|z|)\left(1-|z|^2|\phi'(0)|^2\right)} \\
&= \frac{\left(1-|z|^2\right)(1-|z||\phi'(0)|)}{(1-|z|)\left(1-|z|^2|\phi'(0)|^2\right)} \\
&= \frac{1+|z|}{1+|z||\phi'(0)|}.
\end{aligned}$$

Passing to the angular limit in the last inequality yields

$$\left|\phi'(1)\right| \geq \frac{2}{1+|\phi'(0)|}.$$

Since

$$\left|\phi'(0)\right| = \frac{|f'(0)|}{2\cos\beta}$$

and

$$\left|\phi'(1)\right| = \frac{2}{\cos\beta}\left|f'(1)\right|$$

we take

$$\frac{2}{\cos\beta}\left|f'(1)\right| \geq \frac{2}{1+\frac{|f'(0)|}{2\cos\beta}} = \frac{4\cos\beta}{2\cos\beta+|f'(0)|}$$

and

$$\left|f'(1)\right| \geq \frac{2\cos^2\beta}{2\cos\beta+|f'(0)|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{1+ze^{-2i\beta}}{1-z}.$$

Then, we have

$$\left|f'(-1)\right| = \frac{\cos\beta}{2}.$$

On the other hand whereas, we obtain

$$\begin{aligned} 1+c_1z+c_2z^2+... &= \frac{1+ze^{-2i\beta}}{1-z}, \\ c_1z+c_2z^2+... &= \frac{1+ze^{-2i\beta}}{1-z}-1=\frac{z\left(1+e^{-2i\beta}\right)}{1-z} \end{aligned}$$

and

$$c_1+c_2z+...=\frac{1+e^{-2i\beta}}{1-z}.$$

Passing to limit ($z\to 0$) in the last equality yields $|c_1|=2\cos\beta$. Therefore, we obtain

$$\frac{2\cos^2\beta}{2\cos\beta+|f'(0)|}=\frac{\cos\beta}{2}.$$

□

In the following theorem, inequality (2.2) has been strengthened by adding the consecutive terms c_1 and c_2 of $f(z)$ function.

Theorem 2.3. *Let $f(z)\in\mathcal{H}$. Assume that, for some $c\in\partial\mathtt{E}$, f has an angular limit $f(c)$ at c, $f(c)=\frac{1-e^{-2i\beta}}{2}$. Then we have the inequality*

$$\left|f'(c)\right|\geq\frac{\cos\beta}{2}\left(1+\frac{2\left(2\cos\beta-|c_1|\right)^2}{4\cos^2\beta-|c_1|^2+2\left|\left(1+e^{-2i\beta}\right)c_2-c_1^2\right|}\right). \tag{2.3}$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$h(z)=\frac{\phi(z)}{z}$$

and

$$\vartheta(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

The function $\vartheta(z)$ is analytic in $\mathtt{E}$, $\vartheta(0)=0$, $|\vartheta(z)|<1$ for $|z|<1$ and

$$\vartheta'(0)=\frac{h'(0)}{\left(1-|h(0)|^2\right)}=\frac{\phi''(0)}{2\left(1-|\phi'(0)|^2\right)}.$$

From Rogosinski's Lemma and [8, 9], we have

$$|\phi(z)-a_2|\leq r_2, \tag{2.4}$$

where

$$a_2=\frac{z\,|\phi'(0)|\left(1-\tau^2\right)}{1-\tau^2\,|\phi'(0)|^2},\ r_2=\frac{\tau\,|z|\left(1-|\phi'(0)|^2\right)}{1-\tau^2\,|\phi'(0)|^2},\quad \tau=|z|\,\frac{|z|+|\vartheta'(0)|}{1+|z|\,|\vartheta'(0)|}.$$

Without loss of generality, we will assume that $c=1$. Thus, from (2.4), we obtain

$$
\begin{aligned}
\left|\frac{\phi(z)-1}{z-1}\right| &\geq \frac{1-|a_2|-r_2}{1-|z|}=\frac{1-\frac{|z||\phi'(0)|\left(1-\tau^2\right)}{1-\tau^2|\varphi'(0)|^2}-\frac{\tau|z|\left(1-|\phi'(0)|^2\right)}{1-\tau^2|\phi'(0)|^2}}{1-|z|} \\
&= \frac{1-\tau^2\left|\phi'(0)\right|^2-|z|\left|\phi'(0)\right|\left(1-\tau^2\right)-\tau|z|\left(1-\left|\phi'(0)\right|^2\right)}{(1-|z|)\left(1-\tau^2\left|\phi'(0)\right|^2\right)} \\
&= \frac{\left(1-\tau\left|\phi'(0)\right|\right)\left(1+\left|\phi'(0)\right|-|z|\left|\phi'(0)\right|-\tau|z|\right)}{(1-|z|)\left(1-\tau^2\left|\phi'(0)\right|^2\right)} \\
&= \frac{1+\tau\left|\phi'(0)\right|-|z|\left|\phi'(0)\right|-\tau|z|}{(1-|z|)\left(1+\tau\left|\phi'(0)\right|\right)}.
\end{aligned}
$$

Since $\tau=|z|\frac{|z|+|\vartheta'(0)|}{1+|z||\vartheta'(0)|}$, we take

$$
\begin{aligned}
\left|\frac{\phi(z)-1}{z-1}\right| &\geq \frac{1+|\phi'(0)||z|\frac{|z|+|\vartheta'(0)|}{1+|z||\vartheta'(0)|}-|z||\varphi'(0)|-|z||z|\frac{|z|+|\vartheta'(0)|}{1+|z||\vartheta'(0)|}}{(1-|z|)\left(1+|\phi'(0)||z|\frac{|z|+|\vartheta'(0)|}{1+|z||\vartheta'(0)|}\right)} \\
&= \frac{1-|z|^3+|z||\vartheta'(0)|(1-|z|)-|\phi'(0)||z|(1-|z|)+|z||\phi'(0)||\vartheta'(0)|(1-|z|)}{(1-|z|)\left(1+|z||\vartheta'(0)|+|\phi'(0)||z|^2+|z||\phi'(0)||\vartheta'(0)|\right)} \\
&= \frac{1+|z|+|z|^2+|z||\vartheta'(0)|-|\phi'(0)||z|+|z||\phi'(0)||\vartheta'(0)|}{1+|z||\vartheta'(0)|+|\phi'(0)||z|^2+|z||\phi'(0)||\vartheta'(0)|}.
\end{aligned}
$$

Passing to the angular limit in the last inequality yields

$$
\begin{aligned}
\left|\phi'(1)\right| &\geq \frac{3+\left|\vartheta'(0)\right|-\left|\phi'(0)\right|+\left|\phi'(0)\right|\left|\vartheta'(0)\right|}{1+\left|\vartheta'(0)\right|+\left|\phi'(0)\right|+\left|\phi'(0)\right|\left|\vartheta'(0)\right|} \\
&= \frac{3+\left|\vartheta'(0)\right|-\left|\phi'(0)\right|+\left|\phi'(0)\right|\left|\vartheta'(0)\right|}{\left(1+\left|\vartheta'(0)\right|\right)\left(1+\left|\phi'(0)\right|\right)}.
\end{aligned}
$$

A little manipulation gives

$$
\begin{aligned}
\left|\phi'(1)\right| &\geq 1+\frac{2\left(1-\left|\phi'(0)\right|\right)^2}{\left(1+\left|\vartheta'(0)\right|\right)\left(1-\left|\phi'(0)\right|^2\right)} \\
&= 1+\frac{4\left(1-\left|\phi'(0)\right|\right)^2}{2\left(1-\left|\phi'(0)\right|^2\right)+\left|\phi''(0)\right|}.
\end{aligned}
$$

Since

$$
\left|\phi'(0)\right|=\frac{|c_1|}{2\cos\beta},
$$

$$
\left|\phi''(0)\right|=\frac{\left|\left(1+e^{-2i\beta}\right)c_2-c_1^2\right|}{2\cos^2\beta}
$$

and

$$
\left|\phi'(1)\right|=\frac{2}{\cos\beta}\left|f'(1)\right|,
$$

we obtain

$$\begin{aligned}\left|f'(1)\right| &\geq \frac{\cos\beta}{2}\left(1+\frac{4\left(1-\frac{|c_1|}{2\cos\beta}\right)^2}{2\left(1-\left(\frac{|c_1|}{2\cos\beta}\right)^2\right)+\frac{\left|\left(1+e^{-2i\beta}\right)c_2-c_1^2\right|}{2\cos^2\beta}}\right)\\ &= \frac{\cos\beta}{2}\left(1+\frac{2\left(2\cos\beta-|c_1|\right)^2}{4\cos^2\beta-|c_1|^2+2\left|\left(1+e^{-2i\beta}\right)c_2-c_1^2\right|}\right).\end{aligned}$$

□

If $f(z)-1$ have zeros different from $z=0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{H}$. Assume that, for some $c\in\partial\mathtt{E}$, f has an angular limit $f(c)$ at c, $f(c)=\frac{1-e^{-2i\beta}}{2}$. Let $a_1,a_2,...,a_n$ be zeros of the function $f(z)-1$ in $\mathtt{E}$ that are different from zero. Then we have the inequality*

$$|f'(c)|\geq\frac{\cos\beta}{2}\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|c-a_i|^2}\right.$$

(2.5)

$$\left.+\frac{2\left(2\cos\beta\prod_{i=1}^{n}|a_i|-|c_1|\right)^2}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2+\prod_{i=1}^{n}|a_i|\left|(1+e^{-2i\beta})\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}\right).$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1 and $a_1,a_2,...,a_n$ be zeros of the function $f(z)-1$ in $\mathtt{E}$ that are different from zero. Let

$$B(z)=z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

$B(z)$ is an analytic function in $\mathtt{E}$ and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in\mathtt{E}$, we have $|\phi(z)|\leq|B(z)|$. Consider the function

$$h(z)=\frac{\phi(z)}{B(z)}=\frac{f(z)-1}{f(z)+e^{-2i\beta}}\frac{1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}$$

$$\begin{aligned} &= \frac{c_1 z + c_2 z^2 + ...}{1 + e^{-2i\beta} + c_1 z + c_2 z^2 + ...} \frac{1}{z \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}} \\ &= \frac{c_1 + c_2 z + ...}{1 + e^{-2i\beta} + c_1 z + c_2 z^2 + ...} \frac{1}{\prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}}. \end{aligned}$$

$h(z)$ is analytic in $\mathtt{E}$ and $|h(z)| < 1$ for $z \in \mathtt{E}$. In particular, we have

$$|h(0)| = \frac{|c_1|}{2\cos\beta \prod_{i=1}^{n} |a_i|}$$

and

$$|h'(0)| = \frac{\left| \left(1 + e^{-2i\beta}\right) \left(c_2 + c_1 \sum_{i=1}^{n} \frac{1 - |a_i|^2}{a_i} \right) - c_1^2 \right|}{4\cos^2\beta \prod_{i=1}^{n} |a_i|}.$$

In addition, with the simple calculations, we take

$$\frac{c\phi'(c)}{\phi(c)} = |\phi'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)}$$

and

$$|B'(c)| = 1 + \sum_{i=1}^{n} \frac{1 - |a_i|^2}{|c - a_i|^2}.$$

The composite function

$$m(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)} h(z)}$$

is analytic in the unit disc $\mathtt{E}$, $m(0) = 0$, $|m(z)| < 1$ for $z \in \mathtt{E}$ and $|m(c)| = 1$ for $c \in \partial\mathtt{E}$. From (1.2), we obtain

$$\begin{aligned} \frac{2}{1 + |m'(0)|} &\leq |m'(c)| = \frac{1 + |h(0)|^2}{\left|1 - \overline{h(0)} h(b)\right|^2} |h'(c)| \\ &\leq \frac{1 + |h(0)|}{1 - |h(0)|} \left\{ |\phi'(c)| - |B'(c)| \right\}. \end{aligned}$$

Since

$$\left|m'(0)\right| = \frac{|h'(0)|}{1-|h(0)|^2} = \frac{\dfrac{\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}{4\cos^2\beta\prod_{i=1}^{n}|a_i|}}{1-\left(\dfrac{|c_1|}{2\cos\beta\prod_{i=1}^{n}|a_i|}\right)^2}$$

$$= \prod_{i=1}^{n}|a_i|\frac{\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2},$$

we get

$$\frac{2}{1+\prod_{i=1}^{n}|a_i|\dfrac{\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2}} \le$$

$$\frac{1+\dfrac{|c_1|}{2\cos\beta\prod_{i=1}^{n}|a_i|}}{1-\dfrac{|c_1|}{2\cos\beta\prod_{i=1}^{n}|a_i|}}\left\{\frac{2}{\cos\beta}|f'(c)|-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|c-a_i|^2}\right\},$$

$$\frac{2\left(4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2\right)}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2+\prod_{i=1}^{n}|a_i|\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}$$

$$\le \frac{2\cos\beta\prod_{i=1}^{n}|a_i|+|c_1|}{2\cos\beta\prod_{i=1}^{n}|a_i|-|c_1|}\left\{\frac{2}{\cos\beta}|f'(c)|-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|c-a_i|^2}\right\},$$

$$\frac{2\left(2\cos\beta\prod_{i=1}^{n}|a_i|-|c_1|\right)^2}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2+\prod_{i=1}^{n}|a_i|\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}$$

$$\leq \tfrac{2}{\cos\beta}\left|f'(c)\right|-1-\sum_{i=1}^{n}\frac{1-|a_i|^2}{|c-a_i|^2}$$

and

$$|f'(c)|\geq \tfrac{\cos\beta}{2}\left(1+\sum_{i=1}^{n}\frac{1-|a_i|^2}{|c-a_i|^2}\right.$$

$$\left.+\frac{2\left(2\cos\beta\prod_{i=1}^{n}|a_i|-|c_1|\right)^2}{4\cos^2\beta\left(\prod_{i=1}^{n}|a_i|\right)^2-|c_1|^2+\prod_{i=1}^{n}|a_i|\left|\left(1+e^{-2i\beta}\right)\left(c_2+c_1\sum_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right)-c_1^2\right|}\right). \quad \square$$

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